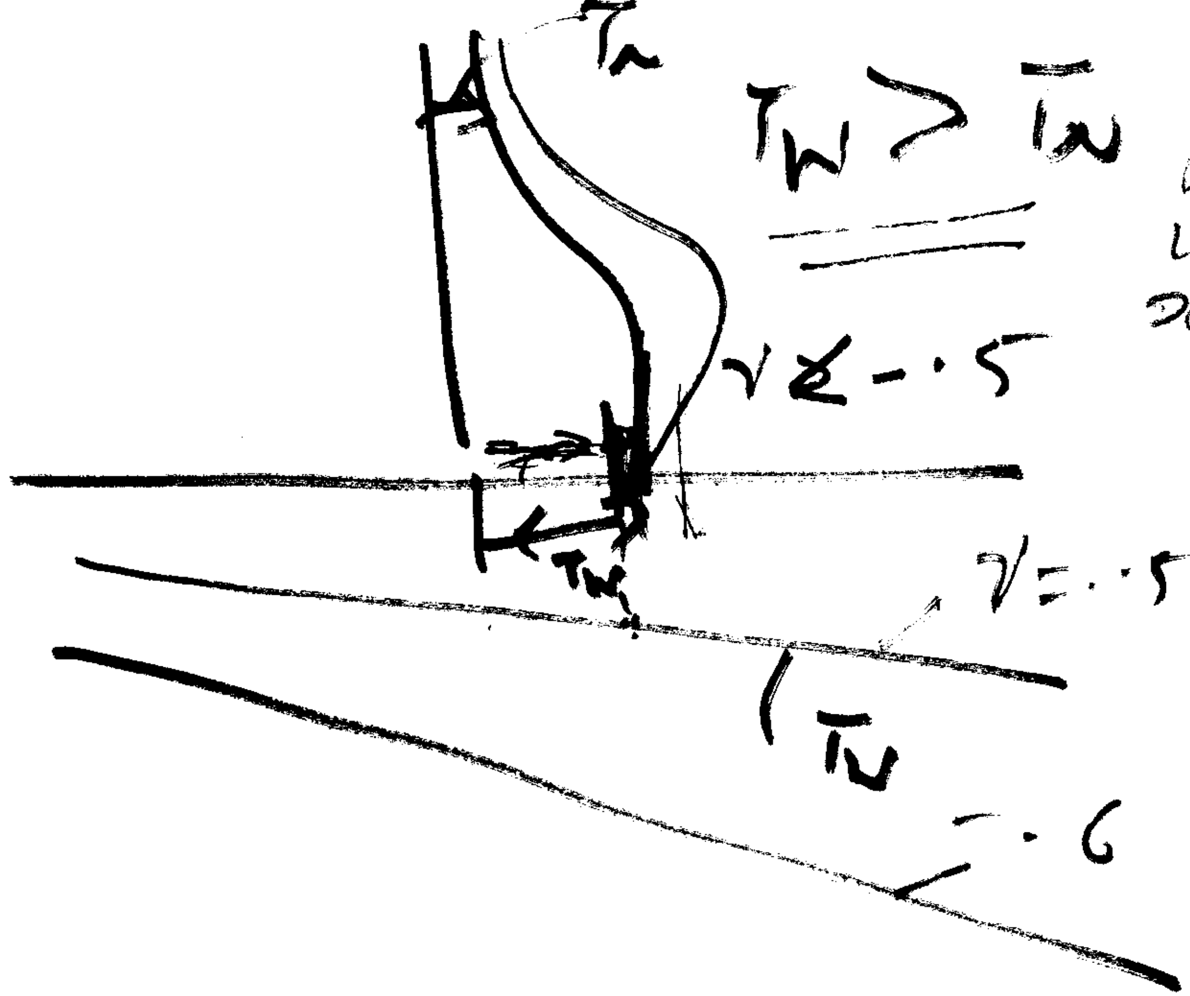
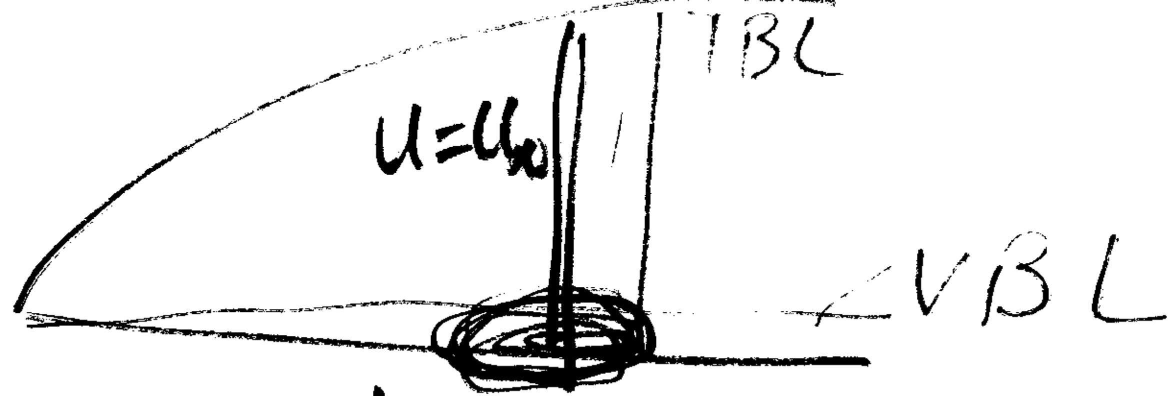
Prof. D.W. Date.

