ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-29 PREDICTION OF TURBULENT FLOWS

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1. Prediction of $C_{f,x}$ (Ext Bls)
    1. Integral Method
    2. Complete Laminar-Transition-Turbulent BL
    3. Similarity Method
2. Prediction of f (Internal Flows)- Use of Law of the wall

Integral Method - Ext BLs - 1 L29($\frac{1}{19}$)

1. The Integral Momentum Eqn (IME) is applicable to laminar, trnasition and turbulent BLs (lecture 10)

$$\frac{d\,\delta_2}{d\,x} + \frac{1}{U_\infty}\frac{d\,U_\infty}{d\,x}(2\,\delta_2 + \delta_1) = \frac{C_{f,x}}{2} + \frac{V_w}{U_\infty}$$

$$\delta_1 = \int_0^\delta (1 - \frac{u}{U_\infty})\,d\,y \quad \text{and} \quad \delta_2 = \int_0^\delta \frac{u}{U_\infty}(1 - \frac{u}{U_\infty})\,d\,y$$

2. In each flow regime appropriate profiles of u/U_∞ must be specified.
3. We consider Fully turbulent boundary layer starting from x = 0 (leading edge) or from $x = x_{te}$ (end of transition)

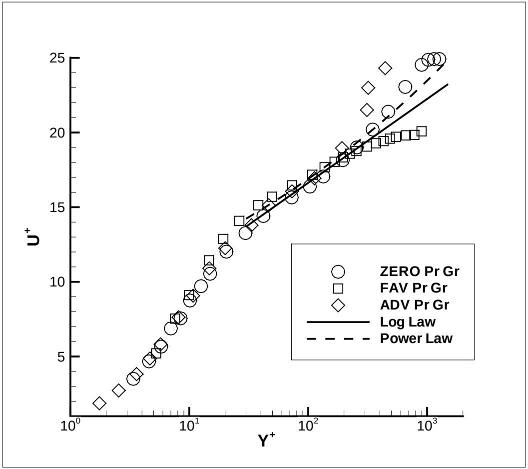
Power Law Assumption - L29($\frac{2}{19}$)

1. Evaluations of δ_1 and δ_2 are negligibly affected in TBLs when Laminar sub-layer and transition layers are ignored.
2. Then, only fully turbulent vel profile (universal logarithmic law (inner) + law of the wake (outer)) suffices.
3. However, integration as well as evaluation of $C_{f,x} = \tau_w/(\rho\, U_\infty^2)$ becomes extremely involved.
4. Hence, for an impermeable smooth wall ($v_w = 0$), a power-law is assumed

$$u^+ = a\, {y^+}^b \quad a \simeq 8.75 \;\text{ and }\; b = 1./7.$$

5. This 1 / 7th power law fits the logarithmic law well upto $y^+ \simeq 1500$ and also fits the exptl data in the wake-region better than log-law (see next slide)

Comparison with Expt Data - L29($\frac{3}{19}$)



Use of power law - $v_w = 0$ - L29($\frac{4}{19}$)

1. Then, it follows that

$$\frac{u}{U_\infty} = (\frac{y}{\delta})^{1/7} \text{ integration gives}$$
$$\frac{\delta_1}{\delta} = 0.125, \quad \frac{\delta_2}{\delta} = \frac{7}{72} = 0.097, \quad H = \frac{\delta_1}{\delta_2} = 1.29$$

2. Now, unlike in laminar flows, $\tau_w = \rho\, u_\tau^2$ is evaluated from $(U_\infty / u_\tau) = 8.75\, (\delta\, u_\tau / \nu)^{1/7}$ giving

$$\frac{C_{f,x}}{2} = \frac{\tau_w}{\rho\, U_\infty^2} = 0.0225\, (\frac{U_\infty\, \delta}{\nu})^{-0.25} = 0.0125\, (\frac{U_\infty\, \delta_2}{\nu})^{-0.25}$$

3. Both expressions are very good approximations to mildly adv pr gr through to highly fav. pr. gr and upto $Re_x \simeq 10^7$.

