Module 8: Angular Momentum - II

8.1 $L_{\pm} = L_x \pm iL_y = \hbar e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta}+i\cot\frac{\partial}{\partial\phi}\right]; \quad Y_{1,-1} = C\ L_{-}Y_{1,0}$. The constant C is given by

(a) $C=\frac{1}{\hbar\sqrt{2}}$

(b) $C=\frac{1}{\sqrt{2}}$

(c) $C=\frac{1}{\hbar\sqrt{6}}$

(d) $C=\frac{1}{\sqrt{6}}$

[Answer (a)]

8.2 $L_{\pm} = L_x \pm iL_y = \hbar e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta}+i\cot\frac{\partial}{\partial\phi}\right]; \quad Y_{3,2} = C\ L_{+}Y_{3,1}$. The constant C is given by

(a) $C=\frac{1}{\hbar\sqrt{10}}$

(b) $C=\frac{1}{\sqrt{10}}$

(c) $C=\frac{1}{\hbar\sqrt{12}}$

(d) $C=\frac{1}{\sqrt{12}}$

[Answer (a)]

8.3 For $j=\frac{1}{2}$; $J_x=\frac{1}{2}\hbar\sigma_x$, $J_y=\frac{1}{2}\hbar\sigma_y$ and $J_z=\frac{1}{2}\hbar\sigma_z$ where σ_x,σ_y and σ_z are Pauli spin matrices.

Evaluate $\langle 2|J_x|1\rangle$

(a) $\langle 2|J_x|1\rangle = -\frac{1}{2}\hbar$

(b) $\langle 2|J_x|1\rangle = \frac{1}{2}\hbar$

(c) $\langle 2|J_x|1\rangle = -\hbar$

(d) $\langle 2|J_x|1\rangle = +i\hbar$

[Answer (b)]

8.4 $|j,m>$ are simultaneous eigenkets of J^2 and J_z. Let $|1\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle$ and $|2\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle$. Evaluate $\langle 1|J_y|2\rangle$.

(a) $\langle 1|J_y|2\rangle = +\frac{i\hbar}{2}$

(b) $\langle 1|J_y|2\rangle = -\frac{i\hbar}{2}$

(c) $\langle 1|J_y|2\rangle = +\frac{\hbar}{2}$

(d) $\langle 1|J_y|2\rangle = -\frac{\hbar}{2}$

[Answer (b)]

8.5 $|j,m>$ are simultaneous eigenkets of J^2 and J_z. Let $|1\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle$ and $|2\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle$. Evaluate $\langle 2|J^2|2\rangle$.

(a) $\langle 2|J^2|2\rangle = \frac{3}{4}\hbar^2$

(b) $\langle 2|J^2|2\rangle = -\frac{3}{4}\hbar^2$

(c) $\langle 2|J^2|2\rangle = \frac{1}{2}\hbar^2$

(d) $\langle 2|J^2|2\rangle = -\frac{1}{2}\hbar^2$

[Answer (a)]

8.6 For $j=\frac{1}{2}$; $J_x=\frac{1}{2}\hbar\sigma_x$, $J_y=\frac{1}{2}\hbar\sigma_y$ and $J_z=\frac{1}{2}\hbar\sigma_z$ where σ_x,σ_y and σ_z are Pauli spin matrices. What are the eigenvalues of J_y?

(a) The eigenvalues of J_y are $\pm\hbar$

(b) The eigenvalues of J_y are $\pm\frac{i}{2}\hbar$

(c) The eigenvalues of J_y are $\pm\frac{1}{2}\hbar$

(d) The eigenvalues of J_y are $\pm i\hbar$

[Answer (c)]

8.7 Assume $j=\frac{3}{2}$. The kets $|1\rangle=\left|\frac{3}{2},\frac{3}{2}\right\rangle$, $|2\rangle=\left|\frac{3}{2},\frac{1}{2}\right\rangle$, $|3\rangle=\left|\frac{3}{2},-\frac{1}{2}\right\rangle$ and $|4\rangle=\left|\frac{3}{2},-\frac{3}{2}\right\rangle$ are simultaneous eigenkets of J^2 and J_z. Which one of the following answers would be completely correct

(a) $J^2|1\rangle=\frac{3}{2}\hbar^2|1\rangle$ **&** $J_z|1\rangle=\frac{3}{2}\hbar|1\rangle$

