ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-16 FULLY-DEVELOPED LAMINAR FLOWS-2

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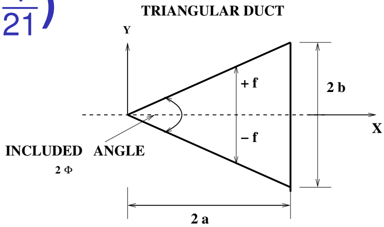
1. Friction Factor - Triangular Cross Section
2. Friction Factor - Arbitrary Cross-sections

Triangular Duct - L16($\frac{1}{21}$)



Governing Eqn

$$\frac{\partial^2 u^*}{\partial {x^*}^2} + \frac{\partial^2 u^*}{\partial {y^*}^2} = 1 \qquad (1)$$

$$u^* = \frac{u}{(4\, a^2 \frac{1}{\mu}\, \frac{d\,p}{d\,z})}$$

$$x^* = \frac{x}{2a} \quad y^* = \frac{y}{2a}$$

with BCs

$u^* = 0$ at $x^* = 0$ and 1

$u^* = 0$ at $y^* = \pm\, f(x^*) = \pm\, m\, x^*$

Solution is obtained by Variational Method due to Kantarovich. Thus let,

$u^* = (f^2 - {y^*}^2)\, F(x^*)$

The objective is to find $F(x^*)$.

where m = tan Φ. We restrict attention to $2\Phi < 90^0$, so that $m < 1$.

Variational Method - L16($\frac{2}{21}$)

The variational

$$\delta\, I = \int_0^1 \int_{-f}^{f} (\frac{\partial^2 u^*}{\partial {x^*}^2} + \frac{\partial^2 u^*}{\partial {y^*}^2} - 1)\, \delta\, u^*\, dx^*\, dy^* = 0 \qquad (2)$$

Note that df / dx^* = m = tan Φ = const. Hence, in the present case, $d^2 f / d{x^*}^2 = 0$. Then, letting dF / dx^* = F' etc,

$$\begin{aligned} \frac{\partial^2 u^*}{\partial {x^*}^2} &= (f^2 - {y^*}^2)\, F'' + 4\, m\, f\, F' + 2\, m^2\, F \\ \frac{\partial^2 u^*}{\partial {y^*}^2} &= -2\, F \end{aligned}$$

Substitution for u^* and the derivatives and, further carrying out the integrations gives (see next slide)

Solution-1 - L16($\frac{3}{21}$)

$$\delta I = \frac{4}{3} f^3 \delta \int_0^1 \left[\frac{4}{5}\, f^2\, F^{''} + \left\{ 4\, m\, f\, F^{'} + 2(m^2 - 1) \right\}\, F - 1 \right] F dx^* = 0$$

This implies that terms in the square bracket equal zero. Or,

$$\frac{4}{5}\, f^2\, F^{''} + \left\{ 4\, m\, f\, F^{'} + 2(m^2 - 1) \right\}\, F - 1 = 0$$

Define $F^* = F - 0.5\,(m^2 - 1)^{-1}$. Then, since, f = m x^*,

$$x^{*^2}\, F^{*^{''}} + 5\, x^*\, F^{*^{'}} + \frac{5}{2}\,(\frac{m^2 - 1}{m^2})\, F^* = 0$$

Solution-2 - L16($\frac{4}{21}$)

The last eqn can be transformed to read as

$$\frac{1}{{x^*}^3}\,\frac{d}{dx^*}\,\left[{x^*}^5\,\frac{dF^*}{dx^*}\right]+\frac{5}{2}\,(\frac{m^2-1}{m^2})\,F^*=0$$

The solution is: $F^* = F - 0.5\,(m^2-1)^{-1} = A\,{x^*}^{R_1} + B\,{x^*}^{R_2}$. Or,

$F = 0.5\,(m^2-1)^{-1} + A\,{x^*}^{R_1} + B\,{x^*}^{R_2}$. Or,

$u^* = (m^2\,{x^*}^2 - {y^*}^2)\,\left\{0.5\,(m^2-1)^{-1} + A\,{x^*}^{R_1} + B\,{x^*}^{R_2}\right\}$

where $R_1 = 0.5\,\left[-4+\left\{16-10\,(\frac{m^2-1}{m^2})\right\}^{0.5}\right]$

and $R_2 = 0.5\,\left[-4-\left\{16-10\,(\frac{m^2-1}{m^2})\right\}^{0.5}\right]$

Constants A and B are to be determined from the boundary condition $u^* = 0$ at $x^* = 0$ and 1.