Lec-8.9.

Date. 30-3-10





$$f'(\eta) \simeq 1.0$$

$$f(\eta) = \eta$$

$$Re_x Pr = \frac{U_\infty x}{\cancel{\nu}} \frac{\cancel{\nu}}{\alpha}$$

Reynold

Prandtl

In liquid metal viscosity has no influence

Peclet no

Let $\dfrac{Pr\,(m+1)\,f''(0,m)}{12} = A$

Define $A\eta^3 = x \qquad 3A\eta^2\,d\eta = dx.$

$$\eta^2 = (\eta^3)^{2/3} = \left(\frac{x}{A}\right)^{2/3}$$

$$d\eta = \frac{dx}{3A\eta^2}$$

$$= \frac{dx}{3A\left(\frac{x}{A}\right)^{2/3}}$$

$$= \frac{1}{3A^{1/3}}\cdot\frac{dx}{x^{2/3}}$$

$$\text{Int} = \int_0^\infty \exp(-A\eta^3)\,d\eta$$

$$= \int_0^\infty \frac{\exp(-x)}{3A^{1/3}}\cdot x^{-2/3}\cdot dx$$

$$= \frac{1}{3A^{1/3}}\int_0^\infty x^{-2/3}\exp(-x)\,dx$$

$$n-1 = -2/3$$

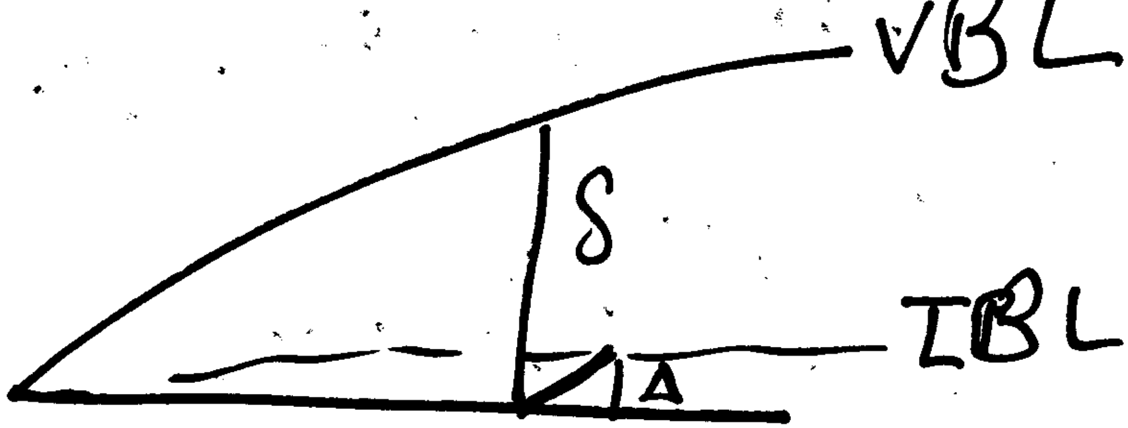
$$n = 1 - 2/3 = 1/3$$

$$\text{Int} = \frac{1}{3A^{1/3}}\,\Gamma(1/3) = \frac{\Gamma(4/3)}{A^{1/3}}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$\Gamma(1/3) = \frac{1}{1/3}\cdot\Gamma(4/3)$$

