ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-22 SUSTAINING MECHANISM OF TURBULENCE-1

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1. Continuum Approximation
2. Turbulent Kinetic Energy Equation
3. Scale Analysis
4. Spatial and Auto Correlation Coefficient

Continuum Approximation L22($\frac{1}{20}$)

1. In spite of irregular motion, is continuum approximation valid in turbulent flows ? In other words, can the fluctuations split the fluid ?
2. In turbulent flows, the scales of velocity fluctuations vary from as high as that of the mean flow (1 cm/s to 1 m/s, say) to *very low scales* (that are governed by the presence of molecular viscosity). The associated length scales vary from as high as the mean flow dimension (BL thickness δ or pipe radius R) to a very small fraction of the mean dimension.
3. The molecular velocity in air, for example, is of the order of 50 m/s and the mean-free-path-length is of the order of 10^{-4} mm $<< \delta$ or R.
4. Also, turbulence frequencies vary between 1 and 10000 sec^{-1} whereas molecular frequencies are about 5×10^9 sec^{-1}

Main Postulate - L22($\frac{2}{20}$)

1. The numbers of the previous slide suggest that the fluid viscosity will continue to influence events in a turbulent flow in two ways.
   1. Firstly, by causing diffusion of the transported property
   2. Secondly, through dissipation of energy of the fluctuations (to heat) since the turbulent fluctuations are indeed killed by the action of viscosity and fluid continuum is maintained.
2. A mechanism must therefore exist that feeds energy from the mean motion to sustain turbulence.
3. Study of this mechanism reveals that in vigourously turbulent flow, the diffusive role of viscosity is marginal, the viscosity plays its principal role through energy dissipation.
4. This is in contrast to what occurs in laminar flow where the diffusive influence dominates over the dissipative one unless the fluid viscosity was high.

Instantaneous KE Eqn - 1 - L22($\frac{3}{20}$)

The Eqn for IKE $\hat{E} \equiv \hat{u}_i\, \hat{u}_i / 2$ is derived from the N-S Eqns by first multiplying instantaneous momentum equations by $\hat{u}_i$ and then adding the three equations.

$$
\begin{aligned}
\rho\, \frac{D\hat{E}}{Dt} &= -\frac{\partial}{\partial x_i}\left(\hat{p}\hat{u}_i\right) + \frac{\partial}{\partial x_i}\left(\hat{u}_j \hat{\tau}_{ij}\right) - \mu\hat{\Phi}_v \\
\hat{\tau}_{ij} &= \mu\, \hat{S}_{ij} = \mu \left[\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}\right] \\
\mu\hat{\Phi}_v &= \hat{\tau}_{ij}\, \frac{\partial \hat{u}_i}{\partial x_j}
\end{aligned}
$$

Mean KE Eqn - 2 - L22($\frac{4}{20}$)

The Eqn for MKE (E $\equiv u_i\, u_i / 2$) is derived from time-averaged N-S Eqns by first multiplying by u_i and then adding the three equations.

$$\begin{aligned}
\rho \frac{DE}{Dt} &= -\frac{\partial}{\partial x_i}(p u_i) + \frac{\partial}{\partial x_j}(\tau_{ij}\, u_i) \\
(a) & \qquad\qquad (b) \qquad\qquad\qquad (c) \\
&+ \frac{\partial}{\partial x_j}(-\,\rho\overline{u_i' u_j'}\, u_i) - \mu\Phi_v - (-\rho\,\overline{u_i' u_j'}\,\frac{\partial u_i}{\partial x_j}) \\
& \qquad\qquad (d) \qquad\qquad\qquad (e) \qquad\qquad\qquad (f)
\end{aligned}$$

The total rate of change of mean E (a) = the rate of work done by pressure forces (b) + by viscous stresses (c) + by turbulent stresses (d) - the rate of energy *dissipated by viscous action* (e) - the rate of energy *transferred to turbulence by the mean motion* (f).

