ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-15 FULLY-DEVELOPED LAMINAR FLOWS-1

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1. Definition
2. Friction Factor - Circular Cross-section
3. Friction Factor - Annular Cross-section
4. Friction Factor - Rectangular and Annular Sectors

Definition - L15($\frac{1}{15}$)

1. Fully-developed flow region occupies greater part of the tube length in ducts of large L / (D * Re).
2. Fully-developed flow friction factors f_{fd} provide the lower bounds to the apparent f_{app} and local f_l friction factors.
3. In laminar flows, $f_{fd} \times Re$ = const for the given duct
4. f_{fd} is evaluated from force balance

$$\Delta p \times A_c = \overline{\tau}_w \times P \times \Delta x$$

where $\overline{\tau}_w$ is average wall shear stress. Thus,

$$f_{fd} = \frac{\overline{\tau}_w}{\rho\,\overline{u}^2/2} = \frac{1}{2}\left|\frac{dp}{dx}\right| \frac{D_h}{\rho\,\overline{u}^2} \qquad D_h = \frac{4 \times A_c}{P}$$

5. This is called the Fanning's Friction Factor

Circular Tube - 1 - L15($\frac{2}{15}$)

1. When flow is fully-developed, $v_r = v_\theta = \partial u / \partial x$ =0 and dp / dx = const (negative)
2. Hence, the axial momentum equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = \frac{1}{\mu}\frac{d\,p}{d\,x} = \text{Constant} \qquad (1)$$

   with boundary conditions, u = 0 at r = R (tube wall) and $\partial u/\partial r = 0$ at r = 0 (symmetry).
3. Integrating equation 1 twice with respect to r and using bcs,

$$u = -\frac{R^2}{4\,\mu}\frac{d\,p}{d\,x}(1 - \frac{r^2}{R^2}) \qquad (2)$$

4. Hence,

$$\overline{u} = \frac{\int_0^R u\,r\,d\,r}{\int_0^R r\,d\,r} = -\frac{R^2}{8\,\mu}\frac{d\,p}{d\,x} \quad \text{or} \quad \frac{u}{\overline{u}} = 2\,(1 - \frac{r^2}{R^2}) \qquad (3)$$

Circular Tube - 2 - L15($\frac{3}{15}$)

Further, wall shear stress is evaluated as

$$\tau_w = -\mu \left(\frac{\partial u}{\partial r}\right)_{r=R} = -\frac{R}{2}\frac{d\,p}{d\,x} = \frac{4\,\mu\,\overline{u}}{R} \tag{4}$$

Hence,

$$f_{fd} = \frac{\tau_w}{\rho\,\overline{u}^2/2} = \frac{1}{2}\left|\frac{d\,p}{d\,x}\right|\frac{D}{\rho\,\overline{u}^2} = \frac{16}{Re} \tag{5}$$

Note that $f_{fd} \times Re$ = 16 = const. Also, for a circular tube, D_h = D and τ_w is circumferentially uniform.

Annulus - 1 - L15($\frac{4}{15}$)

1. For the annulus, equation 1 again applies with No-slip (u = 0) bcs at $r = r_i$ and $r = r_o$.
2. Integrating twice

$$u = \frac{1}{\mu}\frac{d\,p}{d\,x}\frac{r^2}{4} + C_1 \ln(r) + C_2 \quad (6)$$

$$C_1 = -\frac{1}{\mu}\frac{d\,p}{d\,x}\frac{r_m^2}{2} \quad C_2 = -\frac{1}{\mu}\frac{d\,p}{d\,x}\left[\frac{r_o^2 \ln r_i - r_i^2 \ln r_o}{2\ln(r_i/r_o)}\right] \quad (7)$$

$$\overline{u} = -\frac{1}{\mu}\frac{d\,p}{d\,x}\left[\frac{r_o^2 + r_i^2}{8} - \frac{r_m^2}{4}\right] \quad r_m^2 = \frac{r_i^2 - r_o^2}{2\ln(r_i/r_o)} \quad (8)$$

$$\frac{u}{\overline{u}} = 2\left[\frac{r_o^2 - r^2 + 2\,r_m^2 \ln(r/r_o)}{r_o^2 + r_i^2 - 2\,r_m^2}\right] \quad (9)$$

where r_m radius of maximum axial velocity or the location of $\partial u/\partial r = 0$.

Annulus - 2 - L15($\frac{5}{15}$)

Further, based on hydraulic diameter,

$$f_{fd} \times Re = (\frac{1}{2}\,|\frac{dp}{dx}|\,\frac{D_h}{\rho\,\overline{u}^2}) \times (\frac{\rho\,\overline{u}\,D_h}{\mu})$$

Hence, it can be shown that

$$f_{fd} \times Re = \frac{-\,16\,(1 - r^*)^2}{2\,r_m^{*\,2} - 1 - r^{*2}}$$

where $r^* = r_i/r_o$, $r_m^* = r_m/r_o$ and $D_h = 2\,(r_o - r_i)$.

