ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-13 SUPERPOSITION THEORY & APPLICATION

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1. Develop the Theory
2. Obtain Solutions with Arbitrary Variation of T_w (x) using unheated starting length (x_0) solution for a flat plate

$$St_x = \frac{3\alpha}{2\Delta\, U_\infty} = 0.331\, Re_x^{-0.5}\, Pr^{-0.66} \left[1 - (\frac{x_0}{x})^{0.75}\right]^{-0.33}$$

Theory of Superposition - I - L13($\frac{1}{9}$)

1. For constant fluid properties, The energy equation is a linear, homogeneous equation so that a sum of solutions is also a solution. This property can be exploited to derive St_x results for arbitrary variation of T_w(x) knowing the solution for T_w = const.
2. Define $\theta(x, y) = (T_w - T)/(T_w - T_\infty)$
3. Thus, let $\theta(x, y, x_0)$ be the unheated starting length solution for T_w = const for $x \geq x_0$. Then
$T - T_\infty = [1 - \theta(x, y, x_0)]\ (T_w - T_\infty)$
4. The response of T to infinitisimal change $d\ T_w$ then is:
$d\,(T - T_\infty) = [1 - \theta(x, y, x_0)\,]\ d\,(T_w - T_\infty)$
5. Similarly, the response to discrete change $\Delta\ T_w$ is:
$\Delta\,(T - T_\infty) = [1 - \theta(x, y, x_0)]\ \Delta\,(T_w - T_\infty)$

Theory of Superposition - II - L13($\frac{2}{9}$)

1. Therefore, for continuous and discrete changes, one may write the total solution as:

$$T - T_\infty = \int_{x_0=0}^{x_0=x} [1 - \theta(x, y, x_0)] \; d\,T_w + \sum_{i=1}^{i=I} [1 - \theta(x, y, x_0)] \; \Delta\,(T_w - T_\infty)_i$$

2. But, for continuous change $d\,T_w = (d\,T_w / d\,x_0)\,d\,x_0$. Hence,

$$T - T_\infty = \int_{x_0=0}^{x_0=x} [1 - \theta(x, y, x_0)] \; \frac{d\,T_w}{d\,x_0}\,d\,x_0 + \sum_{i=1}^{i=I} [1 - \theta(x, y, x_0)] \; \Delta\,(T_w - T_\infty)_i$$

Theory of Superposition - III - L13($\frac{3}{9}$)

1. Now, $q_{w,x} = -k\,\partial T/\partial y|_{y=0}$. Hence, $h(x, x_0) = q_{w,x}/(T_w - T_\infty) = -k\,\partial\theta/\partial y|_{y=0}$

$$q_{w,x} = \int_0^x h(x, x_0)\,\frac{d\,T_w}{d\,x_0}\,d\,x_0 + \sum_{i=1}^{i=I} h(x, x_0)\,\Delta\,(T_w - T_\infty)_i$$

where for Flat Plate and Pr ≥ 1, $h(x, x_0)$ is evaluated from

$$St_x = \frac{h(x, x_0)}{\rho\,Cp\,U_\infty} = 0.331\,Re_x^{-0.5}\,Pr^{-0.66}\,\left[1 - (\frac{x_0}{x})^{0.75}\right]^{-0.33}$$

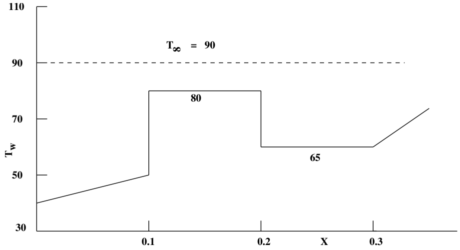
and $Nu_x = St_x\,Re_x\,Pr$

An Application L13($\frac{4}{9}$)

Consider flat plate boundary layer in which the surface temperature varies as follows.
$0 < x < x_1$, $T_w = 40 + 100\,x$.
$x_1 < x < x_2$, $T_w = 80$.
$x_2 < x < x_3$, $T_w = 65$
$x > x_3$, $T_w = 65 + 200\,(x - x_3)$.

$x_1 = 0.1$ m, $x_2 = 0.2$ m,
$x_3 = 0.3$ m.

Determine $q_{w,x}$ and Nu_x



$T_\infty = 90$, U_∞ = 7.5 m / s
$\nu = 18.97 \times 10^{-6}\ m^2/s$,
k = 0.029 W/m-K
Pr = 0.696.