Turbulent KE Eqn - 3 - L22($\frac{5}{20}$)

The Eqn for TKE ($e \equiv \overline{u_i'\, u_i'}/\,2$) is derived by first time-averaging Eqn for IKE ($\hat{E}$).

$$\rho\,\frac{D(E+e)}{Dt} + \frac{\partial}{\partial x_j}\,(u_i\,\rho\overline{u_i' u_j'}) + \frac{\partial}{\partial x_j}\,(\rho\overline{u_j' u_i'\, u_i'}/2) =$$

$$-\frac{\partial}{\partial x_i}\,(p u_i + \overline{p' u_i'}) + \frac{\partial}{\partial x_j}\,(\tau_{ij}\, u_i + \overline{\tau_{ij}'\, u'_i}) - \tau_{ij}\,\frac{\partial u_i}{\partial x_j} - \overline{\tau_{ij}'\,\frac{\partial u_i'}{\partial x_j}}$$

Then, the Eqn for MKE (E) is subtracted

$$\rho\,\frac{De}{Dt} \;=\; -\frac{\partial}{\partial x_j}\,\overline{u_j'(p' + \rho\, u_i'\, u_i'/2)} + (-\rho\overline{u_i' u_j'}\,\frac{\partial u_i}{\partial x_j})$$

$$(A) \qquad\qquad (B) \qquad\qquad (C)$$

$$+ \quad \frac{\partial}{\partial x_j}\,(\overline{u_j'\,\tau_{ij}'}) - \overline{\tau_{ij}'\,\frac{\partial u_i'}{\partial x_j}}$$

$$(D) \qquad\qquad (E)$$

(next slide)

Comments on TKE - 1 - L22($\frac{6}{20}$)

The rate of change of TKE e (A) =

+ the rate of (convective) diffusion of total fluctuating pressure energy ($p' + \rho u_i' u_i'/2$) by velocity fluctuation (B)

+ the rate of energy is *transferred from mean motion to turbulence* by the turbulent stresses (C)

+ the rate of work done by viscous turbulent stresses (D)

- the rate of *dissipation of energy by the turbulent motion* (E).

1. Eqn of MKE (E) shows that E is lost in two ways
    1. Firstly, by viscous dissipation (term e)
    2. Secondly, by (term f) which appears as a positive contributor to TKE via (term C). Hence, term C is called *Production or Generation* term
2. In a laminar flow, E is directly dissipated into heat. In a turbulent flow, E is *first transferred to sustain turbulence* before finally *dissipating to heat through (term E)*

Comments on TKE - 2 - L22($\frac{7}{20}$)

1. Besides dissipation, MKE (E) and TKE (e) experiences convective-diffusion of energy through terms b, c, d, B and D. These terms merely redistribute (spatially) energy but make zero net contribution to the integral energy balance as shown below.
2. If the turbulent flow bounded by walls ($u_i = u_i^{'} = 0$) or by a wall and a symmetry plane ($\tau_{ij} = \tau_{ij}^{'} = 0$) is considered and equation for TKE is integrated over flow cross-section

$$\rho \frac{D}{Dt} \int_V e \, dV = \int_V \left\{ (-\rho \overline{u_i^{'} u_j^{'}} \frac{\partial u_i}{\partial x_j}) - (\overline{\tau_{ij}^{'} \frac{\partial u_i^{'}}{\partial x_j}}) \right\} dV$$

$$\text{Net change} = \text{Net (Production - Dissipation)}$$

3. Net change $>$ 0 when Production $>$ Dissipation and *vice-versa*. The *near-balance* represents Transition

Scale Analysis - 1 - L22($\frac{8}{20}$)

