ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-4 SCALAR TRANSPORT EQUATIONS

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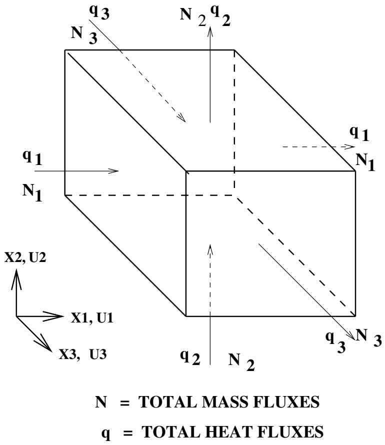
1. Law of Mass Conservation for a Specie in a Mixture
   (Mass Transfer Equation)
2. 1st Law of Thermodynamics
   (Energy Equation)

Mass Transfer Equation - L4($\frac{1}{17}$)

For Specie k in a Mixture

Rate of accumulation of mass ($\dot{M}_{k,ac}$) =
Rate of mass in ($\dot{M}_{k,in}$)
- Rate of mass out ($\dot{M}_{k,out}$)
+ Rate of generation within CV ($R_{k,cv}$)



$$\dot{M}_{k,ac} = \frac{\partial(\rho_k \Delta V)}{\partial t}$$

$$\dot{M}_{k,in} = N_{1,k}\ \Delta A_1 \mid_{x_1} + N_{2,k}\ \Delta A_2 \mid_{x_2} + N_{3,k}\ \Delta A_3 \mid_{x_3}$$

$$\dot{M}_{k,out} = N_{1,k}\ \Delta A_1 \mid_{x_1+\Delta x_1} + N_{2,k}\ \Delta A_2 \mid_{x_2+\Delta x_2} + N_{3,k}\ \Delta A_3 \mid_{x_3+\Delta x_3}$$

ρ_k = Specie Density

Substitute, Divide each term by ΔV and Let $\Delta x_1, \Delta x_2, \Delta x_3 \rightarrow 0$

Mass Transfer Equation - I L4($\frac{2}{17}$)

$$\frac{\partial(\rho_k)}{\partial t} + \frac{\partial(N_{1,k})}{\partial x_1} + \frac{\partial(N_{2,k})}{\partial x_2} + \frac{\partial(N_{3,k})}{\partial x_3} = R_k \qquad (1)$$

Now, the total mass transfer flux $N_{i,k}$ in direction i
is the sum of *Convective Flux* ($\rho_k \, u_i$) due to bulk fluid motion
and *Diffusion* Flux ($m^{''}_{i,k}$) due to density difference. Thus,

$$N_{i,k} = \rho_k \, u_i + m^{''}_{i,k} \qquad (2)$$

where

$$m^{''}_{i,k} = -\, D \, \frac{\partial \rho_k}{\partial x_i} \qquad (3)$$

where D (m^2/s) is the mass-diffusivity

Mass Transfer Equation - II L4($\frac{3}{17}$)

Substituting for $N_{i,k}$ gives

$$\frac{\partial(\rho_k)}{\partial t} + \frac{\partial(\rho_k\, u_1)}{\partial x_1} + \frac{\partial(\rho_k\, u_2)}{\partial x_2} + \frac{\partial(\rho_k\, u_3)}{\partial x_3}$$

$$= \frac{\partial}{\partial x_1}(D\,\frac{\partial \rho_k}{\partial x_1}) + \frac{\partial}{\partial x_2}(D\,\frac{\partial \rho_k}{\partial x_2}) + \frac{\partial}{\partial x_3}(D\,\frac{\partial \rho_k}{\partial x_3}) + R_k \qquad (4)$$

Define Mass Fraction ω_k

$$\omega_k = \frac{\rho_k}{\rho_m} \qquad \sum_{\text{all species}} \omega_k = 1 \ \text{ and } \ \sum \rho_k = \rho_m \qquad (5)$$

$$\frac{\partial(\rho_m\, \omega_k)}{\partial t} + \frac{\partial(\rho_m\, u_j\, \omega_k)}{\partial x_j} = \frac{\partial}{\partial x_j}(\rho_m\, D\, \frac{\partial \omega_k}{\partial x_j}) + R_k \qquad (6)$$

Summation gives Bulk Mass Conservation because $\sum \omega_k$ = 1 and hence, $\sum R_k = 0$. Also, $\sum N_{i,k} = \rho_m u_i + \sum \dot{m}''_{i,k} = \rho_m u_i$

Mass Diffusivity L4($\frac{4}{17}$)

1. The mass diffusivity is *defined* only for a Binary Mixture of two fluids 1 and 2 as D_{12}.
2. In Multicomponent Gaseous Mixtures , however, diffusivities for pairs of species are nearly equal and a single symbol D suffices for all species.
3. Incidentally, in turbulent flows , this assumption of equal (effective) diffusivities holds even greater validity as will be shown later

1st Law of Thermodynamics L4($\frac{5}{17}$)

In Rate Form (W / m^3), the 1st Law of Thermodynamics reads as

$$\dot{E} = \dot{Q}_{conv} + \dot{Q}_{cond} + \dot{Q}_{gen} - \dot{W}_s - \dot{W}_b \tag{7}$$

where

$$\begin{aligned}
\dot{E} &= \text{Rate of Change of Energy of the CV} \\
\dot{Q}_{conv} &= \text{Net Rate of Energy transferred by Convection} \\
\dot{Q}_{cond} &= \text{Net Rate of Energy transferred by Conduction} \\
\dot{Q}_{gen} &= \text{Net Rate of Volumetric Heat Generation within CV} \\
\dot{W}_s &= \text{Net Rate of Work Done by Surface Forces} \\
\dot{W}_b &= \text{Net Rate of Work Done by Body Forces}
\end{aligned}$$

Each term will now be represented by a mathematical expression.

