ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-11 INTEGRAL SOLNS TO LAMINAR VEL BL

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1. Solution Procedure
2. Solutions with Effects of Pressure Gradient and Sucrion/Blowing

Solution Procedure - L11($\frac{1}{18}$)

Integral Momentum Eqn (IME)

$$\frac{d\,\delta_2}{d\,x} + \frac{1}{U_\infty}\frac{d\,U_\infty}{d\,x}\left(2\,\delta_2 + \delta_1\right) = \frac{C_{f,x}}{2} + \frac{V_w}{U_\infty} \qquad (1)$$

1. In the Similarity Method , 3rd order Similarity Eqn was solved with appropriate Boundary Conditions to obtain Velocity Profile . The Integral Parameters δ_1, δ_2 and $C_{f,x}$ were then recovered from the profiles.
2. In contrast, in Integral Method , the Velocity Profile is assumed (usually a polynomial in y / δ) such that it satisfies the Boundary Conditions
3. Then, Integral Parameters δ_1, δ_2 and $C_{f,x}$ are evaluated and substituted in the Integral Momentum Eqn
4. The IME is then solved to obtain $\delta_2(x)$ and hence, all other parameters as functions of x

Typical Velocity Profile - L11($\frac{2}{18}$)

Let $$\frac{u}{U_\infty} = a + b\,\eta + c\,\eta^2 + d\,\eta^3 + e\,\eta^4 \quad \eta = \frac{y}{\delta} \qquad (2)$$

At y = 0 (Wall)

$$u = 0 \qquad (3)$$

$$\nu\,\frac{\partial^2 u}{\partial y^2} = -U_\infty \frac{d\,U_\infty}{d\,x} + V_w\,\frac{\partial u}{\partial y} \qquad (4)$$

2nd BC derived from PDE

At y = δ (Edge of BL)

$$u = U_\infty \qquad (5)$$

$$\frac{\partial u}{\partial y} = 0 \qquad (6)$$

$$\frac{\partial^2 u}{\partial y^2} = 0 \qquad (7)$$

3rd BC ensures assymptotic behaviour as $y \to \delta$

Five BCs give 5 coefficients a, b, c, d and e

Derived Velocity Profile - L11($\frac{3}{18}$)

Coefficients are:

1. a = 0
2. b = 3 - e
3. c = 3 (e -1)
4. d = 1 - 3e
5. $e = \frac{3\,V_w^* - \lambda + 6}{6 + V_w^*}$

ALERT:

$(V_w^*, \lambda)_{min,max}$
must be such that
$u/U_\infty \leq 1$
for $\eta \leq 1$

Hence, the Vel Profile

$$\frac{u}{U_\infty} = \left(\frac{6}{6 + V_w^*}\right)\left(F_1 + V_w^*\,F_2 + \lambda\,F_3\right) \quad (8)$$

$$F_1 = 2\,\eta - 2\,\eta^3 + \eta^4 \quad (9)$$

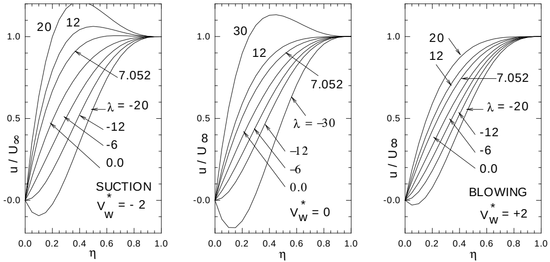
$$F_2 = \frac{1}{6}(6\,\eta^2 - 8\,\eta^3 + 3\,\eta^4) \quad (10)$$

$$F_3 = \frac{1}{6}(\eta - 3\,\eta^2 + 3\,\eta^3 - \eta^4) \quad (11)$$

$$V_w^* = \frac{V_w\,\delta}{\nu} \quad \text{Suc/Blow Param} \quad (12)$$

$$\lambda = \frac{\delta^2}{\nu}\,\frac{d\,U_\infty}{d\,x} \quad \text{Pr Gr Param} \quad (13)$$

Velocity Profiles $-2 < V_w^* < 2$ - L11($\frac{4}{18}$)



For $V_w^* \leq 0$, $\frac{u}{U_\infty} > 1$ for $\lambda > 12$
For $V_w^* > 0$, $\frac{u}{U_\infty} \leq 1$ for $\lambda \leq 20$