1. Thus, turbulence derives its sustenance by drawing energy from mean motion. How does this transfer take place ?
2. A laminar BL is characterised by two length scales δ and distance x such that $(\delta/x) \propto Re_x^{-0.5} << 1$. The relevant time scale is $t = x/\,U_\infty$. Therefore, $\delta \propto (\nu\, t)^{0.5}$. More importantly, δ could be *discovered* only because of the inclusion of the transverse diffusion term ($\mu\, \partial^2 u/\partial y^2$). In other words, the smaller length scale δ is *associated* with the effect of viscosity.
3. In a turbulent BL , motions of several scales occur simultaneously. Let V'_{mean} represent velocity fluctuation in the direction y away from the wall . Then, transverse momentum is carried out by $-\rho \overline{u' V'_{mean}} >> \mu\, \partial u/\partial y$. And, $d\,\delta/d\,t \propto V'_{mean}$ or, $\delta \propto V'_{mean}\, t = V'_{mean}\, x/\,U_\infty$. Thus, diffusion time scale $\delta/\,V'_{mean} \simeq$ mean time scale $(x/\,U_\infty\,)$ and $V'_{mean} \equiv$ large scale.

Scale Analysis - 2 - L22($\frac{9}{20}$)

1. Recall that the dissipation process ($\rho\,\epsilon = \overline{\tau_{ij}^{'}\,\partial u_i^{'}/\partial x_j}$) essentially *kills or smoothens out* velocity fluctuations due to action of viscosity and $l_\epsilon << l_{mean}$ and $t_\epsilon << t_{mean}$.
2. At such very small scales, turbulent fluctuations in all three directions could be considered statistically equal. That is, $\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$ and their spatial variations are also small.
3. Such a small scale turbulence structure is called *homogeneous and isotropic*. It is characterised in association with ϵ by *Kolmogorov Scales*.

$$u_\epsilon^{'} = v_\epsilon^{'} = w_\epsilon^{'} \quad = \quad (\nu\,\epsilon)^{0.25} \ \text{(velocity scale)}$$

$$t_\epsilon = (\frac{\nu}{\epsilon})^{0.50} \ \text{(time scale)} \qquad l_\epsilon = (\frac{\nu^3}{\epsilon})^{0.25} \ \text{(length scale)}$$

$$Re_{t,\epsilon} \equiv \frac{(l\,v^{'})_\epsilon}{\nu} = 1 \quad << \quad Re_{t,mean} = \frac{(l\,V^{'})_{mean}}{\nu} \simeq O(100)$$

Scale Analysis - 3 - L22($\frac{10}{20}$)

1. Thus, we have provided relative estimates of largest and smallest fluctuations.
2. Whenever large scale fluctuations are present, small scale motions are automatically created so that viscosity can play its major role via energy dissipation.
3. The creation of small scale motions is believed to be caused by the non-linear convective terms in the N-S equations.
4. That this creation is not a one-step process but takes place in a large number of continuous steps will be shortly demonstrated.
5. The large scale fluctuations thus create smaller scale fluctuations which in turn transfer their energy to produce even smaller scale fluctuations and so on till the scales are so small that non-linear terms become unimportant and viscosity takes over to produce an isotropic structure of turbulence.

Spatial Correlation - 1 - L22($\frac{11}{20}$)

In turbulence literature, the idea of scales is often expressed through the notion of an *eddy*. Whenever a fluctuation occurs, it can be expected to influence events over a zone that extends both spatially and in time. The **eddy**, notionally represents the *size of this zone.*

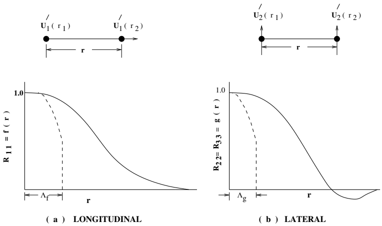


Figure: Here $\lambda_f = l_f$ and $\lambda_g = l_g$

Consider two points at positions $\vec{r_1}$ and $\vec{r_2}$ with $\vec{r} = \vec{r_2} - \vec{r_1}$. Then, let u_i' at $\vec{r_1}$ (x_1, x_2, x_3) and u_j' at $\vec{r_2}(x_1 + r, x_2, x_3)$ be the velocity fluctuations at the *same time instant*