Solution-3 - L16($\frac{5}{21}$)

Condition at $x^* = 1$ gives, A + B = $-0.5 * (m^2 - 1)^{-1}$.
Now, for $m < 1$, $R_2 < 0$. Therefore, condition at $x^* = 0$ gives, B = 0. Hence, the final solutions is:
$u^* = -0.5\,(m^2 - 1)^{-1}\,(m^2\,{x^*}^2 - {y^*}^2)\,({x^*}^{R_1} - 1)$.

Integration gives

$$\overline{u}^* = \frac{\int_0^1 \int_{-f}^{f} u^*\,dx^*\,dy^*}{\int_0^1 \int_{-f}^{f} dx^*\,dy^*} = \frac{1}{6}\,(\frac{m^2}{m^2 - 1})\,(\frac{R_1}{R_1 + 1})$$

Further, it can be shown that $D_h/(2a) = 2\,m\,(m + \sqrt{m^2 + 1})^{-1}$.
Hence,

$$f_{fd}\,Re = \frac{1}{2\,\overline{u}^*}\,(\frac{D_h}{2a})^2 = \frac{12\,(m^2 - 1)}{(m + \sqrt{m^2 + 1})^2}\,(\frac{4}{R_1} + 1)$$

Parametric Solutions - L16($\frac{6}{21}$)

2Φ	m	R_1	D_h / 2a	f_{fd} Re
85	0.9163	0.11598	0.80639	13.219
75	0.7673	0.39708	0.7568	13.288
60	0.5773	1.00	0.6667	13.333
50	0.4663	1.60517	0.59414	13.308
40	0.3640	2.5135	0.50971	13.2267
30	0.26795	4.0266	0.4112	13.073
20	0.1763	7.0503	0.29591	12.8309
10	0.08749	16.114	0.16034	12.4808
5	0.04366	34.234	0.08359	12.258

2Φ = 60 degrees corresponds to an Equilateral Triangle . Methods of this type are not general. For different ducts such as ellpitical or triangular with rounded corners, different strategies must be invoked. Therefore, we seek a general method applicable to all types of complex ducts. (see next slide)

Arbitrary Cross-Sections - L16($\frac{7}{21}$)

Governing Eqn

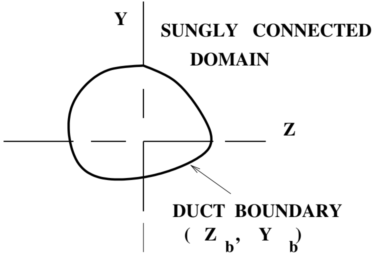
$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{d\,p}{d\,x} = \text{Const}$$

Define

$$\frac{u}{-\frac{1}{\mu}\frac{d\,p}{d\,x}} = u^* - \left(\frac{z^2 + y^2}{4}\right)$$

Hence, Laplace Eqn

$$\frac{\partial^2 u^*}{\partial z^2} + \frac{\partial^2 u^*}{\partial y^2} = 0$$



No-slip condition implies

$$u_b^* = \left(\frac{z_b^2 + y_b^2}{4}\right)$$

Soln of Laplace Eqn - L16($\frac{8}{21}$)

Soln is given by

$$u^*(z, y) = \sum_{i=1}^{N} c_i \, g_i(z, y)$$

where c_i are coefficients to be determined and the functions g_i are prescribed by exploiting the following property of the Laplace equation:

For any positive integer n, the real and imaginary parts of the complex variable $(z + i\,y)^n$ are each exact solutions ($g_n(z, y)$) of the Laplace's equation.

