ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-6 SIMILARITY METHOD

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1. Notion of Similarity of Profiles
2. Condition for Existence of Similarity Solutions
3. Similarity Equation and Boundary Conditions

2D BL Eqns - L6($\frac{1}{12}$)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = -\frac{dp_\infty}{d\,x} + \mu \frac{\partial^2 u}{\partial y^2} \quad (2)$$

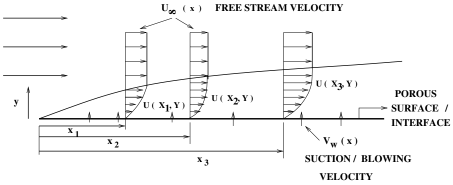
$$-\frac{dp_\infty}{d\,x} = \rho U_\infty \frac{d\,U_\infty}{d\,x} \quad (3)$$

Boundary Conditions:

1. at y = 0, u = 0, v = V_w (x) (Suction/Blowing Velocity)
2. as y $\rightarrow \infty$, u = U_∞ (x) (Free Stream Velocity)

Notion of Similarity of Profiles - L6($\frac{2}{12}$)

1. The term *similarity* is associated with the possibility that under certain conditions , the velocity profiles at different streamwise locations (x_1, x_2, x_3 say) in the boundary layer will be similar in shape.
2. Then, actual magnitudes of u at same y at different locations may differ by a stretching factor s (x) that is a function of the streamwise distance x only.



$$u(x,y) = u(\overline{\eta}) \quad (4)$$
$$v(x,y) = v(\overline{\eta}) \quad \text{and,} \quad (5)$$
$$\overline{\eta} = y \times S(x) \quad (6)$$

$\overline{\eta}$ is called Similarity Variable

Search for Similarity Condition - I - L6($\frac{3}{12}$)

1. The relations suggest that if similarity exists then velocity profiles u (x, y) and v (x, y) at any streamwise location can be *collapsed* on a single curve
2. This is because u and v that were functions of two independent variables x and y are now functions of a single variable $\overline{\eta}$ only
3. The *Partial* Differential Equations (PDEs) of the boundary layer can therefore be reduced to *Ordinary* Differential Equations. (ODEs)
4. Such a reduction is however possible only when $U_\infty(x)$, $V_w(x)$ and $S(x)$ assume certain restricted forms known as *similarity conditions*.

Search for Sim Cond - II - L6($\frac{4}{12}$)

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \rho U_\infty \frac{d\, U_\infty}{d\, x} + \mu \frac{\partial^2 u}{\partial y^2} \quad (7)$$

Define Stream Function ψ (x, y) such that continuity equation $\partial u/\partial x + \partial v/\partial y = 0$ is satisfied.

$$\frac{\partial \psi}{\partial y} \equiv u \quad ; \quad \frac{\partial \psi}{\partial x} \equiv - v \quad (8)$$

Substitution gives ψ Equation

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U_\infty \frac{d\, U_\infty}{d\, x} + \nu \frac{\partial^3 \psi}{\partial y^3} \quad (9)$$

Search for Sim Cond - III - L6($\frac{5}{12}$)

Define

$$\psi(x,y) \equiv n(x)\ Z(\overline{\eta}) \quad (10)$$

$$\frac{u}{U_\infty} \equiv \frac{dZ}{d\,\overline{\eta}} \quad (11)$$

Then

$$\frac{u}{U_\infty} = \frac{1}{U_\infty}\frac{\partial\psi}{\partial y}$$

$$\frac{dZ}{d\,\overline{\eta}} = \frac{1}{U_\infty}\frac{\partial\psi}{\partial\overline{\eta}} \times \frac{\partial\overline{\eta}}{\partial y}$$

$$\frac{dZ}{d\,\overline{\eta}} = \frac{n}{U_\infty} \times \frac{dZ}{d\,\overline{\eta}} \times \frac{\partial\overline{\eta}}{\partial y}$$