Note that as $r^* \rightarrow 1$, $f_{fd} \times Re \rightarrow 24.0$ (that is, flow between parallel plates)

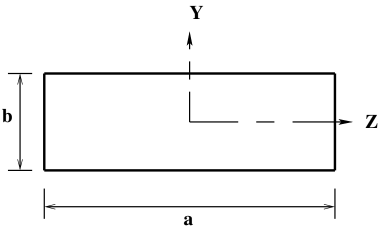
Rectangular Ducts - L15($\frac{6}{15}$)

In the F D state, v = w = $\partial u/\partial x$ =0. Hence, axial mom eqn reduces to

$$\frac{\partial^2 u^*}{\partial z^2} + \frac{\partial^2 u^*}{\partial y^2} = -1 \qquad (10)$$

$$u^* = u/(-\frac{1}{\mu}\frac{d\,p}{d\,x}) \qquad (11)$$

with bcs $u^* = 0$ at $z = \pm\, a/2$ and $u^* = 0$ at $y = \pm\, b/2$



RECTANGULAR DUCT

The Poisson's eqn can be solved by employing double Fourier series with the method of undetermined coefficients.

Method of Solution - L15($\frac{7}{15}$)

In the most general case, both sides of the Poisson's equation are multiplied by $F_1(z) \times F_2(y)$ where

$$F_1(z) = A_m \cos(\frac{m\,\pi\,z}{a}) + B_m \sin(\frac{m\,\pi\,z}{a})$$

$$F_2(y) = C_n \cos(\frac{n\,\pi\,y}{b}) + D_n \sin(\frac{n\,\pi\,y}{b})$$

But, in the present case, BCs require that terms containing SINE functions vanish. Hence,

$$u^* = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} C_{mn}\, F(y,z)$$

where

$$F(y,z) = \cos(\frac{m\,\pi\,z}{a}) \cos(\frac{n\,\pi\,y}{b})$$

Solution Procedure - L15($\frac{8}{15}$)

$$\int_{-\frac{a}{2}}^{+\frac{a}{2}} \int_{-\frac{b}{2}}^{+\frac{b}{2}} (\frac{\partial^2 u^*}{\partial z^2} + \frac{\partial^2 u^*}{\partial y^2}) F(y,z) dy\, dz = -\int_{-\frac{a}{2}}^{+\frac{a}{2}} \int_{-\frac{b}{2}}^{+\frac{b}{2}} F(y,z) dy\, dz$$

Integration by parts gives

$$\text{LHS} = -\pi^2 (\frac{m^2}{a^2} + \frac{n^2}{b^2}) \int_{-\frac{a}{2}}^{+\frac{a}{2}} \int_{-\frac{b}{2}}^{+\frac{b}{2}} u^* F(y,z)\, dy\, dz$$

$$\text{RHS} = -\frac{4ab}{m\, n\, \pi^2} (-1)^{(\frac{m+n}{2} - 1)}$$

Substitute for u^* and equate LHS = RHS to obtain C_{mn}

Determination of C_{mn} and $\overline{u}^*$ - L15($\frac{9}{15}$)

$$C_{mn} = \frac{\frac{4ab}{m\,n\,\pi^2}\,(-1)^{(\frac{m+n}{2}-1)}}{\pi^2\,(\frac{m^2}{a^2}+\frac{n^2}{b^2})\,\int_{-\frac{a}{2}}^{+\frac{a}{2}}\int_{-\frac{b}{2}}^{+\frac{b}{2}} F^2(y,z)\,dy^*dz^*}$$

$$= \frac{16}{m\,n\,\pi^4}\,(\frac{m^2}{a^2}+\frac{n^2}{b^2})^{-1}\,(-1)^{(\frac{m+n}{2}-1)}$$

Hence, $u^* = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} C_{mn}\,F(y,z)$
and average velocity is given by

$$\overline{u}^* = \frac{64}{\pi^6}\,b^2 \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\left\{(mn)^2\,(\gamma^2\,m^2+n^2)\right\}^{-1} \quad \gamma = \frac{b}{a}$$

Final Solution - L15($\frac{10}{15}$)

$$\frac{u^*}{\overline{u}^*} = \frac{\pi^2}{4} \left[\frac{\sum_{m,n=1,3,5}^{\infty} \left\{ mn \left(\gamma^2 m^2 + n^2 \right) \right\}^{-1} (-1)^{\left(\frac{m+n}{2}-1\right)} F(y,z)}{\sum_{m,n=1,3,5}^{\infty} \left\{ (mn)^2 \left(\gamma^2 m^2 + n^2 \right) \right\}^{-1}} \right]$$

$$f_{fd} Re = \frac{1}{2} \frac{D_h^2}{\overline{u}^*} = \frac{\pi^6}{32} (1+\gamma)^{-2} \left[\sum_{m,n=1,3,5}^{\infty} \left\{ (mn)^2 \left(\gamma^2 m^2 + n^2 \right) \right\}^{-1} \right]^{-1}$$

where $D_h/b = 2/(1+\gamma)$ and $\gamma = b/a$

Results - Rect Ducts L15($\frac{11}{15}$)

γ	$u_{max}/\overline{u}$	$f_{fd}\ Re$	Remarks
1.0	2.08	14.261	Sq Duct
0.8	2.086	14.413	
0.6	2.039	15.016	
0.5	1.993	15.586	
0.4	1.925	16.407	
0.2	1.716	19.117	
0.1	1.602	21.220	
0.05	1.550	22.533	
0.0	1.500	24.000	Parallel Pl

Calculations with m = n = 101.