(b) $J^2|2\rangle=\frac{15}{4}\hbar^2|2\rangle$ **&** $J_z|2\rangle=\frac{3}{2}\hbar|2\rangle$

(c) $J^2|3\rangle=\frac{15}{4}\hbar^2|3\rangle$ **&** $J_z|3\rangle=\frac{1}{2}\hbar|1\rangle$

(d) $J^2|4\rangle=\frac{15}{4}\hbar^2|4\rangle$ **&** $J_z|4\rangle=-\frac{3}{2}\hbar|4\rangle$

[Answer (b)]

8.8 Assume $j = \frac{3}{2}$. The kets $|1\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle$, $|2\rangle = \left|\frac{3}{2},\frac{1}{2}\right\rangle$, $|3\rangle = \left|\frac{3}{2},-\frac{1}{2}\right\rangle$ and $|4\rangle = \left|\frac{3}{2},-\frac{3}{2}\right\rangle$ are simultaneous eigenkets of J^2 and J_z. Evaluate $\langle 2|J_y|1\rangle$

(a) $\langle 2|J_y|1\rangle = -\frac{i\sqrt{3}}{2}\hbar$

(b) $\langle 2|J_y|1\rangle = \frac{i\sqrt{3}}{2}\hbar$

(c) $\langle 2|J_y|1\rangle = \frac{\sqrt{3}}{2}\hbar$

(d) $\langle 2|J_y|1\rangle = -\frac{\sqrt{3}}{2}\hbar$

[Answer (b)]

8.9 $\Phi(j_1, j_2, j, m)$ are simultaneous eigenfunctions of J_1^2, J_2^2, J^2 and J_z;
$\Psi(j_1, j_2, m_1, m_2)$ are simultaneous eigenfunctions of J_1^2, J_2^2, J_{1z} and J_{2z}. Now

$$\Phi\left(1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right) = \Psi\left(1,\frac{1}{2},1,\frac{1}{2}\right)$$

Using above, we get

$$\Phi\left(1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right) = C_1 \ \Psi\left(1,\frac{1}{2},0,\frac{1}{2}\right) + C_2 \ \Psi\left(1,\frac{1}{2},-1,\frac{1}{2}\right)$$

(a) $C_1 = \sqrt{\frac{3}{5}}$ and $C_2 = \sqrt{\frac{2}{5}}$

(b) $C_1 = \sqrt{\frac{2}{3}}$ and $C_2 = \sqrt{\frac{1}{3}}$

(c) $C_1 = \sqrt{\frac{1}{2}}$ and $C_2 = \sqrt{\frac{1}{2}}$

(d) $C_1 = \sqrt{\frac{3}{4}}$ and $C_2 = \sqrt{\frac{1}{4}}$

[Answer (b)]

8.10 $\Phi(j_1, j_2, j, m)$ are simultaneous eigenfunctions of J_1^2, J_2^2, J^2 and J_z; $\Psi(j_1, j_2, m_1, m_2)$ are simultaneous eigenfunctions of J_1^2, J_2^2, J_{1z} and J_{2z}. Now

$$\Phi\left(2,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right) = C_1\ \Psi\left(2,\frac{1}{2},0,\frac{1}{2}\right) + C_2\ \Psi\left(2,\frac{1}{2},1,-\frac{1}{2}\right)$$

$j =$	$m_2 = 1/2$	$m_2 = -1/2$
$j_1 + \frac{1}{2}$	$\sqrt{\frac{j_1 + m + 1/2}{2j_1 + 1}}$	$\sqrt{\frac{j_1 - m + 1/2}{2j_1 + 1}}$
$j_1 - \frac{1}{2}$	$-\sqrt{\frac{j_1 - m + 1/2}{2j_1 + 1}}$	$\sqrt{\frac{j_1 + m + 1/2}{2j_1 + 1}}$

Use the Table for Clebsch Gordon coefficients to determine C_1 and C_2.

(a) $C_1 = \sqrt{\frac{3}{5}}$ and $C_2 = \sqrt{\frac{2}{5}}$

(b) $C_1 = -\sqrt{\frac{2}{5}}$ and $C_2 = \sqrt{\frac{3}{5}}$

(c) $C_1 = \sqrt{\frac{1}{2}}$ and $C_2 = \sqrt{\frac{1}{2}}$

(d) $C_1 = \sqrt{\frac{3}{4}}$ and $C_2 = \sqrt{\frac{1}{4}}$

[Answer (b)]