Hence $\frac{\partial\overline{\eta}}{\partial y} = \frac{U_\infty}{n} = S(x)$

$$\frac{\partial\psi}{\partial y} = U_\infty \frac{dZ}{d\overline{\eta}}$$

$$\frac{\partial^2\psi}{\partial y^2} = \frac{U_\infty^2}{n}\frac{d^2Z}{d\overline{\eta}^2}$$

$$\frac{\partial^3\psi}{\partial y^3} = \frac{U_\infty^3}{n^2}\frac{d^3Z}{d\overline{\eta}^3}$$

$$\frac{\partial\psi}{\partial x} = Z\frac{dn}{dx} + n\frac{dZ}{d\overline{\eta}}\frac{\partial\overline{\eta}}{\partial x}$$

$$\frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial}{\partial x}\left(U_\infty \frac{dZ}{d\overline{\eta}}\right)$$

$$= \frac{dZ}{d\overline{\eta}}\frac{d\,U_\infty}{d\,x}$$

$$+ \; U_\infty \frac{d^2Z}{d\overline{\eta}^2}\frac{\partial\overline{\eta}}{\partial x}$$

Search for Sim Cond - IV - L6($\frac{6}{12}$)

Replacing derivatives in the ψ equation 9

$$Z''' + \beta_1 \, Z \, Z'' + \beta_2 \, (1 - Z'^2) = 0 \qquad (12)$$

$$Z' = dZ \, / \, d\overline{\eta} \qquad (13)$$

$$\beta_1 = \frac{n}{\nu \, U_\infty} \, \frac{d \, n}{d \, x} \qquad (14)$$

$$\beta_2 = \frac{n^2}{\nu \, U_\infty^2} \, \frac{d \, U_\infty}{d \, x} \qquad (15)$$

Boundary Conditions:

1. $V_w = -\,(\partial\psi/\partial x)_{y=0} = -\,Z(0)\,dn/dx$. Hence, $Z(0) = -\,V_w(x)/(dn/dx)$
2. $Z'(0) = 0$ (No-Slip Condition)
3. $Z'(\infty) = 1$ (Free Stream Condition)

Search for Sim Con - V - L6($\frac{7}{12}$)

Equation $Z''' + \beta_1 Z Z'' + \beta_2 (1 - Z'^2) = 0$ will be an ODE if β_1, β_2 and Z (0) are absolute constants . Hence, Consider

$$2\beta_1 - \beta_2 = \frac{2n}{\nu U_\infty} \frac{d n}{d x} - \frac{n^2}{\nu U_\infty^2} \frac{d U_\infty}{d x} = \frac{d}{d x} \left[\frac{n^2}{\nu U_\infty} \right]$$

or, integrating from x = 0 to x = x,

$$(2 \beta_1 - \beta_2) x = \frac{n^2}{\nu U_\infty}$$

Multiplying both sides by $U_\infty^{-1} d U_\infty / d x$,

$$\frac{d U_\infty}{U_\infty} = (\frac{\beta_2}{2\beta_1 - \beta_2}) \frac{d x}{x}$$

Integration gives Similarity conditions.

Search for Sim Cond - VI - L6($\frac{8}{12}$)

$$U_\infty = C\, x^{\left(\frac{\beta_2}{2\beta_1-\beta_2}\right)} \tag{16}$$

$$n(x) = \sqrt{\nu\, U_\infty\, (2\,\beta_1 - \beta_2)\, x} \tag{17}$$

$$\overline{\eta} = y\, S(X) = \frac{y\, U_\infty}{n} = y\,\sqrt{\frac{U_\infty}{\nu\,(2\,\beta_1 - \beta_2)\, x}} \tag{18}$$

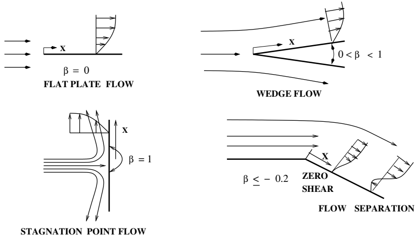
$$\psi = Z\,(\overline{\eta})\,\sqrt{\nu\, U_\infty\, (2\,\beta_1 - \beta_2)\, x} \tag{19}$$

$$Z(0) = -\frac{V_w(x)}{dn/dx} = \text{constant} \tag{20}$$

Useful Deduction - L6($\frac{9}{12}$)