Rate of Change L4($\frac{6}{17}$)

$$\dot{E} = \frac{\partial(\rho_m e^o)}{\partial t} \qquad (8)$$

$$e^o = e_m + \frac{V^2}{2} = h_m - \frac{p}{\rho_m} + \frac{V^2}{2} \qquad (9)$$

where

e_m = Mixture Specific Energy (J / kg)

h_m = Mixture Specific Enthalpy (J / kg)

$V^2 = u_1^2 + u_2^2 + u_3^2$ (J / kg)

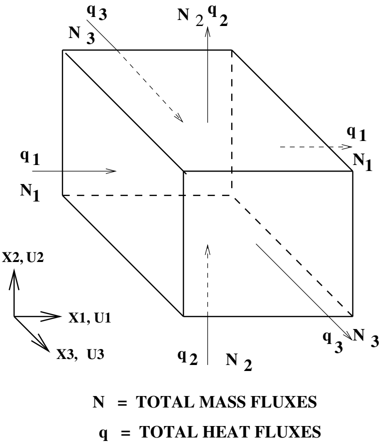
$\rho_m = \sum \rho_k$ = Mixture Density (kg / m^3)

Contributions to e^0 from other forms (Potential, Electro-Magnetic etc) of energy are neglected.

Thermodynamic Convention L4($\frac{7}{17}$)

Convention:

1. Heat Flow into CV is Positive whereas Heat Flow out of CV is Negative.
2. Also, all species are transported at *Mixture Velocity*



Net Convection L4($\frac{8}{17}$)

Since $\sum N_{i,k} = \rho_m u_i$

$$\dot{Q}_{conv} = -\frac{\partial \sum (N_{j,k} e_k^o)}{\partial x_j} = -\frac{\partial}{\partial x_j}\left[(\sum N_{j,k} h_k) + \rho_m\, u_j\, (-\frac{p}{\rho_m} + \frac{V^2}{2})\right] \tag{10}$$

Now, following definition of $N_{j,k}$ and noting $\sum \omega_k\, h_k = h_m$,

$$\begin{aligned} \sum N_{j,k}\, h_k &= \sum (\rho_m\, u_j\, \omega_k + m''_{j,k})\, h_k \\ &= \rho_m\, u_j\, h_m + \sum m''_{j,k}\, h_k \end{aligned} \tag{11}$$

Hence, after some algebra

$$\dot{Q}_{conv} = -\frac{\partial(\rho_m\, u_j\, e^o)}{\partial x_j} - \frac{\partial(\sum m''_{j,k}\, h_k)}{\partial x_j} \tag{12}$$

Net Conduction L4($\frac{9}{17}$)

Similarly, from Fourier's Law of Conduction

$$\dot{Q}_{cond} = -\frac{\partial q_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left[k_m \frac{\partial T}{\partial x_j}\right] \quad (13)$$

where k_m = Mixture Conductivity

Volumetric Generation L4($\frac{10}{17}$)

$$\dot{Q}_{gen} = \dot{Q}_{chem} + \dot{Q}_{rad} \tag{14}$$

1. The Chemical Energy is positive for exothermic reactions and negative for endothermic reactions .
2. Evaluation of $\dot{Q}_{chem}$ depends on the chemical reaction model employed in a particular situation.
3. The $\dot{Q}_{rad}$ term represents the Net Radiation Exchange between the CV and its surroundings . Evaluation of this term, in general, requires solution of integro-differential equations
4. When absorptivity (a) and scattering coefficient (s) are large,

$$\dot{Q}_{rad} = \frac{\partial}{\partial x_j}\left[k_{rad}\,\frac{\partial T}{\partial x_j}\right] \qquad k_{rad} = \frac{16\,\sigma\,T^3}{a+s} \tag{15}$$

where σ is the Stefan-Boltzmann constant

Work-Done by Forces-I L4($\frac{11}{17}$)

Convention: The work done *on the CV* is negative,

$$\begin{aligned}
-\dot{W}_s &= \frac{\partial}{\partial x_1}\left[\sigma_1\, u_1 + \tau_{12}\, u_2 + \tau_{13}\, u_3\right] \\
&+ \frac{\partial}{\partial x_2}\left[\tau_{21} u_1 + \sigma_2\, u_2 + \tau_{23}\, u_3\right] \\
&+ \frac{\partial}{\partial x_3}\left[\tau_{31}\, u_1 + \tau_{32}\, u_2 + \sigma_3\, u_3\right] \qquad (16)
\end{aligned}$$

$$-\dot{W}_b = \rho_m\,(B_1\, u_1 + B_2\, u_2 + B_3\, u_3) \qquad (17)$$

where $\dot{W}_s$ = Stress Work and $\dot{W}_b$ = Body-Force Work
Further use of *differentiation of product* gives (see next slide)