Thus, by successively assigning n = 0, 1, 2, . . . 8 (say), the first seventeen solutions are given by (see next slide)

Functions $g_n(z, y)$ - L16($\frac{9}{21}$)

$$
\begin{aligned}
g_1 &= 1\ (n = 0) \\
g_2 &= z\ (n = 1) \\
g_3 &= y\ (n = 1) \\
g_4 &= z^2 - y^2\ (n = 2) \\
g_5 &= 2\ z\ y\ (n = 2\ etc) \\
g_6 &= z^3 - 3\ z\ y^2 \\
g_7 &= 3\ y\ z^2 - y^3 \\
g_8 &= z^4 + y^4 - 6\ z^2\ y^2 \\
g_9 &= 4\ z^3\ y - 4\ z\ y^3 \\
g_{10} &= z^5 - 10\ z^3\ y^2 + 5\ z\ y^4 \\
g_{11} &= y^5 - 10\ y^3\ z^2 + 5\ y\ z^4
\end{aligned}
$$

$$
\begin{aligned}
g_{12} &= z^6 - 15\ z^4\ y^2 + 15\ z^2\ y^4 \\
&- y^6 \\
g_{13} &= 6\ z^5\ y + 6\ z\ y^5 - 20\ z^3\ y^3 \\
g_{14} &= z^7 - 21\ z^5\ y^2 + 35\ z^3\ y^4 \\
&- 7\ y^6\ z \\
g_{15} &= -y^7 + 21\ y^5\ z^2 \\
&- 35\ y^3\ z^4 + 7\ z^6\ y \\
g_{16} &= z^8 + y^8 - 28\ z^6\ y^2 \\
&- 28\ y^6\ z^2 + 70\ z^4\ y^4 \\
g_{17} &= 8\ z^7\ y - 56\ z^5\ y^2 \\
&+ 56\ z^3\ y^5 - 8\ x\ y^7
\end{aligned}
$$

Coefficients C_i - L16($\frac{10}{21}$)

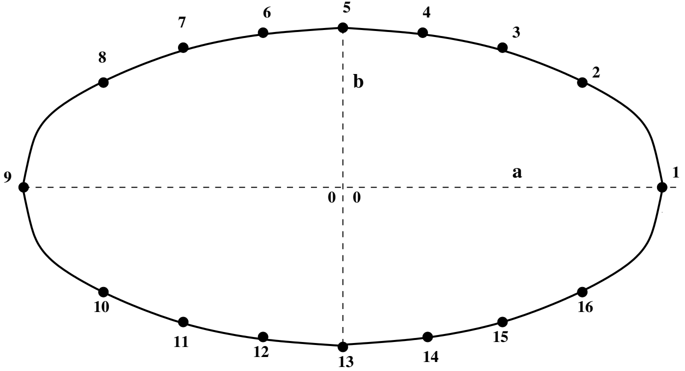
We choose 16 boundary points (say)
The coefficients $c_{i=1,2,...,16}$ are determined from 16 boundary conditions. Thus,

$$u^* \left(z_b, y_b\right) = \left(\frac{z_b^2 + y_b^2}{4}\right) = \sum_{i=1}^{N} c_i \, g_i \left(z_b, y_b\right)$$

The coefficients are determined by LU-decomposition followed by forward elimination and backward substitution Procedure[1]

[1] Another method is to use Gramm-Schmidt Ortho-normalisation method

Example - Elliptical Duct L16($\frac{11}{21}$)



See next slide for the computed data for b =1, a = 2

Coordinates and functions L16($\frac{12}{21}$)

z_b	y_b	RHS_i	c_i
2.0	0.0000	1.0000	0.40
1.5	0.6614	0.6719	0.00
1.0	0.8660	0.4375	0.00
0.5	0.9682	0.2969	0.15
0.0	1.0000	0.2500	0.0
−0.5	0.9682	0.2969	0.0
−1.0	0.8660	0.4375	0.0
−1.5	0.6614	0.6719	0.0
−2.0	0.0000	1.0000	0.0

z_b	y_b	RHS_i	c_i
−1.5	−0.6614	0.6719	0.0
−1.0	−0.8660	0.4375	0.0
−0.5	−0.9682	0.2969	0.0
0.0	−1.0000	0.2500	0.0
0.5	−0.9682	0.2969	0.0
1.0	−0.8660	0.4375	0.0
1.5	−0.6614	0.6719	0.0