Define Spatial correlation coefficient

$$R_{ij} = \frac{B_{ij}}{\sqrt{B_{ii}}\sqrt{B_{jj}}} \quad \rightarrow \quad B_{ij} = \overline{u_i' \, u_j'}$$

Spatial Correlation - 2 - L22($\frac{12}{20}$)

1. R_{ij} has nine components in general. Being a coefficient, R_{ij} will be bounded by -1 and +1.
2. At these two extremes, the correlation between $u_i^{'}$ and $u_j^{'}$ is said to be *perfect*. When $R_{ij} = 0$, no correlation exists between $u_i^{'}$ and $u_j^{'}$ which would understandably be the case as $\vec{r} \rightarrow \infty$.
3. For $0 < |R_{ij}| < 1$, the correlation is said to be moderate.
4. It is tedious to measure R_{ij} for a real non-homogeneous, non-isotropic turbulent flow since nine components must be measured in all directions for different values of $\vec{r}$ and $\vec{r_1}$. Usually, only R_{ij} $(\vec{r_1}, 0, 0)$ are measured.

Spatial Correlation - 3 - L22($\frac{13}{20}$)

1. In homogeneous turbulence , all statistical correlations $\partial(\overline{\phi_1'\,\phi_2'})/\partial x_i = 0$ but, $\partial(\overline{\phi_1})/\partial x_i$ and $\partial(\overline{\phi_2})/\partial x_i$ can be finite.
2. *Isotropic* turbulence implies that any relation between turbulence quantities must be constant (invariant) under rotation of the coordinate system and under reflection with respect to the coordinate system. As such, turbulence cannot be isotropic unless it was also homogeneous.
3. For a homogeneous and isotropic turbulence , only R_{11}, R_{22} and, R_{33} are finite since $R_{ij} = 0$ for $i \neq j$. For 180^o rotation about x_1-axis, from reflection condition, $\overline{u_1'\,u_2'} = \overline{u_1'\,(-u_2')} = -\,\overline{u_1'\,u_2'}$. This is true only if $\overline{u_1'\,u_2'} = 0$.
4. Further, $R_{22} = R_{33}$ since the coordinate system is invariant under rotation about x_1-axis (say). The $R_{11} \equiv f(r)$ coefficient parallel to x_1 is called the *longitudinal coefficient* and coefficient $R_{22} = R_{33} \equiv g(r)$ is called the *lateral coefficient*. (see slide 11)

Spatial Correlation - 4 - L22($\frac{14}{20}$)

1. Both f (r) and g (r) decline to zero as $r \to \infty$. For a given r, f(r) is greater than g(r).
2. The coefficient curves are nearly parabolic near $r = 0$ and therefore symmetric about $r = 0$. Expanding f(r) and g(r) in Taylor's series about $r = 0$

$$f(r) \simeq 1 - (\frac{r}{l_f})^2 + \ldots \qquad g(r) \simeq 1 - (\frac{r}{l_g})^2 + \ldots$$

$$l_f^2 = -2\,(\frac{\partial^2 f}{\partial r^2}\,|_{r=0})^{-1} \qquad l_g^2 = -2\,(\frac{\partial^2 g}{\partial r^2}\,|_{r=0})^{-1}$$

3. Hence, l_f and l_g in Taylor's micro-scales range,

$$l_f \equiv \left[\frac{2\,\overline{(u_1')^2}}{\overline{(\partial u_1'/\partial x_1)^2}}\right]^{0.5} \qquad l_g \equiv \left[\frac{2\,\overline{(u_2')^2}}{\overline{(\partial u_2'/\partial x_1)^2}}\right]^{0.5}$$

Micro & Integarl Length Scales - L22($\frac{15}{20}$)