$$= 3\,\Gamma(4/3)$$



$$f'(\eta) = f''(0,m)\cdot \eta$$

$$f(\eta) = f''(0,m)\cdot \frac{\eta^2}{2} + C$$

But $f(0)=0 \quad \therefore C=0$

$$f(\eta) = f''(0,m)\cdot \frac{\eta^2}{2}$$

$$Nu_x = \frac{h_x x}{k} = C_{(Pr)} Re_x^{1/2}$$

$$h_x \propto \frac{k}{x} \cdot \left(\frac{U_\infty x}{\nu}\right)^{1/2}$$

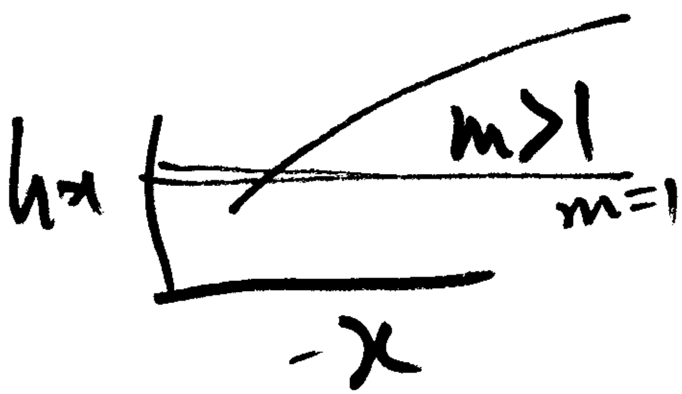
$$\propto \frac{k}{x} \left(\frac{c}{\nu}\right)^{1/2} \cdot x^{\frac{m+1}{2}}$$

$$Nu_x = C\, Re_x^{1/2} \qquad \propto k \left(\frac{c}{\nu}\right)^{1/2} x^{\frac{m-1}{2}}$$

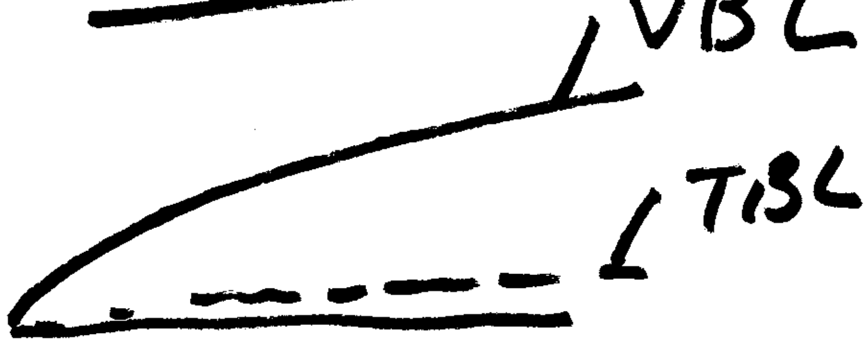
$$C = f(Pr, m)$$

$$.7 < Pr < 25$$

$$-.085 < m < 4$$



Oils ($Pr \gg 1$)



$\Delta \ll \delta$

Liquid Metals ($Pr \ll 1$)

$\Delta \gg \delta$

Flat Plate — $m = 0$

No Suction/Bl. $\rightarrow B_f = 0$

Const Wall Tem $\rightarrow \gamma = 0$

No viscous Diss $\rightarrow Ec = 0$

$Pr = 1$

$$\theta'' + \frac{Pr f \theta'}{2} = 0$$

$$f''' + \frac{1}{2} f f'' = 0 \qquad f(0) = f'(0) = 0$$

$$f'(\infty) = 1$$

$$\theta'' + \frac{1}{2} f \theta' = 0 \qquad \theta(0) = 1$$

$$\theta(\infty) = 0$$

$$\theta(\eta) = 1 - \underline{\underline{f'}}(\eta) \qquad Pr = \frac{\nu}{\alpha} = 1$$

$$Nu_x = \frac{h_x \cdot x}{k} = -\frac{x}{\cancel{k}} \cdot \cancel{k}\, \theta'(0) \sqrt{\frac{U_\infty}{\nu x}}$$