For all V_w^*, $\frac{u}{U_\infty} < 0$ for $\lambda < -12$
For $\lambda = -12$, $\frac{\partial u}{\partial y}|_{y=0} = 0$
(Separation)

Evaluation of Thicknesses - 1 - L11($\frac{5}{18}$)

In the Integral method, we evaluate 3 Thicknesses

$$\delta_1 = \int_0^\infty (1 - \frac{u}{U_\infty})\, d\, y = \int_0^\delta (1 - \frac{u}{U_\infty})\, d\, y \quad (14)$$

$$\delta_2 = \int_0^\infty \frac{u}{U_\infty} (1 - \frac{u}{U_\infty})\, d\, y = \int_0^\delta \frac{u}{U_\infty} (1 - \frac{u}{U_\infty})\, d\, y \quad (15)$$

$$\delta_4 = \frac{\mu\, U_\infty}{\tau_{w,x}} \quad \text{Shear Thickness} \quad (16)$$

Note that $\int_0^\infty = \int_0^\delta + \int_\delta^\infty$.

But, in the region $\delta < y < \infty$, u = U_∞.

Hence, $\int_\delta^\infty = 0$

Evaluation of Thicknesses - 2 - L11($\frac{6}{18}$)

$$\frac{\delta_1}{\delta} = \int_0^1 (1 - \frac{u}{U_\infty}) d\eta = \frac{1}{4}(1 + \frac{e}{5}) = \frac{1}{4}\left[1 + \frac{1}{5}\left\{\frac{3\,V_w^* - \lambda + 6}{6 + V_w^*}\right\}\right] \quad (17)$$

$$\frac{\delta_2}{\delta} = \int_0^1 \frac{u}{U_\infty}(1 - \frac{u}{U_\infty})\,d\eta = \frac{3}{28} + \frac{e}{70} - \frac{e^2}{252} = \frac{3}{28} + \frac{1}{70}(\frac{3\,V_w^* - \lambda + 6}{6 + V_w^*}) - \frac{1}{252}(\frac{3\,V_w^* - \lambda + 6}{6 + V_w^*})^2 \quad (18)$$

$$\frac{\delta_4}{\delta} = \frac{1}{3 - e} = \frac{6 + V_w^*}{\lambda + 12} \quad (19)$$

Note that $\delta_4 = \mu\, U_\infty / \tau_{w,x} = \infty$ when $\lambda = -12$. Hence, separation will occur.

Reorganisation of IME - L11($\frac{7}{18}$)

$$\frac{d\,\delta_2}{d\,x} + \frac{1}{U_\infty}\,\frac{d\,U_\infty}{d\,x}\,(2\,\delta_2 + \delta_1) = \frac{C_{f,x}}{2} + \frac{V_w}{U_\infty} \qquad (20)$$

Multiply by $U_\infty\,\delta_2/\nu$

$$\frac{U_\infty}{\nu}\frac{d\,\delta_2^2}{d\,x} = 2\left[S + V_w^+ - \kappa\,(2 + H)\right] = F(\kappa) \qquad (21)$$

$$S = \frac{\delta_2}{\delta_4} \quad \text{(Shear factor)} \quad H = \frac{\delta_1}{\delta_2} \quad \text{(Shape Factor)}$$

$$\kappa = \frac{\delta_2^2}{\nu}\,\frac{d\,U_\infty}{d\,x} = \lambda\,(\frac{\delta_2}{\delta})^2 \quad \text{(Pr Gr Param)} \qquad (22)$$

$$V_w^+ = \frac{V_w\,\delta_2}{\nu} = V_w^*\,(\frac{\delta_2}{\delta}) \quad \text{(Suc/Blow Param)} \qquad (23)$$

Eqn 21 is a Universal Relationship

Important Deductions - L11 ($\frac{8}{18}$)