1. In l_f and l_g, derivatives of velocity fluctuations are difficult to measure
2. None-the-less, if this local spatial change is *imagined* to have been caused by the smallest scales of motion, then l_f and l_g can be regarded as the average dimensions of the *range of small scale motions*.
3. Similarly, we can define Integral scales as

$$l_{int,f} = \int_0^\infty f(r)\,dr \quad \text{and} \quad l_{int,g} = \int_0^\infty g(r)\,dr$$

4. Thus, we have 4 length scales l_f, l_g, $l_{int,f}$ and $l_{int,g}$ in a simple homogeneous isotropic turbulence besides l_ϵ at the smallest Kolmogorov scales where viscosity kills turbulence and isotropy prevails.

Estimate of ϵ - L22($\frac{16}{20}$)

1. From slide 9, $l_\epsilon = (\nu^3/\epsilon)^{0.25}$. The dissipation rate ϵ can be estimated by noting that in isotropic turbulence[1],

$$\overline{(\frac{\partial u_1'}{\partial x_1})^2} = \overline{(\frac{\partial u_2'}{\partial x_2})^2} = \frac{1}{2}\,\overline{(\frac{\partial u_1'}{\partial x_2})^2} = \frac{1}{2}\,\overline{(\frac{\partial u_2'}{\partial x_1})^2} = \text{etc.}$$

2. Hence, l_f and l_g are related (see slide 14) to ϵ as

$$\begin{aligned} \rho\,\epsilon &= \mu\,\overline{\left[\partial u_i'/\partial x_j + \partial u_j'/\partial x_i\right]^2} \quad \text{(definition)} \\ \epsilon &= 15\,\nu\,\overline{(\frac{\partial u_1'}{\partial x_1})^2} = 30\,\nu\,(\frac{u_1'}{l_f})^2 = 15\,\nu\,(\frac{u_2'}{l_g})^2 \end{aligned}$$

[1] Hinze J O - *Turbulence, an Introduction to its Mechanism and Theory* , McGraw-Hill, New York, 1959

Ratio of Strain Rates - L22($\frac{17}{20}$)

1. To estimate l_{int}, consider *homogeneous pure shear flow* in which the strain rate S_{ij} = constant . Then, from TKE eqn, setting derivatives of all statistical relations to zero

$$-\rho\,\overline{u_i'u_j'}\,\frac{\partial u_i}{\partial x_j} \;=\; \overline{\tau_{ij}'\,\frac{\partial u_i'}{\partial x_j}} = \rho\,\epsilon \quad \text{(Prod = Diss)}$$

$$\text{or,}\quad -\,\overline{u_i'u_j'}\,(S_{ij}/2) \;=\; \nu\,\overline{s_{ij}\,s_{ij}}/2 = \epsilon$$

2. The LHS of this Eqn is associated with large scale motion. Hence

$$\begin{aligned}
-\overline{u_i'\,u_j'}\,(S_{ij}/2) &\simeq (V')^2\,(V'/l_{int}) \simeq (V')^3/l_{int} \\
\epsilon &= \nu\,\overline{s_{ij}\,s_{ij}}/2 \simeq (V')^3/l_{int} \quad \text{(Imp result)} \\
\frac{\overline{s_{ij}\,s_{ij}}}{S_{ij}\,S_{ij}} &\simeq \frac{(V')^3/(l_{int}\,\nu)}{(V'/l_{int})^2} = \frac{V'\,l_{int}}{\nu} = Re_{t,l_{int}} \rightarrow O(100)
\end{aligned}$$

Comparison of Scales - L22($\frac{18}{20}$)