Data for b = 1, a = 2.
$RHS_i = (z_b^2 + y_b^2)/4$

Note that only c_1 and c_4 are non-zero. This is found to be true for all values of a and b

Final Solution - 1 - L16($\frac{13}{21}$)

Hence, the solution for b=1 and a = 2 is

$$\frac{u}{-\frac{1}{\mu}\frac{d\,p}{d\,x}} = 0.4 + 0.15\,(z^2 - y^2) - (\frac{z^2 + y^2}{4})$$

$$= 0.4 - 0.1\,z^2 - 0.4\,y^2$$

$$\frac{\overline{u}}{-\frac{1}{\mu}\frac{d\,p}{d\,x}} = \frac{\int_0^a \int_0^{y_b} u\,dz\,dy}{\int_0^a \int_0^{y_b} dz\,dy} = 0.2$$

$$y_b = b\left[1 - \frac{z_b^2}{a^2}\right]^{0.5} \quad \text{(Eqn of Ellipse)}$$

$$\frac{u}{\overline{u}} = 2 - 0.5\,z^2 - 2\,y^2$$

Final Solution - 2 - L16($\frac{14}{21}$)

Generalisation gives:

$$\frac{u}{-\frac{1}{\mu}\,\frac{d\,p}{d\,x}} = c_1 + (c_4 - 0.25)\,z^2 - (c_4 + 0.25)\,y^2$$

$$\frac{\overline{u}}{-\frac{1}{\mu}\,\frac{d\,p}{d\,x}} = c_1 + 0.25\,(c_4 - 0.25)\,a^2 - 0.25\,(c_4 + 0.25)\,b^2$$

$$f_{fd}\,Re = D_h^2/(2 \times \frac{\overline{u}}{-\frac{1}{\mu}\,\frac{d\,p}{d\,x}})$$

$$D_h = 4\,(\frac{A}{P}) \quad \text{and} \quad \mathrm{A} = a\,b\,\pi$$

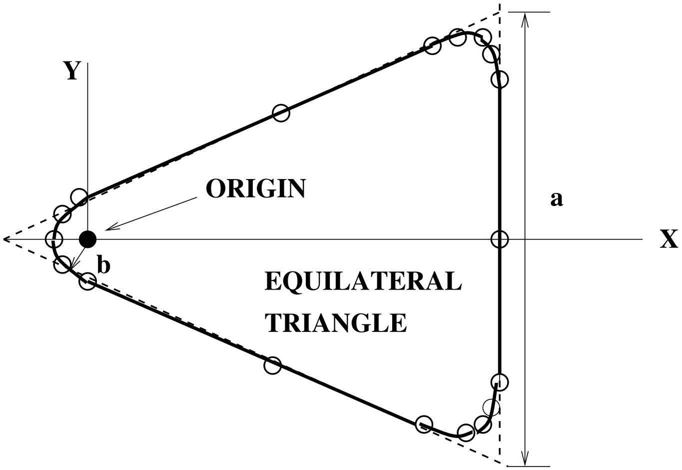
$$\mathrm{P} = \pi\,(a+b)\left[1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{64} + \frac{\lambda^6}{256} + \frac{25\,\lambda^8}{16384}\right] \quad \lambda = \frac{a-b}{a+b}$$

Results - Ellipse - L16($\frac{15}{21}$)

b	a	C_1	C_4	A / b^2	P/b	D_h/b	f_{fd} Re
1	1	0.25	0.00	π	2 π	2.0	16.00
1	1.25	0.3049	0.0549	3.927	7.0904	2.254	16.098
1	1.67	0.3676	0.1176	5.236	8.059	2.461	16.479
1	2	0.4	0.15	6.283	9.688	2.594	16.823
1	2.5	0.431	0.181	7.854	11.506	2.730	17.294
1	5.0	0.4808	0.2308	15.708	21.008	2.991	18.605
1	10.0	0.495	0.245	31.416	40.623	3.0934	19.329

a = 1 and b = 1 corresponds to circular tube

Triangle with Rounded Corners - L16($\frac{16}{21}$)



