$A \in \mathbb{R}^{n \times n}$ Diagonalizable?

<u>Crucial Notion</u>

Eigenpairs (λ, φ)

$\lambda \in \mathbb{R}$, $\varphi \in \mathbb{R}^n$ s.t.

$$\boxed{A\varphi = \lambda \varphi} \longrightarrow \varphi \neq \theta_n$$

λ : Eigenvalue

φ : Eigenvector corresp. to λ

For diagonalizability of $A \in \mathbb{R}^{n \times n}$

We need n eigenpairs

$(\lambda_1, \varphi_1), (\lambda_2, \varphi_2), \ldots, (\lambda_n, \varphi_n)$

s.t. $\varphi_1, \varphi_2, \ldots, \varphi_n$ are l.i.

Note: $\lambda_1, \lambda_2, \ldots, \lambda_n$ need not be distinct.

Examples:

$$A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

$$\lambda_1 = 4, \quad \varphi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$A\varphi_1 = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$$

$$= 4\varphi_1$$

φ_1 is an eigenvector corresp. to eigenvalue $\lambda_1 = 4$

(λ_1, φ_1) i.e. $\left(4, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$ is an eigenpair for A

Similarly we can verify that if $\lambda_2 = 2$ and $\varphi_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Then $A\varphi_2 = 2\varphi_2 = \lambda_2 \varphi_2$

. $(2, \varphi_2)$ i.e. $\left(2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$ is an eigenpair for A

Similarly we can also verify

$\lambda_3 = -2$, $\varphi_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ satisfy

$$A\varphi_3 = 2\varphi_3 = \lambda_3 \varphi_3$$

$\therefore$ (λ_3, φ_3) i.e. $\left(-2, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$ is an eigenpair for A

We have three eigenpairs

(λ_1, φ_1), (λ_2, φ_2), (λ_3, φ_3)

are eigenpairs for A

$\varphi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\varphi_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\varphi_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

are clearly l.i.

Since A has 3 eigenpairs (λ_1, φ_1), (λ_2, φ_2), (λ_3, φ_3), where $\varphi_1, \varphi_2, \varphi_3$ are l.i.,

we have A is diagonalizable over $\mathbb{R}$

We construct

$$P = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Then P is invertible and

$$P^{-1}AP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

<u>Example:</u> $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$

$\lambda_1 = 4$, $\varphi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ Verify $A\varphi_1 = \lambda_1 \varphi_1 = 4\varphi_1$

$\lambda_2 = 4$, $\varphi_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ Verify $A\varphi_2 = 4\varphi_2 = \lambda_2 \varphi_2$

$\lambda_3 = 2$; $\varphi_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ Verify $A\varphi_3 = 2\varphi_3 = \lambda_3 \varphi_3$

$\therefore$ We have three eigenpairs

$\left(4, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$, $\left(4, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$, $\left(2, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$

and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ are l.i

$\therefore$ A is diagonalizable

<u>Construct</u> $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

P is invertible and

$$P^{-1}AP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Summarize

If $A \in \mathbb{R}^{n \times n}$ has n eigenpairs (λ_j, φ_j), $j = 1, 2, \cdots, n$

where $\varphi_1, \cdots, \varphi_n \in \mathbb{R}^n$ are l.i

then A is diagonalizable.

If $P = [q_1 \; q_2 \; \cdots \; q_n]$

then P^{-1} exists and

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix}$$

Thus the answer to the diagonalization question crucially depends on finding these eigenpairs

Question: Where do we look for these eigenpairs?

Eigenvalues

Suppose $A \in \mathbb{R}^{n \times n}$
$\lambda \in \mathbb{R}$ is an eigenvalue of A

$\Longrightarrow \exists\, u \in \mathbb{R}^n,\ u \neq \theta_n$ s.t. $Au = \lambda u$

$\Longrightarrow (\lambda I - A)u = \theta_n, \quad (u \neq \theta_n)$

$\Rightarrow \quad A_\lambda u = \theta_n \quad , (u \neq \theta_n)$

$\Rightarrow$ The system $\qquad (A_\lambda \in \mathbb{R}^{n \times n})$

$$A_\lambda x = \theta_n \quad \Leftarrow$$

has a nontrivial sol. u

$\Rightarrow A_\lambda$ is not invertible

$\Rightarrow \det(A_\lambda) = 0$

$\Rightarrow \det(\lambda I - A) = 0$

<u>Conclusion:</u>

$A \in \mathbb{R}^{n \times n}$

$\lambda \in \mathbb{R}$ is an eigenvalue of A

$\Longrightarrow \det(\lambda I - A) = 0$

Conversely

Suppose $\det(\lambda I - A) = 0$

$\Longrightarrow (\lambda I - A)$ is not invertible

$\Longrightarrow$ The system

$$(\lambda I - A)x = \theta_n$$

must have a nontrivial sol

$u \in \mathbb{R}^n$, $(u \neq \theta_n)$

$\Rightarrow \quad Au = \lambda u$

$\Rightarrow$ λ is an eigenvalue (with eigenvector u)