Solving Int Mom Eqn - L29($\frac{5}{19}$)

1. Substituting for δ_1 and $C_{f,x}$ with $v_w = 0$ gives

$$\frac{d\,\delta_2}{dx} = 0.0125\,(\frac{U_\infty\,\delta_2}{\nu})^{-0.25} - 3.29\,\frac{\delta_2}{U_\infty}\,\frac{d\,U_\infty}{dx} \quad \text{or}$$

$$\frac{d}{dx}\left[U_\infty^{4.11}\,\delta_2^{1.25}\right] = 0.0156\,\nu^{0.25}\,U_\infty^{3.86} \quad \text{integration gives}$$

$$U_\infty^{4.11}\,\delta_2^{1.25}\,|_x = U_\infty^{4.11}\,\delta_2^{1.25}\,|_{x_{in}} + 0.0156\,\nu^{0.25}\int_{x_{in}}^{x} U_\infty^{3.86}\,dx$$

2. If TBL originates at the leading edge ($x_{in} = 0$)

$$\delta_2 = \frac{0.036\,\nu^{0.2}}{U_\infty^{3.29}}\,(\int_0^x U_\infty^{3.86}\,dx)^{0.8} \rightarrow C_{f,x} = 0.025\,(\frac{U_\infty\,\delta_2}{\nu})^{-0.25}$$

δ_2 and $C_{f,x}$ can be evaluated for any arbitrary variation of U_∞ from mildly adv pr gr through to highly fav. pr. gr

For U_∞ = const, $C_{f,x,dpdx=0} = 0.0574\,(\frac{U_\infty\,x}{\nu})^{-0.2}$

Highly Adv Pr Gr & v_w - L29($\frac{6}{19}$)

1. For these cases, IME is again written as

$$\frac{d\,\delta_2}{d\,x} + \frac{\delta_2}{U_\infty}\,\frac{d\,U_\infty}{d\,x}\,(2 + H) = \frac{C_{f,x}}{2} + \frac{V_w}{U_\infty}$$

2. Now, H and $C_{f,x}$ are modeled as

$$H = \left[1 - G\sqrt{C_{f,x}/2.0}\right]^{-1}$$

$$G \simeq 6.2\,(1.43 + \beta + B)^{0.482}, \quad \beta = \frac{\delta_1}{\tau_w}\,\frac{dp}{dx}, \quad B = \frac{v_w/U_\infty}{C_{f,x}/2}$$

$$C_{f,x} = C_{f,x,dpdx=0} \times (1 + 0.2\,\beta)^{-1} \quad \text{(Crawford and Kays), or}$$

$$C_{f,x} = 0.246 \times 10^{-0.678\,H} \times Re_{\delta_2}^{-0.268} \quad \text{(Ludwig and Tilman)}$$

$$C_{f,x} = 0.336 \times \{\ln\,(854.6\,\delta_2/y_{re})\}^{-2} \quad \text{(rough surface)}$$

Valid for $-1.43 < \beta + B < 12$. Iterative soln of IME is required.

Complete BL Prediction - 1 - L29($\frac{7}{19}$)

Laminar Regime

1. For given $U_\infty(x)$ and $v_w(x)$, evaluate $\delta_{2,l}(x)$
2. Hence, evaluate $\kappa = (\delta_{2,l}^2/\nu)\, dU_\infty/dx$, H = $\delta_{1,l}/\delta_{2,l}$ and S = $\delta_{2,l}/\delta_{4,l}$.
3. Hence evaluate $C_{f,x,l}$ - subscript l for laminar
4. Continue calculations until Onset of transition using Cebeci or Fraser and Milne criterion (lecture 28).
   Note the values of $x_{t,s}$ and End of transition ($x_{te} - x_{ts}$)

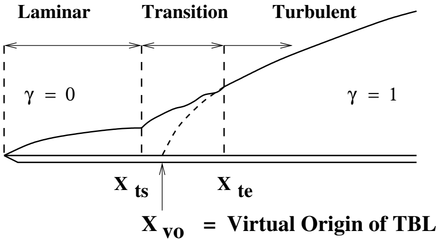
Complete BL Prediction - 2 - L29($\frac{8}{19}$)