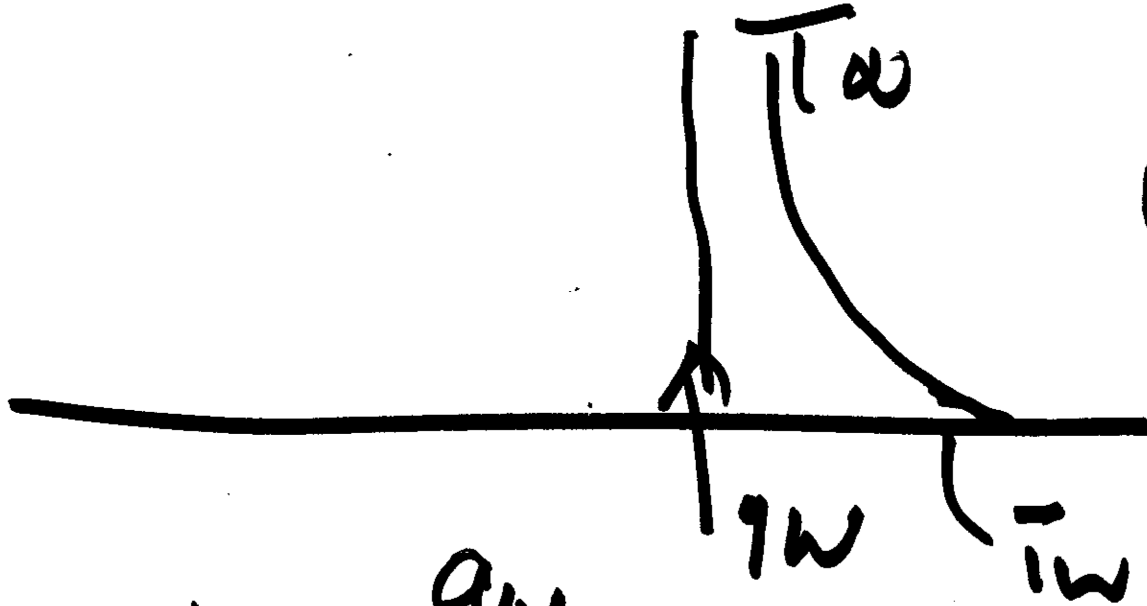
$$= -\theta'(0) \sqrt{\frac{U_\infty x}{\nu}}$$