17 points are chosen. b = rounding radius, a = unrounded side

Coefficients c_i - L16($\frac{17}{21}$)

RHS_i	c_i
0.0006	0.756E-02
0.0006	0.198E+00
0.0006	-0.371E-01
0.0427	0.680E+00
0.1592	-0.763E+00
0.1795	-0.838E+01
0.1860	0.384E+01
0.1908	0.317E+02
0.1894	-0.804E+01
0.1467	-0.580E+02

RHS_i	c_i
0.1467	-0.580E+02
0.1894	0.890E+01
0.1908	0.585E+02
0.1860	-0.528E+01
0.1795	-0.321E+02
0.1592	0.139E+01
0.0427	0.761E+01
0.0006	0.224E-01

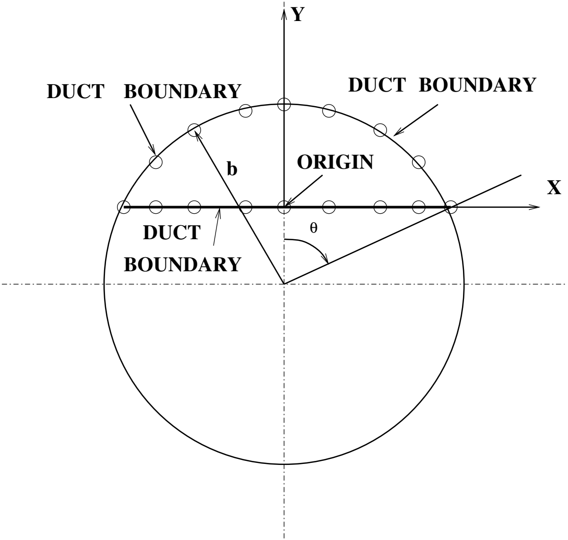
b = 0.05, a = 1. z_b and y_b are not listed.

Triangle with Rounded Corners - L16($\frac{18}{21}$)

b	a	A / a^2	P/a	D_h/a	f_{fd} Re
0.0	1.0	0.433	3.0	0.57735	13.33
0.05	1.0	0.4142	2.794	0.59287	14.91
0.10	1.0	0.4031	2.588	0.62277	15.66
0.167	1.0	0.400	2.316	0.69207	15.74

b=0 corresponds to equilateral triangle with sharp corners

Circular Segment Cross-Section - L16($\frac{19}{21}$)



16 points are chosen. b = radius, θ = Apex angle

Coefficients and f_{fd}Re - L16($\frac{20}{21}$)

θ	D_h/b	c_3	c_4	c_7	c_{11}	c_{15}	f_{fd}Re
90	1.223	.426	.250	-.0816	-.0083	-.001	15.765
60	0.6422	.231	.250	-.0785	-.0104	-.0031	15.69
45	0.3825	.140	.250	-.0789	-.0115	-.005	15.643
30	0.177	.0655	.250	-.0805	-.0132	-.008	15.598
10	0.0202	.0076	.250	-.083	-.0133	-.0889	15.56

$c_1, c_2, c_5, c_6, c_8, c_9, c_{10}, c_{12}, c_{13}, c_{14}, c_{16} \to 0.$

$\theta = 90$ corresponds to a duct of semi-circular cross section.

Conclusions - L16($\frac{21}{21}$)

The method developed for Ducts of Arbitrary Cross-sections is most general. It can be applied to any Singly Connected Duct Cross Section . In lecture 18, we shall apply this method to FD heat transfer.

Important References:

1. Sparrow E M and Haji-Sheikh A *Flow and Heat Transfer in Ducts of Arbitrary Shape with Arbitrary Thermal Boundary Conditions*, Trans ASME Jnl of Heat Transfer, pp 351 - 358, Nov (1966)
2. Shah R K and London A L *Laminar Forced Convection in Ducts*, Advances in Heat Transfer, vol 15, Academic Press, New York (1978)