Conclusion 2

$\lambda \in \mathbb{R}$, $\det(\lambda I - A) = 0 \Rightarrow \lambda$ is an eigenvalue of A

Result (Theorem)

$A \in \mathbb{R}^{n \times n}$; $\lambda \in \mathbb{R}$

λ eigenvalue of A

$$\Longleftrightarrow \quad \det(\lambda I - A) = 0$$

<u>Therefore</u> we must look for the eigenvalues as the roots of the fun.

$$\underline{D(\lambda)} = \underline{\underline{\det(\lambda I - A)}}$$

So the search for eigenvalues depends on our analysis of this fun

$$D(\lambda) = \det(\lambda I - A)$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ - & - & - & - \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

This is

1) a polynomial in λ,

2) with real coefficients,

3) of degree n,

4) with leading coefficient as 1
("monic")

This polynomial is called the
CHARACTERISTIC POLYNOMIAL
of A and denoted by $C_A(\lambda)$

$C_A(\lambda)$ · is a monic poly
of degree n with real
coefficients

The eigenvalues of A are the (real) roots of this polynomials

The equation

$$C_A(\lambda) = 0$$

is called the characteristic equation of A

Example:

$$A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}$$

$$C_A(\lambda) = \det(\lambda I - A)$$

$$= \begin{vmatrix} \lambda-1 & 3 & -3 \\ 2 & \lambda & -2 \\ -1 & 1 & \lambda-3 \end{vmatrix}$$

$$= (\lambda-4)(\lambda-2)(\lambda+2) \rightarrow \left\{ \begin{array}{l} \text{monic} \\ \text{Poly of degree 3} \\ \text{with real} \\ \text{coeffs} \end{array} \right.$$

Therefore the roots are

$$\lambda_1 = 4, \quad \lambda_2 = 2, \quad \lambda_3 = -2$$

Example 2

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$C_A(\lambda) = \det(\lambda I - A)$$

$$= \begin{vmatrix} \lambda-3 & 1 & -1 \\ 1 & \lambda-3 & -1 \\ 0 & 0 & \lambda-4 \end{vmatrix}$$

$$= \underline{(\lambda-4)^2}\,\underline{(\lambda-2)}$$

monic poly of degree 3 with real coeffs

Roots $\lambda_1 = 4$ repeated twice

$\lambda_2 = 2$

These are the eigenvalues of A

Example $\quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$

$$c_A(x) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix}$$

$$= \lambda^2 + 1$$

Has no real roots

$\therefore$ No real eigenvalues

$\therefore$ No eigen pairs

$\therefore$ No diagonalizability over $\mathbb{R}$

Even though the above matrix is real it does not have any real eigenvalues.

This difficulty arises since we are seeking eigenvalues for $A \in \mathbb{R}^{n \times n}$ as roots of a polynomial with real coefficients But in general a polynomial of degree n with real coefficients,

i) may not have real roots or

ii) if it has real roots it may not have n real roots

In order to avoid this difficulty we shall allow complex eigenvalues and hence complex eigenvectors if necessary.

Therefore we look at the problem as that of diagonalizing $A \in \mathbb{C}^{n \times n}$

over $\mathbb{C}$ i.e. to find

$$P \in \mathbb{C}^{n \times n}, \text{ invertible}$$

s.t. $P^{-1} A P = D$, a diagonal matrix in $\mathbb{C}^{n \times n}$

What is the advantage?

Now we get to treat

$$C_A(\lambda)$$

as a monic polynomial of degree n with complex coefficients.

By Fundamental Theorem of Algebra, $C_A(\lambda)$ will have n

roots (complex), (some of them may be repeated)

If in particular A is real the complex roots will appear in conjugate pairs

Hence we will be able to "successfully" search for n eigenvalues of A as the n roots of the characteristic polynomial

Moral If we allow complex roots

$$C_A(\lambda) = (\lambda - \lambda_1)^{a_1} (\lambda - \lambda_2)^{a_2} \cdots (\lambda - \lambda_k)^{a_k}$$

$\lambda_1, \ldots, \lambda_k$ dist roots

$a_1, \ldots, a_k$ repetitions