$$= -\theta'(0)\, Re_x^{1/2}$$

$$\therefore \underline{\underline{Nu_x\, Re_x^{-1/2} = -\theta'(0)}}$$

Nu — Nusselt

St = Stanton



$$\theta = \frac{T - T_\infty}{T_w - T_\infty}$$

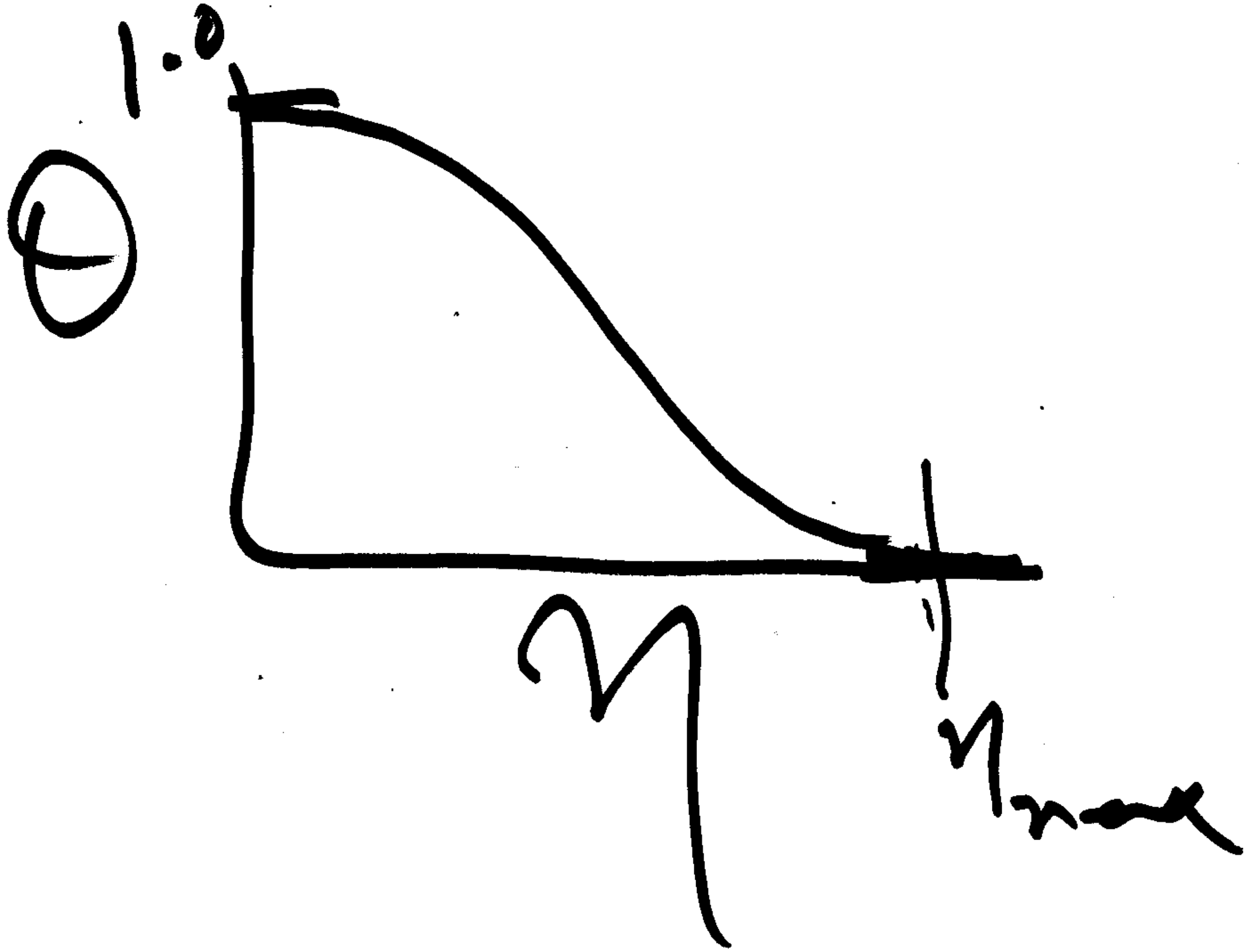
$$h_x = \frac{q_w}{T_w - T_\infty} = \frac{-k \left.\frac{\partial T}{\partial y}\right|_{y=0}}{T_w - T_\infty}$$

$$= -k \left.\frac{\partial \theta}{\partial y}\right|_{y=0}$$

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

$$= -k \left.\frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right|_{y=0}$$

$$= -k\,\theta'(0) \cdot \sqrt{\frac{U_\infty}{\nu x}}$$


1.0
Θ
η
η_{max}

$$E_{cx} = \frac{V_\infty^2/2}{c_p\,(T_w - T_\infty)} \quad a. \quad = \text{const}$$

$$= \frac{c^2\, x^{2m}}{2\, c_p \cdot \Delta T_{ref}\, x^{\nu}}$$

$$\nu = 2m \text{ if } E_{cx} \neq 0$$

$$\frac{x}{G}\frac{dG}{dx} = \nu$$

$$\frac{dG}{G} = \nu \cdot \frac{dx}{x}$$

$$\boxed{G = \Delta T_{ref} \cdot x^{\nu}}$$

$$U_\infty = c x^m$$

$$\eta = \sqrt{\nu U_\infty x}$$

$$= \sqrt{c\nu x^{m+1}}$$

$$= \sqrt{c\nu} \cdot x^{\frac{m+1}{2}}$$

$$\frac{d\eta}{dx} = \sqrt{c\nu} \cdot \left(\frac{m+1}{2}\right) x^{\frac{m-1}{2}}$$

$$\frac{x}{\eta}\frac{d\eta}{dx} = \cancel{\frac{x}{\sqrt{c\nu}\, x^{\frac{m+1}{2}}}} \cdot \cancel{\sqrt{c\nu}} \cdot \left(\frac{m+1}{2}\right) \cancel{x^{\frac{m-1}{2}}}$$

$$= \left(\frac{m+1}{2}\right)$$

$$\text{LHS}\ [U_\infty f'][\theta' \frac{\partial\eta}{\partial x}] - [f' \cdot \sqrt{\nu U_\infty x} \cdot \frac{\partial\eta}{\partial x} + f \frac{\partial\eta}{\partial x}]\theta' \sqrt{\frac{U_\infty}{\nu x}}$$