In the Transition regime

$$
\begin{aligned}
(\frac{u}{U_\infty})_{tr} &= (1-\gamma)\,(\frac{u}{U_\infty})_l + \gamma\,(\frac{u}{U_\infty})_t \\
\gamma &= 1 - \exp\left(-\,5\,\xi^3\right) \quad \xi = (x - x_{ts})/(x_{te} - x_{ts}) \\
\delta_{1,tr} &= (1-\gamma)\,\delta_{1,l} + \gamma\,\delta_{1,t} \\
\delta_{2,tr} &= (1-\gamma)\ \{(1-\gamma)\,\delta_{2,l} - \gamma\,\delta_{1,l}\} \\
&+ \gamma\ \{\gamma\,\delta_{2,t} - (1-\gamma)\,\delta_{1,t}\} \\
&+ 2\,\gamma\,(1-\gamma)\int_0^\delta \left[1 - (\frac{u}{U_\infty})_l\,(\frac{u}{U_\infty})_t\right] dy \\
H_{tr} &= \delta_{1,tr}/\delta_{2,tr} \\
C_{f,x,tr} &= (1-\gamma)\,C_{f,x,l} + \gamma\,C_{f,x,t} \\
(\frac{u}{U_\infty})_t &= (\frac{y}{\delta_t})^{1/n} \ \rightarrow\ n = \frac{2}{H_t - 1} \ \rightarrow\ \delta_t = \delta_{2,t}\,\frac{H_t\,(H_t+1)}{H_t - 1}
\end{aligned}
$$

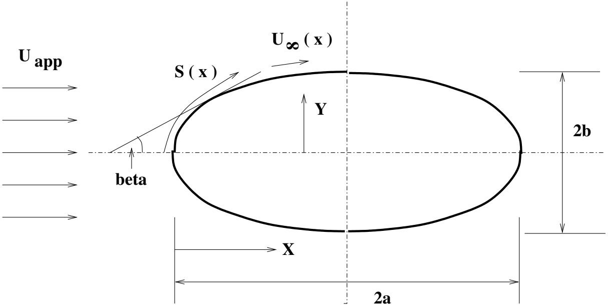
Complete BL Prediction - 3 - L29($\frac{9}{19}$)

1. To compute in Turbulent regime , we define $x_{vo} - x_{ts} = 0.126\,(x_{te} - x_{ts})$
2. Define $x' = x - x_{vo}$ and commence soln of turbulent IME where at x' = 0, arbitrarily, $\delta_{2,t} = 0.2\ \delta_{2,l}$, $H_t = 1.5$ and $C_{f,x,t} = 0.99\ C_{f,x,l}$
3. At $x'_{te} = x_{te} - x_{vo}$, the appropriate specifications are $\delta_{2,t} = \delta_{2,tr}$, $H_t = H_{tr}$ and $C_{f,x,t} = C_{f,x,tr}$ and laminar calculations are stopped.



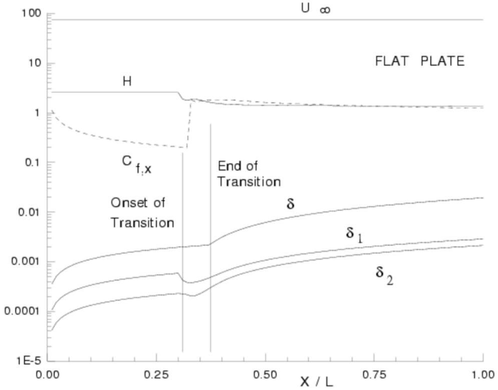
For $x' > x'_{te}$, turbulent IME is solved iteratively as described in slide 5. With $\Delta x' = 0.25\ \delta_{2,t}$, convergence is obtained in ≤ 4 iterations.

Solns for Ellipse Family - L29($\frac{10}{19}$)



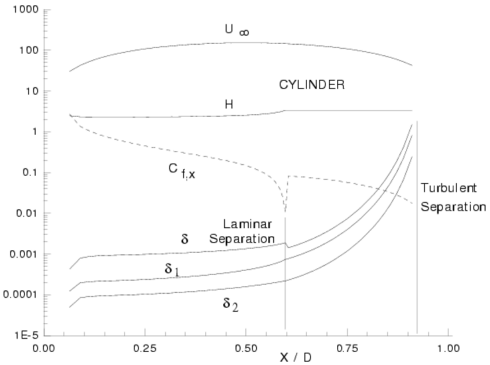
1. a, b, U_{app}, ρ and μ are specified. Re = $(\rho\ U_{app}\ 2a)/\mu$
2. $U_\infty = U_{app} \times (1 + b/a) \times \cos(\beta)$ where β is function of x
3. S (x) is distance along the surface.
4. (b / a) $>$ 0 (Ellipse), = 1.0 (cylinder), = 0 (flat plate)

Flat Plate - L29($\frac{11}{19}$)