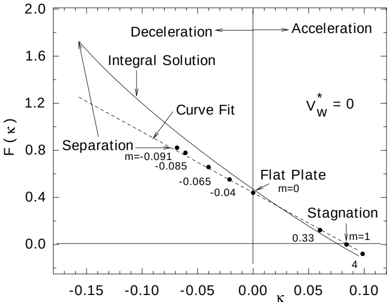
1. Universal means it is applicable to all types of variations of U_∞ (x) including $U_\infty = Cx^m$
2. $\kappa \;\; \propto \frac{d\,U_\infty}{d\,x} = 0$ implies Flat Plate Solution because U_∞ = Const.
3. $F(\kappa) = \frac{U_\infty}{\nu} \frac{d\,\delta_2^2}{d\,x} = 0$ implies Stagnation Point Solution because recall that all thicknesses are constant for m = 1 (see below)
4. S = $\frac{\delta_2}{\delta_4}$ = 0 implies Separation because $\tau_{w,x} = 0$
5. All values of V_w^+ and λ for which $\delta_1, \delta_2, \delta_4 < 0$ must be discarded because $u/U_\infty > 1$ or $u/U_\infty < 0$
6. For $U_\infty = Cx^m$, it is easy to derive that $\kappa = m\,\delta_2^{*2}$, F (κ) = (1 - m) δ_2^{*2}, S = $f''(0)\,\delta_2^*$ and $V_w^+ = B_f\,\delta_2^*$ where δ_2^* (m) are available from Similarity method

Solution $V_w^* = 0$ - L11 ($\frac{9}{18}$)

λ	κ	δ_1/δ	δ_2/δ	S=δ_2/δ_4	H=δ_1/δ_2	F (κ)
12.0	0.095	0.200	0.089	0.356	2.250	-0.095
9.0	0.088	0.225	0.099	0.347	2.273	-0.061
7.5	0.080	0.237	0.103	0.336	2.299	-0.017
6.0	0.069	0.250	0.107	0.321	2.333	0.046
3.0	0.039	0.275	0.113	0.283	2.427	0.226
0.0	0.000	0.300	0.117	0.235	2.554	0.470
-3.0	-0.043	0.325	0.120	0.179	2.716	0.764
-6.0	-0.086	0.350	0.120	0.120	2.921	1.088
-7.5	-0.107	0.363	0.119	0.089	3.041	1.253
-9.0	-0.125	0.375	0.118	0.059	3.176	1.417
-12.0	-0.157	0.400	0.114	0.000	3.500	1.724

λ = 7.052 or κ = 0.07824 represents Stagnation Point Solution.
Recall Sim Solns: $\delta^*_{2,m=0}$ = 0.663 and $\delta^*_{2,m=1}$ = 0.292.
Hence, $\kappa_{m=1}$ = 0.0841 and F ($\kappa_{m=0}$) = 0.44

F (κ) vs κ ($V_w = 0$) - L11($\frac{10}{18}$)



$$F(\kappa) \;=\; [a - b\,\kappa] \rightarrow \quad a = \delta^{*2}_{2,m=0}, \;\; b = a/\delta^{*2}_{2,m=1} \qquad (24)$$

$$\frac{U_\infty}{\nu}\frac{d\,\delta_2^2}{d\,x} \;=\; 0.44 - 5.17\,\frac{\delta_2^2}{\nu}\,\frac{d\,U_\infty}{d\,x} \quad \text{Thwaite's Curve-Fit} \; (25)$$

Integral Soln deviates from Sim Soln for $\kappa < 0$ but, reasonably accurate for $\kappa > 0$

Closed Form Soln $V_w^* = 0$ - L11($\frac{11}{18}$)

$$\frac{U_\infty}{\nu}\frac{d\,\delta_2^2}{d\,x} = 0.44 - 5.17\,\frac{\delta_2^2}{\nu}\frac{d\,U_\infty}{d\,x} \quad (26)$$

$$\frac{d}{d\,x}(\delta_2^2\,U_\infty^{5.17}) = 0.44\,\nu\,U_\infty^{4.17} \quad (27)$$

$$(\delta_2^2\,U_\infty^{5.17})_x - (\delta_2^2\,U_\infty^{5.17})_{x=0} = 0.44\,\nu\int_0^x U_\infty^{4.17}\,d\,x \quad (28)$$

1. Soln applicable to any U_∞(x)
2. Calculate δ_2 from eqn 28. $\delta_{2,x=0}^2$ (known)
3. Evaluate κ from $d\,U_\infty/d\,x$.
4. For this value of κ, evalute λ and hence S and δ_4
5. Hence obtain $C_{f,x} = 2\,\nu/(\delta_4\,U_\infty)$.