Annulus Sectors - L15($\frac{12}{15}$)

Governing Eqn

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\mu}\frac{d\,p}{d\,x} = \text{Const}$$

Define

$z = ln\,(r/r_o)$ & $u^* = u/(-\frac{r_o^2}{\mu}\,\frac{d\,p}{d\,x})$

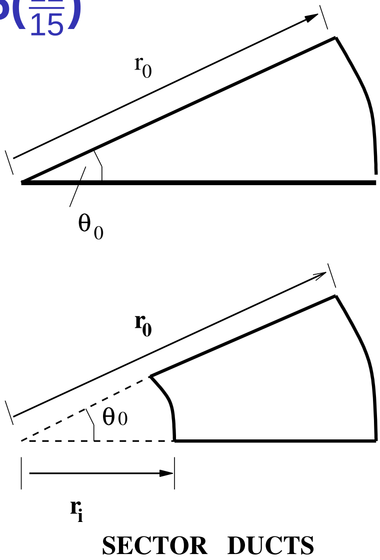
Hence,

$$\frac{\partial^2 u^*}{\partial z^2} + \frac{\partial^2 u^*}{\partial \theta^2} = -e^{2\,z}$$

BCs: $u^* = 0$ at $z_i = ln\,(r_i/r_o)$,

$z_o = 0$, $\theta = \pm\,\theta_0/2$

Sectoral ducts are formed in slots (eg. stampings) or smallest symm sector of an internally finned annulus.



Solution - 1 - L15($\frac{13}{15}$)

Solution Procedure is same as before.

$$u^* = \sum_{m=1,3,5}^{\infty} \sum_{n=1,2,3}^{\infty} F_{mn} \cos\left(\frac{m\,\pi\,\theta}{\theta_0}\right) \sin\left(\frac{n\,\pi\,z}{z_i}\right) \qquad (12)$$

$$F_{mn} = F_1/F_2$$

$$F_1 = \frac{2}{\pi^2\,z_i^2}\left(\frac{n}{m}\right)(-1)^{\frac{m-1}{2}}\left\{1-(-1)^n\,e^{2\,z_i}\right\} \qquad (13)$$

$$F_2 = \left(1+\frac{n^2\,\pi^2}{4\,z_i^2}\right)\left(\frac{n^2}{z_i^2}+\frac{m^2}{\theta_0^2}\right) \qquad (14)$$

Solution - 2 - L15($\frac{14}{15}$)

$$\overline{u^*} = \sum_{m=1,3,5}^{\infty} \sum_{n=1,2,3}^{\infty} \frac{F_3}{F_4} \tag{15}$$

$$F_3 = -F_{mn} \left(\frac{n}{m}\right) (-1)^{\frac{m-1}{2}} \left\{1 - (-1)^n e^{2 z_i}\right\} \tag{16}$$

$$F_4 = z_i \left\{1 - e^{2 z_i}\right\} \left(1 + \frac{n^2 \pi^2}{4 z_i^2}\right) \tag{17}$$

$$f_{fd} \times Re = \left(\frac{D_h}{r_o}\right)^2 / \left(2 \overline{u^*}\right) \tag{18}$$

$$\frac{D_h}{r_o} = \frac{2 \theta_0 \left\{1 - e^{2 z_i}\right\}}{\theta_0 \left\{1 + e^{z_i}\right\} + 2 \left\{1 - e^{z_i}\right\}} \tag{19}$$

Annular Sector Results - L15($\frac{15}{15}$)

$r^* = r_i/r_o$

θ_0	$f_{fd}\,Re$	$f_{fd}\,Re$	$f_{fd}\,Re$	$f_{fd}\,Re$
	$(r^* = 0.75)$	$(r^* = 0.5)$	$(r^* = 0.25)$	$(r^* = 0.001)$
180^o	25.006	20.877	17.536	16.0856
90^o	21.827	17.128	15.213	14.949
60^o	19.568	15.481	14.906	14.308
30^o	16.001	14.795	15.538	13.409
20^o	14.821	15.570	16.069	13.025
10^o	15.216	17.609	16.807	12.584
5^o	17.602	19.363	17.274	12.341

In the next lecture, we shall consider ducts of complex cross-section.