$$\text{LHS}\ [U_\infty f'] \cancel{[\theta' \frac{\partial\eta}{\partial x}] - [f'\theta' U_\infty \frac{\partial\eta}{\partial x}} + f\theta' \sqrt{\frac{U_\infty}{\nu x}} \cdot \frac{d\eta}{dx}]$$

$$\boxed{-f\theta' \cdot \sqrt{\frac{U_\infty}{\nu x}} \frac{d\eta}{dx} + \frac{f' U_\infty \theta}{q} \frac{dG}{dx} = \alpha \frac{U_\infty}{\nu x} \theta'' + \frac{\nu}{c_p (T_w - T_\infty)} \frac{U_\infty^3}{\nu x} (f'')^2}$$

$$-f\theta' \sqrt{\frac{\nu x}{U_\infty}} \cdot \frac{d\eta}{dx} + f'$$

$$[U_\infty f'][\theta' \frac{\partial \eta}{\partial x}] - [f' n \frac{\partial \eta}{\partial x} + f \frac{dn}{dx}]\theta' \sqrt{\frac{U_\infty}{\nu x}} + \frac{U_\infty f' \theta}{a} \frac{da}{dx}$$

$$= \alpha \cdot \frac{U_\infty}{\nu x} \cdot \theta'' + \frac{\nu}{c_p (T_\infty - T_0)} \frac{U_\infty^3}{\nu x} (f'')^2$$

$$U_\infty = c x^m \qquad \eta = \sqrt{\frac{U_\infty}{\nu x}}$$

$$n = \sqrt{\nu U_\infty x}$$

$$= \nu x \sqrt{\frac{U_\infty}{\nu x}}$$

$$= \nu x \cdot \eta$$

$$\eta = \underline{\underline{\frac{n}{\nu x}}} = \sqrt{\frac{U_\infty}{\nu x}}$$

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left[T_\infty + \theta\left(T_w - T_\infty\right)\right]$$

$$= \left(T_w - T_\infty\right)\frac{\partial \theta}{\partial x} + \theta \frac{d}{dx} T_w$$

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[T_\infty + \theta\left(T_w - T_\infty\right)\right]$$

$$= \left(T_w - T_\infty\right)\frac{\partial \theta}{\partial y}$$

$$\frac{\partial^2 T}{\partial y^2} = \left(T_w - T_\infty\right)\frac{\partial^2 \theta}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

$$\left(T_w - T_\infty\right)\left[u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\right] + \theta u\left(\frac{dT_w}{dx}\right) = \alpha\left(T_w - T_\infty\right)\frac{\partial^2 \theta}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + \frac{u\theta}{G}\frac{dG}{dx} = \alpha\frac{\partial^2 \theta}{\partial y^2} + \frac{\nu}{c_p\left(T_w - T_\infty\right)}\left(\frac{\partial u}{\partial y}\right)^2$$

$$\frac{dT_w}{dx} = \frac{dG}{dx}$$

Derivations for Lecture 8

Slide 2

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

$y=0 \quad T=T_w(x) \qquad y=\infty \quad T=T_\infty$

$$T = T_\infty + \theta\,(T_w(x) - T_\infty)$$

$$\therefore u\frac{\partial T}{\partial x} = u(T_w - T_\infty)\frac{\partial\theta}{\partial x} + u\cdot\theta\frac{dT_w}{dx}$$

$$v\frac{\partial T}{\partial y} = v(T_w - T_\infty)\frac{\partial\theta}{\partial y}$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\frac{\partial^2\theta}{\partial y^2}$$

$$\therefore (T_w - T_\infty)\left[u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right] + u\theta\frac{dT_w}{dx} = \alpha(T_w - T_\infty)\frac{\partial^2\theta}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

Divide by $(T_w - T_\infty) = G(x)$

$$u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + \frac{u\cdot\theta}{G}\frac{\partial G}{\partial x} = \alpha\frac{\partial^2\theta}{\partial y^2} + \frac{\nu}{c_p(T_w - T_\infty)}\left(\frac{\partial u}{\partial y}\right)^2$$

$\eta = \sqrt{\nu u_\infty x} \qquad \sqrt{\frac{u_\infty}{\nu x}} = \frac{\sqrt{u_\infty}\sqrt{u_\infty}}{\eta}$