Flate Plate Soln $\lambda = \kappa = 0$ - L11($\frac{12}{18}$)

V_w^*	δ_1/δ	δ_2/δ	S=δ_2/δ_4	H=δ_1/δ_2	V_w^+	F (κ)
5.0	0.35	0.12	0.13	2.88	0.60	1.46
4.0	0.34	0.12	0.14	2.83	0.48	1.25
3.0	0.33	0.12	0.16	2.78	0.36	1.04
2.0	0.33	0.12	0.18	2.72	0.24	0.84
1.0	0.31	0.12	0.20	2.64	0.12	0.65
0.0	0.30	0.12	0.23	2.55	0.00	0.47
-1.0	0.28	0.11	0.27	2.45	-0.11	0.32
-2.0	0.25	0.11	0.32	2.33	-0.21	0.21
-3.0	0.20	0.09	0.36	2.25	-0.27	0.18
-4.0	0.10	0.03	0.17	3.50	-0.11	0.11
-4.2	0.07	0.00	0.01	47.25	-0.01	0.01
-4.4	0.03	-0.04	-0.28	-0.67	0.16	-0.23

Feasible Solutions for $\lambda > -4.2$ only

Solution $V_w^* = -2.0$ (Suction) - L11($\frac{12}{18}$)

λ	κ	δ_1/δ	δ_2/δ	S=δ_2/δ_4	H=δ_1/δ_2	V_w^+	F (κ)
15.0	0.00	0.06	0.00	-0.02	-28.00	0.00	-0.02
14.0	0.00	0.07	0.01	0.06	8.79	-0.02	0.05
10.0	0.02	0.12	0.05	0.26	2.68	-0.09	0.12
6.0	0.04	0.17	0.08	0.35	2.28	-0.15	0.08
2.0	0.02	0.22	0.10	0.35	2.27	-0.20	0.13
0.0	0.00	0.25	0.11	0.32	2.33	-0.21	0.21
-2.0	-0.03	0.28	0.11	0.28	2.43	-0.23	0.34
-6.0	-0.09	0.32	0.12	0.18	2.72	-0.24	0.69
-10.0	-0.14	0.38	0.12	0.06	3.18	-0.24	1.09
-12.0	-0.16	0.40	0.11	0.00	3.50	-0.23	1.27
-13.0	-0.16	0.41	0.11	-0.03	3.69	-0.22	1.34

Feasible Solutions for $-12 \leq \lambda \leq 14.5$ only
For m = λ = 0, F (κ) = 0.21 = ${\delta_2^*}^2$ and V_w^+ = -0.21 Hence,
$B_f = -0.21/\sqrt{0.21} \simeq -0.458$ Note that for all $\lambda < 2$, $V_w^+ \sim$ const

Solution $V_w^* = 2.0$ (Blowing)- L11($\frac{13}{18}$)

λ	κ	δ_1/δ	δ_2/δ	S=δ_2/δ_4	H=δ_1/δ_2	V_w^+	F (κ)
16.0	0.16	0.22	0.10	0.35	2.27	0.20	-0.25
12.0	0.14	0.25	0.11	0.32	2.33	0.21	-0.12
8.0	0.10	0.28	0.11	0.28	2.43	0.23	0.11
4.0	0.06	0.30	0.12	0.23	2.55	0.23	0.44
0.0	0.00	0.32	0.12	0.18	2.72	0.24	0.84
-4.0	-0.06	0.35	0.12	0.12	2.92	0.24	1.28
-8.0	-0.11	0.38	0.12	0.06	3.18	0.24	1.74
-12.0	-0.16	0.40	0.11	0.00	3.50	0.23	2.18
-13.0	-0.17	0.41	0.11	-0.01	3.59	0.23	2.28
-16.0	-0.19	0.43	0.11	-0.05	3.92	0.22	2.56

Feasible Solutions for $\lambda \geq -12$ only.
F (κ) = 0 for $\lambda = 9.67$ or $\kappa = 0.117$.
For m = λ = 0, F (κ) = 0.84 = $\delta_2^{*^2}$ and V_w^+ = 0.24 Hence,
$B_f = 0.24/\sqrt{0.84} \simeq 0.2619$ Note that for all λ, $V_w^+ \sim$ const

Closed form Solns for V_w^* and λ - L11($\frac{14}{18}$)

