(Suitable)
Finding bases for four subspaces of $A$ involves "eigenvalues & eigenvectors"

Problem of diagonalization:

Given $A \in \mathbb{R}^{n\times n}$ Can we find an invertible matrix $P \in \mathbb{R}^{n\times n}$ s.t.

$$P^{-1}AP$$

is a diagonal matrix?

Example 1

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

$$\underline{\underline{P}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

$|P| = -2 \neq 0$ Hence P is $\underline{\underline{\text{invertible}}}$

$$P^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$AP = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= PD \quad , \text{ where } D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Rightarrow$

$\underline{P^{-1}AP = D}$, a diagonal matrix

Thus there exists an invertible $P \in \mathbb{R}^{2\times 2}$ s.t $P^{-1}AP$ is a diagonable matrix

## Example 2

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

We will show that there is no matrix $P \in \mathbb{R}^{2\times 2}$ which is invertible and s.t. $P^{-1}AP$ is a diagonal matrix

— We shall show this by contradiction

Suppose $\exists$ an invertible $P \in \mathbb{R}^{2\times 2}$ s.t. $P^{-1}AP$ is diagonal matrix

$$\Longrightarrow \quad P^{-1}AP = \underbrace{\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}}_{D}$$

$$\implies A = P\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}P^{-1}$$

$$\implies A^2 = P\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}P^{-1}P\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}P^{-1}$$

$$= P\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}P^{-1}$$

& hence

$$\underline{A^2 = P\begin{pmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{pmatrix}P^{-1}} \quad \text{-- -- (1)}$$

On the other hand

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad A^2 = \underline{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \quad \cdots \quad (2)$$

By (1) & (2)

$$P \begin{pmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{pmatrix} = P^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} P$$

$$\Rightarrow \quad \begin{pmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow d_1 = d_2 = 0$

$\Rightarrow D = O_{2\times 2}$

$\therefore P^{-1} A P = D \Rightarrow P^{-1} A P = O_{2\times 2}$

$\Rightarrow A = O_{2\times 2}$

— contradiction $\because A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq O_{2\times 2}$

$\therefore$ There does not exist any invertible $P \in \mathbb{R}^{2\times 2}$ s.t. $P^{-1}AP$ is a diagonal matrix

Hence we have two examples in one of which we were able to find a P s.t. $P^{-1}AP$ is diagonal & in the other there does not exist any such P.

Hence it becomes necessary to look for some conditions which assure that such a P exists

<u>Definition:</u>

$A \in \mathbb{R}^{n \times n}$ is said to be DIAGONALIZABLE over $\mathbb{R}$ if $\exists$ an invertible $P \in \mathbb{R}^{n\times n}$

s.t $P^{-1}AP$ is a diagonal matrix

The matrix $A$ in Example 1 is diagonalizable over $\mathbb{R}$ & the matrix $A$ in Example 2 is not diagonalizable over $\mathbb{R}$

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Let us first consider a matrix

$$A \in \mathbb{R}^{n\times n}$$

which is diagonalizable over $\mathbb{R}$

By definition this means there is an invertible $P \in \mathbb{R}^{n \times n}$

s.t. $$P^{-1}AP = D, \text{ a diagonal matrix}$$

$$\Rightarrow \boxed{AP = PD \quad \ldots \quad (\mathrm{I})}$$

P is an $n \times n$ real matrix

$\therefore$ Each column of P is an $n \times 1$ matrix

i.e. each column $\in \mathbb{R}^n$

Let us denote these columns as

$$P_1, P_2, \ldots, P_n$$

Then P can be written as

$$P = [P_1 \quad P_2 \quad \cdots \quad P_n]$$

The LHS of (I) becomes

$$AP = A[P_1 \quad P_2 \quad \cdots \quad P_n]$$

$$= [AP_1 \quad AP_2 \quad \cdots \quad AP_n] \quad \text{--- (LHS)}$$

The RHS of (I) becomes

$$PD = [P_1 \quad P_2 \quad \cdots \quad P_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$= [\lambda_1 P_1 \quad \lambda_2 P_2 \quad -- \quad \lambda_n P_n] \quad --- (RHS$$

$\therefore$ LHS = RHS $\Rightarrow$

$$[AP_1 \quad AP_2 \quad -- \quad AP_n] = [\lambda_1 P_1 \quad --- \quad \lambda_n P_n]$$

$$\Rightarrow \left\{ \begin{array}{l} AP_1 = \lambda_1 P_1 \\ AP_2 = \lambda_2 P_2 \\ \vdots \\ AP_n = \lambda_n P_n \end{array} \right\} \quad \begin{array}{l} \lambda_1, \ldots, \lambda_n \in \mathbb{R} \\ P_1, \ldots, P_n \in \mathbb{R}^n \end{array}$$

$\Rightarrow$ There exist $n$ real number $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ (not necessarily distinct) and $n$ l.i. vectors

$p_1, p_2, \ldots, p_n \in \mathbb{R}^n$

s.t. $A p_j = \lambda_j p_j$ , $1 \le j \le n$

## CONCLUSION 1

$A \in \mathbb{R}^{n \times n}$ is diagonalizable over $\mathbb{R}$

$\Longleftrightarrow$ $\exists$ $n$ real numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (not necessarily distinct)

and $n$ l-i vectors

$p_1, p_2, \ldots, p_n \in \mathbb{R}^n$

s.t

$A p_j = \lambda_j p_j$ , for $1 \le j \le n$

# Illustration for a 2x2 matrix

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$. $A$ is diagonalizable over $\mathbb{R}$