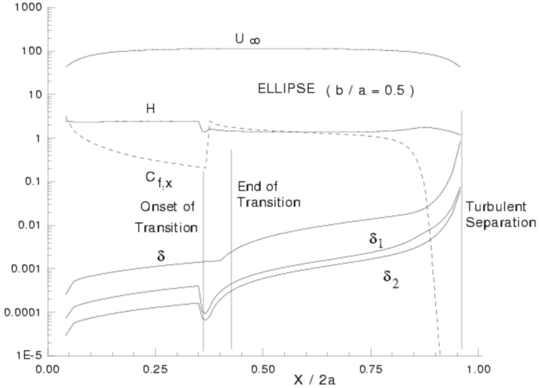
1. L = 2a, $Re_L = 10^7$, $C_{f,x} \times 500$ are plotted.
2. $(x_{ts}/L) = 0.31$, $(x_{te}/L) = 0.4342$, $(x_{vo}/L) = 0.325$

Cylinder - L29($\frac{12}{19}$)



1. D = 2a, $Re_D = 10^7$, $C_{f,x} \times 500$ are plotted.
2. Laminar Separation at (x_{sep}/D) = 0.597
   $\equiv$ Turbulent Reattachment assumed
3. Turbulent Separation at (x_{sep}/D) = 0.812

Ellipse ($\frac{b}{a}$ - 0.5) - L29($\frac{13}{19}$)



1. $Re_{2a} = 10^7$, $C_{f,x} \times 500$ are plotted.
2. $(x_{ts}/2a)$ = 0.3588, $(x_{te}/2a)$ = 0.4284, $(x_{vo}/2a)$ = 0.3676
3. Turbulent Separation at $(x_{sep}/2a)$ = 0.958

Similarity Method for TBL - L29($\frac{14}{19}$)

1. The differential eqn governing TBL can be written as

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = U_\infty\,\frac{dU_\infty}{dx} + \nu\,\frac{\partial}{\partial y}\,\left[(1+\nu_t^+)\,\frac{\partial u}{\partial y}\right]$$

where $\nu_t^+ = \nu_t/\nu$ and ν_t is given by Prandtl's mixing length.

2. To convert this eqn to an ODE, we invoke following similarity variables

$$\eta \equiv y \times \frac{U_\infty}{\sqrt{2\,\nu\,L\,V\,\xi}} \qquad \Psi \equiv \sqrt{2\,\nu\,L\,V\,\xi} \times f(\xi,\eta)$$

$$\xi \equiv \frac{1}{L\,V}\int_0^x U_\infty\,dx \qquad \beta \equiv \frac{2}{U_\infty^2}\,\frac{d\,U_\infty}{dx}\int_0^x U_\infty\,dx$$

$$\frac{d}{d\eta}\,\left[(1+\nu_t^+)\,f''\right] + f\,f'' \;+\; \beta\,(1-f'^2) = 2\,\xi\,(f'\,\frac{d\,f'}{d\,\xi} - f''\,\frac{d\,f}{d\,\xi})$$

with $f\,(\xi,0) = f'\,(\xi,0) = 0,\;\; f'\,(\xi,\infty) = 1.0$. $U_\infty\,(x)$ is prescribed arbitrary variation. L and V - reference scales.

Sim Meth for Eq. BLs - L29($\frac{15}{19}$)

1. The Eqn of previous slide can be used for flow over an ellipse, for example, with $U_\infty = U_{app} \times (1 + b/a) \times \cos(\beta)$ and $\nu_t^+ = 0$ (Lam) and $\nu_{tr}^+ = (1-\gamma) + \gamma\, \nu_t^+$ (Trans)
2. When $U_\infty = C\, x^m$, (Equilibrium BLs) , we have

$$\begin{aligned}
\eta &= y \times \sqrt{(\frac{U_\infty}{\nu\, x})\,(\frac{m+1}{2})} \\
\psi &= \sqrt{(\frac{2}{m+1})\,(U_\infty\, \nu\, x)} \times f\,(x, \eta) \\
\frac{d}{d\eta}\left[(1+\nu_t^+)\, f^{''}\right] &+ f\, f^{''} + (\frac{2m}{m+1})\,(1 - f^{'2}) \\
&= x\,(f^{'}\, \frac{df^{'}}{dx} - f^{''}\, \frac{df}{dx})
\end{aligned}$$

with $f\,(x,0) = f^{'}\,(x,0) = 0,\;\; f^{'}\,(x,\infty) = 1.0.$

Soln Procedure - L29($\frac{16}{19}$)