$\therefore \sqrt{\nu x} = \frac{\eta}{\sqrt{u_\infty}}$

Substitution gives:

$$u_\infty f'\cdot\theta'\frac{\partial\eta}{\partial x} - \left[f'\eta\frac{\partial\eta}{\partial x} + f\frac{\partial\eta}{\partial x}\right]\cdot\theta'\cdot\sqrt{\frac{u_\infty}{\nu x}} + f'u_\infty\frac{\theta}{G}\frac{dG}{dx}$$

$$= \alpha\cdot\frac{u_\infty}{\nu x}\cdot\theta'' + \frac{\nu}{c_p(T_w - T_\infty)}\cdot\frac{u_\infty^3}{\nu x}(f'')^2$$

$$\therefore u_\infty f'\theta'\cancel{\frac{\partial\eta}{\partial x}} - \frac{\theta'}{\eta}\left[f'\cancel{\eta\frac{\partial\eta}{\partial x}} + f\frac{\partial\eta}{\partial y}\right]\cdot u_\infty + f'u_\infty\frac{\theta}{G}\frac{dG}{dx}$$

$$= \frac{u_\infty}{Pr\,x}\theta'' + \frac{u_\infty^3}{c_p(T_w - T_\infty)x}(f'')^2$$

Divide by $\frac{u_\infty}{x}$

$$\frac{\theta''}{Pr} + \frac{u_\infty^2}{c_p(T_w - T_\infty)}(f'')^2 + f\theta'\left(\frac{x}{\eta}\frac{d\eta}{dx}\right) - f'\theta\left(\frac{x}{G}\frac{dG}{dx}\right) = 0$$

Now $\eta = \sqrt{\nu}\cdot\sqrt{u_\infty x}$

$$\frac{d\eta}{dx} = \sqrt{\nu}\cdot\frac{1}{2}\,\frac{x\frac{du_\infty}{dx} + u_\infty}{\sqrt{u_\infty x}}$$

$$\therefore \frac{x}{\eta}\frac{d\eta}{dx} = \frac{\cancel{\sqrt{\nu}}\,\frac{1}{2}\,x}{\cancel{\sqrt{\nu}}\sqrt{u_\infty x}}\cdot\frac{x\frac{du_\infty}{dx} + u_\infty}{\sqrt{u_\infty x}}$$

$$= \frac{x}{2}\,\frac{x\frac{du_\infty}{dx} + u_\infty}{u_\infty x}$$

$$= \frac{x}{2}\left[\frac{1}{u_\infty}\frac{du_\infty}{dx} + \frac{1}{x}\right]$$

$$= \frac{x}{2}\left[\frac{m}{x} + \frac{1}{x}\right]$$

$$= \frac{m+1}{2}$$

$u_\infty = cx^m$

$\therefore \frac{du_\infty}{dx} = cmx^{m-1}$

$\therefore \frac{1}{u_\infty}\frac{du_\infty}{dx} = \frac{cmx^{m-1}}{cx^m} = \frac{m}{x}$

$$\therefore \frac{\theta''}{Pr} + 2Ec(f'')^2 + f\theta'\left(\frac{m+1}{2}\right) - f'\theta\cdot\frac{x}{G}\frac{dG}{dx} = 0$$

$$\text{r}\ \theta'' + Pr\left[f\theta'\left(\frac{m+1}{2}\right) - f'\theta\cdot\left(\frac{x}{G}\frac{dG}{dx}\right) + 2Ec(f'')^2\right] = 0$$

$$\theta'' + Pr\left[f\theta'\left(\frac{m+1}{2}\right) - \gamma f'\theta + 2Ec(f'')^2\right] = 0$$

r $Pr = 1 \quad \gamma = Ec = m = 0 \qquad \theta'' + \frac{1}{2}f\theta' = 0$

$f''' + 0.5f''f = 0$

$\theta = 1 - f'$

$\theta' = -f''$

$\theta'' = -f'''$

$\therefore -f''' - .5ff'' = 0$

Derivations of Lecture 9

Slide 3.

$$\theta'' + Pr\left(\frac{m+1}{2}\right) f \theta' = 0$$

$$\frac{d\theta'/d\eta}{\theta'} = -Pr\left(\frac{m+1}{2}\right) f$$

Integration gives

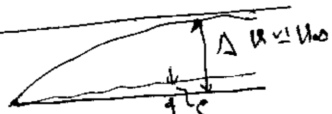
$$\theta' = \theta'(0)\exp\left[-Pr\left(\frac{m+1}{2}\right)\int_0^\eta f\,d\eta\right]$$

$$\theta = \theta'(0)\int_0^\eta \exp\left[-Pr\left(\frac{m+1}{2}\right)\int_0^\eta f\,d\eta\right]d\eta + C_1$$