$\Rightarrow \exists\ P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathbb{R}^{2\times 2}$, $P$ invertible $(pr - qs \neq 0)$,

s.t

$$P^{-1} A P = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$AP = PD$$

$$\underline{\underline{LHS}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

$$P_1 = \begin{pmatrix} p \\ r \end{pmatrix} \qquad P_2 = \begin{pmatrix} q \\ s \end{pmatrix}$$

$$AP_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} ap + br \\ cp + dr \end{pmatrix}$$

III$^{ly}$

$$AP_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ s \end{pmatrix} = \begin{pmatrix} aq + bs \\ cq + ds \end{pmatrix}$$

$$\therefore \ LHS = [AP_1 \quad AP_2]$$

$$\underline{RHS} = PD$$

$$= \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 p & \lambda_2 q \\ \lambda_1 r & \lambda_2 s \end{pmatrix}$$

$1^{st}$ col $= \lambda_1 \begin{pmatrix} p \\ r \end{pmatrix} = \lambda_1 P_1$

$2^{nd}$ col $= \lambda_2 \begin{pmatrix} q \\ s \end{pmatrix} = \lambda_2 P_2$

$$RHS = [\lambda_1 P_1 \quad \lambda_2 P_2]$$

$$LHS \text{ was } = [A P_1 \quad A P_2]$$

$$LHS = RHS \implies \begin{aligned} AP_1 &= \lambda_1 P_1 \\ AP_2 &= \lambda_2 P_2 \end{aligned}$$

Recall Ex 1

We had $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

We found $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

& $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D$

$\underline{P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ $\underline{P_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$, $\lambda_1 = 1$, $\lambda_2 = -1$

Check

$AP_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{1}\, P_1 = \lambda_1 P_1$

$AP_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)P_2 = \lambda_2 P_2$

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Conversely, let $A \in \mathbb{R}^{n \times n}$ be such that

$\exists$ $n$ real numbers

$\lambda_1, \dots, \lambda_n \in \mathbb{R}$ (not necessarily distinct)

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& $n$ l.i. vectors

$p_1, p_2, \dots, p_n \in \mathbb{R}^n$

such that $Ap_j = \lambda_j p_j$ for $1 \le j \le n$ $\longrightarrow (*)$

Now let

$$P = [p_1 \; p_2 \; \dots \; p_n]$$

$P$ is in $\mathbb{R}^{n \times n}$, invertible ($\because$ columns are l.i.)

Then

$$AP = A[p_1 \; p_2 \; \dots \; p_n]$$

$$= [Ap_1 \; Ap_2 \; \dots \; Ap_n]$$

$$= [\lambda_1 p_1 \; \lambda_2 p_2 \; \dots \; \lambda_n p_n] \text{ by } (*)$$

$$= \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}}_{D}$$

$$= PD$$

$\Rightarrow$ $P^{-1}AP = D$, a diagonal matrix

$\Rightarrow$ A is diagonalizable

CONCLUSION 2

$\therefore$ (C) $\Rightarrow$ A is diagonalizable

Conclusion 1

A is diagonalizable over $\mathbb{R}$ $\Rightarrow$ (C)

**Theorem:** $A \in \mathbb{R}^{n\times n}$ is diagonalizable over $\mathbb{R}$

$\Longleftrightarrow$ (C) i.e.

$\exists$ $n$ real numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ (not necessarily distinct)

& $n$ l.i. vectors $P_1, \dots, P_n \in \mathbb{R}^n$

s.t. $\underline{A P_j = \lambda_j P_j}$, for $1 \le j \le n$

**Definition**

$A \in \mathbb{R}^{n\times n}$

A real number $\lambda \in \mathbb{R}$ is said to be an EIGENVALUE (or Characteristic value)

of the matrix $A$ if there exists
a nonzero vector $\phi \in \mathbb{R}^n$ s.t

$$A\phi = \lambda \phi$$

In such a case $\phi$ is called an EIGENVECTOR (or Characteristic vector) of $A$ corresponding to the eigenvalue $\lambda$

$(\lambda, \phi)$ is called an Eigenpair (or Characteristic pair)

Hence by our theorem we get

$A \in \mathbb{R}^{n \times n}$ is diagonalizable over $\mathbb{R}$

$\iff$ There exist $n$ eigenpairs

$(\lambda_1, \phi_1), (\lambda_2, \phi_2), \ldots, (\lambda_n, \phi_n)$

where $\phi_1, \phi_2, \ldots, \phi_n$ are l.i

Search should be for these $n$ eigenpairs

Where do we search for these eigenpairs

— This leads us to the analysis of eigenvalues and eigenvectors

Suppose we have found an eigenvalue $\lambda$

Then we seek $\phi$ s.t. $A\phi = \lambda\phi$

$$(A - \lambda I)\phi = \theta_n$$

$$A_\lambda \phi = \theta_n \qquad (A_\lambda = A - \lambda I)$$

( Sol. for Homog. system corr to $A_\lambda$)

Knowing eigvalue $\lambda$ there is a chance of finding corresponding eigenvector by solving the homog system

$$A_\lambda \phi = \theta_n \quad \text{where } A_\lambda = A - \lambda I$$

So our primary search begin with the search for the eigenvalues of $A$.