1. The presence of axial derivatives on the RHS requires iterative solution.
2. Therefore, at first Δx , Set RHS = 0 and solve 3rd order ODE to predict f, $f^{'}$ and $f^{''}$ as functions of η
3. At subsequent Δx's, evaluate the RHS from df/dx = $(f_x - f_{x-\Delta x})/\Delta x$ etc and solve the 3rd order ODE by Runge-Kutta method.
4. Using the new f, $f^{'}$ and $f^{''}$ distributions, evaluate the RHS and Solve the ODE again
5. Go to step 3 until predicted f-distributions between iterations converge within a tolerance.
6. For further refinements of this method see **Cebeci and Cousteix**, *Modeling and Computation of Boundary-Layer Flows*, 2nd ed, Springer, (2005)

F. D. Pipe Flow - 1 - L29($\frac{17}{19}$)

1. In lecture 26, it was shown that the log-law predicts the vel profile remarkably well upto the pipe center line . Then

$$\overline{u} = \frac{2}{R^2} \int_0^R u\, r\, dr$$

$$\overline{u^+} = \frac{2}{{R^+}^2} \int_0^{R^+} u^+ \left(R^+ - y^+\right) dy^+ \quad \rightarrow \quad y = R - r$$

2. Since R^+ = O (1000), contribution to the integral upto y^+ = 30 is negligible. Writing log-law as $y^+ = E^{-1} \exp\left(\kappa\, u^+\right)$, where E = 9.152 and κ = 0.41,

$$\overline{u^+} = \frac{2\,\kappa}{E{R^+}^2} \int_0^{u_{cl}^+} u^+ \left\{R^+ - E^{-1} \exp\left(\kappa\, u^+\right)\right\} \exp\left(\kappa\, u^+\right) du^+$$

$$= u_{cl}^+ - \frac{3}{2\,\kappa} + \frac{2}{\kappa\, E\, R^+} - \frac{1}{\kappa\, E^2\, {R^+}^2} \simeq u_{cl}^+ - \frac{3}{2\,\kappa}$$

where subscript cl = centerline

F. D. Pipe Flow - 2 - L29($\frac{18}{19}$)

1. The last expression shows that

$$u^+_{cl} = \overline{u^+} + 3.66 = \sqrt{\frac{2}{f}} + 3.66 = \sqrt{\frac{2}{0.046\ Re^{-0.2}}} + 3.66$$

2. Taking Re = 50,000, u^+_{cl} = 23.11 or $(\overline{u}/u_{cl})$ = 1 - 3.66/23.11 = 0.84 or $(u_{cl}/\overline{u}) \simeq 1.19$. u^+_{cl} increases and $(u_{cl}/\overline{u})$ decreases with increase in Re.
3. Further, writing $u^+_{cl} = \kappa^{-1}\ \ln\,(E\ R^+)$, we have

$$\overline{u^+} = \frac{1}{\kappa}\ln\,(\frac{E}{2}\ Re\sqrt{\frac{f}{2}}) - 3.66 \quad \text{or}$$

$$\sqrt{\frac{2}{f}} = \frac{1}{0.41}\ln\,(\frac{9.152}{2}\ Re\sqrt{\frac{f}{2}}) - 3.66 \quad \text{or}$$

$$\frac{f}{2} = 0.168\left[\ln\,(1.021\ Re\sqrt{\frac{f}{2}})\right]^{-2} \quad \text{implicit formula}$$

F. D. Pipe Flow - 3 - L29($\frac{19}{19}$)

1. To derive an explicit formula for f , we use Power law profile $u^+ = a\, {y^+}^b$. Then, evaluating $\overline{u}^+$

$$\frac{f}{2} = \left[\left(\frac{(1+b)\,(2+b)}{2a}\right) \times \left(\frac{2}{Re}\right)^b \right]^{2/(1+b)}$$

2. For a = 8.75 and b = 1/7, f = 0.079 $Re^{-0.25}$ (Re $<$ 50000)
   For a = 10.3 and b = 1/9, f = 0.046 $Re^{-0.2}$ (Re $>$ 50000)
3. For a Rough pipe , log-law is given by (lecture 28)
   $u^+ = \kappa^{-1} \ln(y^+/y^+_{re}) + 8.48 = \kappa^{-1} \ln(29.73\, y^+/y^+_{re})$. Then, carrying out integration as before, it can be shown that

$$\frac{f}{2} = \left[2.5 \ln\left(\frac{D}{y_{re}}\right) + 3.0 \right]^{-2}$$

This eqn is independent of Reynolds number.