But $\theta(0) = 1 \quad \therefore C_1 = 1$

$\theta(\infty) = 0$

$$\therefore \theta'(0) = -\frac{1}{\int_0^\infty \exp\left(-Pr\left(\frac{m+1}{2}\right)\int_0^\eta f\,d\eta\right)d\eta}$$



For $Pr \ll 1 \quad \Delta \gg \delta$

$\therefore f'(\eta) = \frac{u}{u_\infty} = 1.0 \qquad f = \eta + C$

but at $\eta = 0$ $f(0) = 0 \quad \therefore C = 0$

$$\therefore \theta'(0) = -\frac{1}{\int_0^\infty \exp\left\{-\left(\frac{Pr(m+1)}{2}\cdot\frac{\eta^2}{2}\right)\right\}d\eta}$$

Let $A = \frac{Pr(m+1)}{4}$ and $A\eta^2 = x^2 \therefore d\eta = \frac{\cancel{2}x\,dx}{\cancel{2}A\eta} = \frac{x}{\eta A}dx = \frac{x\sqrt{A}\,dx}{x}$

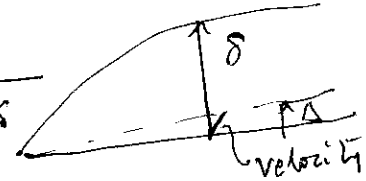
$d\eta = \sqrt{A}\cdot dx$.

$\therefore Int = \int_0^\infty \exp\left(-\frac{Pr(m+1)}{4}\eta^2\right)d\eta = \int_0^\infty \exp(-x^2)\,dx\cdot\sqrt{A}$ and $erf(\infty) = 1$

But $erf(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2}dx$

$\therefore Int = \frac{\sqrt{\pi}}{2}/\sqrt{A}$

or $-\theta'(0) = \frac{2}{\sqrt{\pi}}\sqrt{\frac{Pr(m+1)}{4}} = \sqrt{\frac{Pr(m+1)}{\pi}}$ ✓



for $Pr \gg 1 \quad \Delta \ll \delta$

$\therefore f'(\eta) \approx f''(0)\cdot\eta$

or $f(\eta) = f''(0)\frac{\eta^2}{2} + C$ — but at $\eta = 0$ $f(\eta) = 0 \quad \therefore C = 0$

Hence $f(\eta) = f''(0,m)\cdot\frac{\eta^2}{2}$

$$\therefore \theta'(0) = -\frac{1}{\int_0^\infty \exp\left[-\frac{Pr(m+1)f''(0,m)}{4}\int_0^\eta \eta^2\,d\eta\right]d\eta}$$

$$= -\frac{1}{\int_0^\infty \exp\left[-\frac{Pr(m+1)f''(0,m)}{12}\cdot\eta^3\right]d\eta}$$

Let $\frac{Pr(m+1)f''(0)}{12} = A$

and $A\eta^3 = x \quad \therefore A3\eta^2 d\eta = dx$.

or $d\eta = \frac{dx}{3\eta^2 A} = \frac{dx\,A^{2/3}}{3x^{2/3}\cdot A}$

$= \frac{1}{3A^{1/3}}\cdot\frac{dx}{x^{2/3}}$.

$Int = \int_0^\infty \exp\left[-Pr\frac{(m+1)}{12}f''(0,m)\cdot\eta^3\right]d\eta = \frac{1}{3A^{1/3}}\int_0^\infty x^{-2/3}\exp(-x)\,dx$

Definition: $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ | In our case $n-1 = -2/3$ $\therefore n = 1/3$

Also $\Gamma(n+1) = n\Gamma(n)$

$\therefore \Gamma(4/3) = \frac{1}{3}\Gamma(1/3)$

$\therefore \Gamma(1/3) = 3\Gamma(4/3)$

$\therefore Int = \frac{1}{3A^{1/3}}\cdot 3\Gamma(4/3)$

$= \frac{\Gamma(4/3)}{A^{1/3}}$

$$\therefore -\theta'(0) = \frac{A^{1/3}}{\Gamma(4/3)} = \frac{\left(\frac{Pr(m+1)f''(0,m)}{12}\right)^{1/3}}{\Gamma(4/3)} = 0.893$$