From the results of previous 2 slides,

1. Taylor and Kolmogorov scales are related as

$$t_f = \frac{l_f}{u_1^{'}} = \sqrt{\frac{30\,\nu}{\epsilon}} = \sqrt{30}\,t_\epsilon \qquad \frac{l_f}{l_\epsilon} = \frac{\sqrt{30}\,u_1^{'}}{(\nu\,\epsilon)^{0.25}}$$

$$t_g = \frac{l_g}{u_2^{'}} = \sqrt{\frac{15\,\nu}{\epsilon}} = \sqrt{15}\,t_\epsilon \qquad \frac{l_g}{l_\epsilon} = \frac{\sqrt{15}\,u_2^{'}}{(\nu\,\epsilon)^{0.25}}$$

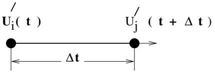
2. Integral and Taylor scales are related as

$$\epsilon \simeq \frac{(V^{'})^3}{l_{int}} \simeq 15\,\nu\,(\frac{u_2^{'}}{l_g})^2 \quad \rightarrow \quad u_2^{'} = A\,V^{'} \text{ (say, with) } A > 1$$

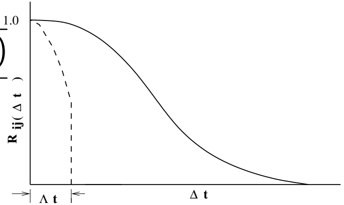
$$\frac{l_g}{l_{int}} \simeq A\,\sqrt{15}\sqrt{(\frac{\nu}{V^{'}\,l_{int}})} = \frac{A\,\sqrt{15}}{(Re_{t,l_{int}})^{0.5}} \quad \rightarrow \quad \frac{t_g}{t_{int}} = \sqrt{\frac{15}{Re_{t,int}}}$$

$$\frac{t_\epsilon}{t_{int}} \simeq (Re_{t,l_{int}})^{-0.5}$$ and $(l_\epsilon < l_{f,g} < l_{int})$ and $(t_\epsilon < t_{f,g} < t_{int})$

Auto-Correlation - L22($\frac{19}{20}$)



To estimate the timescale of a turbulent eddy



$$
\begin{aligned}
R_{ij}\,(x_k\,,\Delta\,t) &= \frac{\overline{u_i^{'}(x_k\,,t)\,u_j^{'}(x_k\,,t+\Delta t)}}{\sqrt{\overline{(u_i^{'})^2}}\,\sqrt{\overline{(u_j^{'})^2}}} \\
\tau_{micro} &= -\,\Big(\frac{1}{2}\,\frac{\partial^2 R}{\partial t^2}\,\Big|_{\Delta\,t=0}\Big)^{-0.5} \\
&= u_i^{'}\,\left[\Big(\frac{\partial u_i^{'}}{\partial t}\Big)_{\Delta\,t=0}\right]^{-1} \\
\tau_{int} &= \int_0^{\infty} R_{ij}\,d(\Delta\,t)
\end{aligned}
$$

The time derivatives of fluctuations at a fixed point are simpler to measure. Taylor's Hypothesis states that if $u_1 >> u_1^{'}$, then $\partial u_1^{'}/\partial t = -\,u_1\,\partial u_1^{'}/\partial x_1$. Hence $R_{11}(x_1)dx_1 = u_1\,R_{11}(\Delta t)\,dt$ and $l_{int} = u_1\,\tau_{int}$

In Reynolds's averaging $t_{max} >> \tau_{int}$ (see previous lecture)

Final comment - L22($\frac{20}{20}$)

1. We have shown how turbulence, once generated, sustains itself by creating fluctuations of ever smaller and smaller length and time scales.
2. This was shown
   1. Firstly from observing terms in Kinetic Energy Equations
   2. Secondly from transverse momentum transfer process in a boundary layer
   3. Thirdly, from scale analysis
3. We have shown that although ϵ is associated with very small scale motions, its magnitude can be estimated from large scale motion . This fact is extensively used in turbulence modeling of RANS equations.
4. Length and time scales of eddies are easier to measure from Auto-correlation.
5. Next lecture deals with spectral analysis and vorticity dynamics .