Solution-I - L13($\frac{5}{9}$)

For $0 < x < x_1$ - ΔT_{wo} = 40 - 90 = -50

$$q_{w,x} = A \left[\int_0^x (\, (1 - \frac{x_0}{x})^{0.75} \,)^{-0.33} \, 100 \, d \, x_0 + \Delta T_{wo} \right] \qquad (1)$$

For $x_1 < x < x_2$ - ΔT_{w1} = 80 - 50 = 30

$$\begin{aligned} q_{w,x} &= A \left[\int_0^{x_1} (\, (1 - \frac{x_0}{x})^{0.75} \,)^{-0.33} \, 100 \, d \, x_0 + \Delta T_{w0} \right] \\ &+ A \left[(\, (1 - \frac{x_1}{x})^{0.75} \,)^{-0.33} \, \Delta T_{w1} \right] \end{aligned} \qquad (2)$$

where $A = 0.3313 \, \frac{k}{x} \, Re_x^{0.5} \, Pr^{0.33}$.

Solution-II - L13($\frac{6}{9}$)

For $x_2 < x < x_3$ - $\Delta T_{w2} = 65 - 80 = -15$

$$\begin{aligned} q_{w,x} &= A\left[\int_0^{x_1} \left(1 - \left(\frac{x_0}{x}\right)^{0.75}\right)^{-0.33} 100\, d\, x_0 + \Delta T_{w0}\right] \\ &+ A\left[\left(\left(1 - \frac{x_1}{x_2}\right)^{0.75}\right)^{-0.33} \Delta T_{w1}\right] \\ &+ A\left[\left(\left(1 - \frac{x_2}{x}\right)^{0.75}\right)^{-0.33} \Delta T_{w2}\right] \end{aligned} \quad (3)$$

where $A = 0.3313\, \frac{k}{x}\, Re_x^{0.5}\, Pr^{0.33}$.

Solution-III - L13($\frac{7}{9}$)

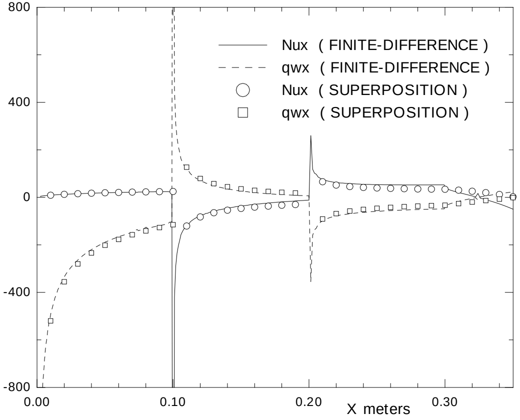
For $x_3 < x$

$$
\begin{aligned}
q_{w,x} &= A \left[\int_0^{x_1} (\,(1 - \frac{x_0}{x})^{0.75}\,)^{-0.33}\, 100\, d\, x_0 + \Delta T_{w0} \right] \\
&+ A \left[(\,(1 - \frac{x_1}{x_2})^{0.75}\,)^{-0.33}\, \Delta T_{w1} + (\,(1 - \frac{x_2}{x_3})^{0.75}\,)^{-0.33}\, \Delta T_{w2} \right] \\
&+ A \left[\int_{x_3}^{x} (1 - \frac{x_3}{x}\,)^{0.75})^{-0.33}\, 200\, d\, x_3 \right] \qquad (4)
\end{aligned}
$$

where $A = 0.3313\, \frac{k}{x}\, Re_x^{0.5}\, Pr^{0.33}$. Note that

$$
\int_0^{x} (\,(1 - \frac{x_0}{x}\,)^{0.75})^{-0.33}\, d\, x_0 = \frac{4}{3}\, \beta\, (\frac{2}{3}, \frac{4}{3})\, x = 1.612\, x
$$

Final Solution L13($\frac{8}{9}$)



Remarkably good Agreement with FD Solutions.

Discussion L13($\frac{9}{9}$)

1. Notice the change in the sign of $q_{w,x}$ and Nu_x although over the entire length $T_\infty > T_w$.
2. Negative $q_{w,x}$ implies heat transfer to the wall and *vice versa*
3. Problems of this type are important in electronics cooling such as the Printed Circuit Boards.
4. Solutions of this type can also be developed for flows with pressure gradient. But, the theory is more involved[1]

[1] Spalding D. B. *Heat transfer from Surfaces of non-uniform Temperature* Jnl of Fluid Mechanics, vol. 4, p 22-32 (1957)