Without loss of generality, we set $\beta_1 = 1$ and $\beta_2 = \beta$. Then

$$
\begin{aligned}
Z''' &+ ZZ'' + \beta(1 - Z'^2) = 0 \\
U_\infty &= C\, x^{\left(\frac{\beta}{2-\beta}\right)} \\
n(x) &= \sqrt{\nu\, U_\infty\, (2-\beta)\, x} \\
\overline{\eta} &= y \sqrt{\frac{U_\infty}{\nu\,(2-\beta)\, x}} \\
\psi &= Z\,(\overline{\eta})\, \sqrt{\nu\, U_\infty\, (2-\beta)\, x} \\
Z(0) &= -\frac{V_w(x)}{dn/dx} = \text{constant}
\end{aligned}
$$



Potential Flow Theory shows that $U_\infty = C\, x^{\left(\frac{\beta}{2-\beta}\right)}$ represents flow over a Wedge of angle $\pi\,\beta$. $\beta < -0.2$ represents Flow Separation . Hence Elliptic Flow (See Next Lecture)

Change of Definitions - L6($\frac{10}{12}$)

For convenience, we redefine parameters as

$$U_\infty = C\, x^m \tag{21}$$

$$m = \frac{\beta}{2-\beta} \text{ or } \beta = \frac{2\, m}{m+1} \tag{22}$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu\, x}} = \overline{\eta}\sqrt{2-\beta} \tag{23}$$

$$f^{'}(\eta) = z^{'}(\overline{\eta}) = \frac{u}{U_\infty} \tag{24}$$

New Similarity Equation - L6($\frac{11}{12}$)

Then, the Z-equation will transform to

$$f''' + (\frac{m+1}{2})\, f\, f'' + m\,(1 - f'^2) = 0 \tag{25}$$

$$\psi = f\,(\eta)\,\sqrt{\nu\, U_\infty\, x} \qquad v = -\frac{\partial \psi}{\partial x} \tag{26}$$

$$\frac{v}{U_\infty}\, Re_x^{0.5} = -\,(\frac{m+1}{2})\left\{f + (\frac{m-1}{m+1})\,\eta\, f'\right\} \tag{27}$$

$$f\,(0) = -\,B_f\,(\frac{2}{m+1}) \tag{28}$$

$$f'\,(0) = 0 \quad \text{and} \quad f'\,(\infty) = 1 \tag{29}$$

$$B_f = \frac{V_w(x)}{U_\infty(x)}\, Re_x^{0.5} \qquad Re_x = \frac{U_\infty\, x}{\nu} \tag{30}$$

B_f is called Blowing Parameter and must be constant for similarity solutions to exist.

Summary - L6($\frac{12}{12}$)

1. We have transformed the 2D Boundary Layer PDE to a 3rd order ODE $f''' + (\frac{m+1}{2})\, f\, f'' + m\,(1 - f'^2) = 0$
2. The ODE is valid for $U_\infty = C\, x^m$ and $(V_w(x)/U_\infty(x))\; Re_x^{0.5} = B_f$ = constant only.
3. The independent similarity variable is $\eta = y\, \sqrt{U_\infty/(\nu\, x)}$
4. For $m > 0$, we have Accelerating Flow and hence the Pressure Gradient dp / dx = - $\rho\, U_\infty\, dU_\infty/dx$ is negative . This is called Favourable Pressure Gradient
5. For $m < 0$, we have De-celerating Flow and Adverse Pressure Gradient .
6. For m = 0, we have U_∞ = constant and dp / dx = 0. It is called Flat Plate Flow. For m = 1, we have Stagnation Point Accelerating Flow
7. Hence, m is called Pressure Gradient Parameter