Work-Done by Forces-II L4($\frac{12}{17}$)

$$
\begin{aligned}
-(\dot{W}_s + \dot{W}_b) &= u_1 \left[\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + \rho_m B_1\right] && (18)\\
&= u_2 \left[\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} + \rho_m B_2\right] && (19)\\
&+ u_3 \left[\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + \rho_m B_3\right] && (20)\\
&+ \sigma_1 \frac{\partial u_1}{\partial x_1} + \sigma_2 \frac{\partial u_2}{\partial x_2} + \sigma_3 \frac{\partial u_3}{\partial x_3} && (21)\\
&+ \tau_{12} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + \tau_{13} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) &&\\
&+ \tau_{23} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) && (22)
\end{aligned}
$$

Complementary stress $\tau_{12} = \tau_{21}$ etc are recognised (see next slide)

Work-Done by Forces-III L4($\frac{13}{17}$)

Multipliers of u_1, u_2, u_3 in equations 18, 19 and 20 are simply RHS of Momentum equations (See Lecture 3, slides 13-14-15). They are replaced by LHS of Momentum equations. Hence,

$$\begin{aligned} \text{Equations } 18,\ 19,\ 20 &= \rho_m \left[u_1 \frac{Du_1}{Dt} + u_2 \frac{Du_2}{Dt} + u_3 \frac{Du_3}{Dt} \right] \\ &= \rho_m \frac{D}{Dt} \left(\frac{V^2}{2} \right) \end{aligned} \tag{23}$$

Work-Done by Forces-IV L4($\frac{14}{17}$)

Similarly, using Stokes's Laws , equations 21, and 22 can be written as

$$\text{Equations } 21,\ 22 = \mu\,\Phi_v - p\ \nabla\ .V \tag{24}$$

where Viscous Dissipation Function is

$$\begin{aligned}\Phi_v &= 2\left[(\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_2})^2 + (\frac{\partial u_3}{\partial x_3})^2\right] \\ &+ (\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1})^2 + (\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3})^2 \end{aligned} \tag{25}$$

Work-Done by Forces-V L4($\frac{15}{17}$)

Hence, from equations 23 and 24, we have

$$-(\dot{W}_s + \dot{W}_b) = \rho_m \frac{D}{Dt}(\frac{V^2}{2}) + \mu\,\Phi_v - p\ \bigtriangledown\ .\,V \qquad (26)$$

where

$$\bigtriangledown\,.\,V = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \qquad (27)$$

Summary L4($\frac{16}{17}$)

$$\dot{E} = \dot{Q}_{conv} + \dot{Q}_{cond} + \dot{Q}_{gen} - \dot{W}_s - \dot{W}_b \tag{28}$$

where

$$\dot{E} = \frac{\partial(\rho_m\, e^o)}{\partial t} \tag{29}$$

$$\dot{Q}_{conv} = -\frac{\partial(\rho_m\, u_j\, e^o)}{\partial x_j} - \frac{\partial(\sum m''_{j,k}\, h_k)}{\partial x_j} \tag{30}$$

$$\dot{Q}_{cond} = \frac{\partial}{\partial x_j}\left[k_m \frac{\partial T}{\partial x_j}\right] \tag{31}$$

$$\dot{Q}_{gen} = \dot{Q}_{chem} + \dot{Q}_{rad} \tag{32}$$

$$-(\dot{W}_s + \dot{W}_b) = \rho_m \frac{D}{Dt}(\frac{V^2}{2}) + \mu\, \Phi_v - p\ \triangledown\ .\, V \tag{33}$$

Final Energy Equation L4($\frac{17}{17}$)

$$\frac{\partial(\rho_m\, e^o)}{\partial t} + \frac{\partial(\rho_m\, u_j\, e^o)}{\partial x_j} = \frac{\partial}{\partial x_j}\left[k_m\,\frac{\partial T}{\partial x_j}\right] - \frac{\partial(\sum m''_{j,k}\, h_k)}{\partial x_j} + \frac{D}{D\,t}\left[\frac{V^2}{2}\right] - p\,\bigtriangledown\,.\,V + \mu\,\Phi_v + \dot{Q}_{chem} + \dot{Q}_{rad} \quad (34)$$

or, using definition of e^o,

$$\rho_m\,\frac{D\,h}{D\,t} = \frac{\partial}{\partial x_j}\left[k_m\,\frac{\partial T}{\partial x_j}\right] - \frac{\partial(\sum m''_{j,k}\, h_k)}{\partial x_j} + \mu\,\Phi_v + \frac{D\,p}{D\,t} + \dot{Q}_{chem} + \dot{Q}_{rad} \quad (35)$$