For simlultaneous variations of V_w^* and λ, Closed Form Solutions can be developed for regions in which V_w^+ is nearly constant. The curve-fit solution is again of the form $F(\kappa) = a - b\,\kappa$, or

$$\frac{U_\infty}{\nu}\frac{d\,\delta_2^2}{d\,x} = a - b\,\frac{\delta_2^2}{\nu}\,\frac{d\,U_\infty}{d\,x} \tag{29}$$

where a and b are functions of V_w^*. Thus, from tabulated values

1. For $V_w^* = -2.0$, a $\simeq$ 0.21, b $\simeq$ 4.2
2. For $V_w^* = +2.0$, a $\simeq$ 0.84, b $\simeq$ 7.4

Solution Procedure - L11($\frac{15}{18}$)

Manipulation gives

$$\frac{d}{d\,x}(\delta_2^2\, U_\infty^b) = a\,\nu\, U_\infty^{b-1} \quad (30)$$

$$(\delta_2^2\, U_\infty^b)_x - (\delta_2^2\, U_\infty^b)_{x=0} = a\,\nu \int_0^x U_\infty^{b-1}\, d\,x \quad (31)$$

1. Soln applicable to any U_∞(x) and V_w(x)
2. Calculate δ_2 from eqn 31. $\delta_{2,x=0}^2$ (known)
3. Evaluate κ from $d\,U_\infty/d\,x$.
4. For this value of κ, evalute λ, V_w^+ and S and hence, δ_4
5. Hence obtain $C_{f,x} = 2\,\nu/(\delta_4\,U_\infty)$.

Prob: Flow over a Cylinder - L11($\frac{16}{18}$)

For flow over an impervious cylinder, with $x^* = x/D$

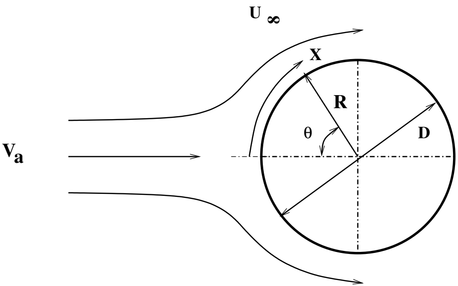
$$\frac{U_\infty}{V_a} = 2\sin(2\,x^*) = F(x^*)$$

Then,

$$\frac{\delta_2^2}{D^2}\,Re_D = \frac{0.44}{F^{5.17}} \int_0^{x^*} F^{4.17}\,dx^*$$

and

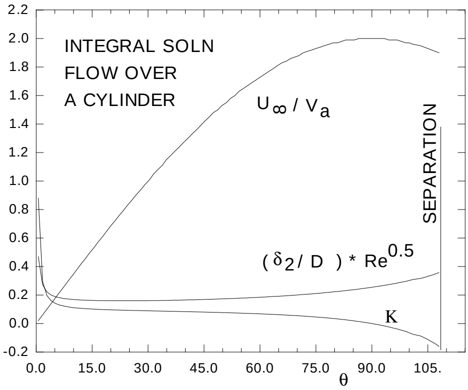
$$\kappa(x^*) = (\frac{\delta_2}{D})^2\,Re_D\,\frac{dF}{dx^*}$$



The objective is to determine point of separation x^*_{sep} corresponding to $\kappa = -0.1567$.

$$Re_D = \frac{V_a\,D}{\nu}$$

Soln: Flow over a Cylinder - L11($\frac{17}{18}$)



Numerical integration gives Separation at $\theta = x/R = 108.3^0$

Effect of Blowing - L11($\frac{18}{18}$)

V_w^*	F (κ)	θ_{sep}	$C_{f,sep}^*$
0.0	$\simeq 0.44 - 5.17\,\kappa$	108.3^0	2.4612
0.5	$\simeq 0.56 - 6.22\,\kappa$	107.0^0	2.3533
1.0	$\simeq 0.65 - 6.50\,\kappa$	105.9^0	2.2280
2.0	$\simeq 0.84 - 7.40\,\kappa$	103.6^0	2.0710

where

$$C_{f,sep}^* = \left(\frac{\overline{\tau}}{\rho\, V_a^2}\right) Re_D^{0.5} \qquad \overline{\tau} = \frac{1}{x_{sep}} \int_0^{x_{sep}} \tau_{w,x}\, dx$$

As expected, Separation Angle is advanced with increase in blowing rate with reduction in average skin-friction due to thickening of the boundary layer