Orthonormal sets

$S \subseteq \mathbb{R}^k$ s t $\quad (\varphi, \psi) = \begin{cases} 0 & \text{if } \varphi \neq \psi \\ 1 & \text{if } \varphi = \psi \end{cases}$

$\varphi, \psi \in S$

Orthonormal Basis

A basis for $\mathbb{R}^k$ which is an o.n. set is called an o n b $\quad \mathcal{B} : \{\varphi_j\}_{j=1}^{k}$ o.n.b

$$x \in \mathbb{R}^k \implies x = \sum_{j=1}^{k} \underline{(x, \varphi_j)}\, \varphi_j . \qquad \left((x, \varphi_j) = \varphi_j^T x\right)$$

$$\implies \|x\|^2 = \sum_{j=1}^{k} (x, \varphi_j)^2$$

Also if $x, y \in \mathbb{R}^k$ then

$$(x, y) = \sum_{j=1}^{k} (x, \varphi_j)(y, \varphi_j)$$

First observe

any o.n. set in $\mathbb{R}^k$ is

either a basis for $\mathbb{R}^k$

OR

it can be extended to an onb for $\mathbb{R}^k$

W a subspace of $\mathbb{R}^k$

Orthogonal complement of W is defined as

$$W^{\perp} = \{ x \in \mathbb{R}^k : (x, w) = 0 \ \forall w \in W \}$$

We observed the following facts.

1 $W^\perp$ is a subspace of $\mathbb{R}^k$

2 If $\mathcal{B}_W$ is a basis for W and
$\mathcal{B}_{W^\perp}$ " " $W^\perp$

Then
$$\underline{\underline{\mathcal{B}}} = \underline{\underline{\mathcal{B}_W}} \cup \underline{\underline{\mathcal{B}_{W^\perp}}}$$
is a basis for $\mathbb{R}^k$

3 $\dim W + \dim W^\perp = \dim \mathbb{R}^k = k$

4. $\mathcal{G}_W$ an o.n.b. for W and
$\mathcal{G}_{W^\perp}$ " " " $W^\perp$ then

$\mathcal{G} = \mathcal{G}_W \cup \mathcal{G}_{W^\perp}$ is an o.n.b. for $\mathbb{R}^k$

5 Any vector $x \in \mathbb{R}^k$ can be decomposed as a sum

$$x = x_W + x_{W^\perp}$$ where

$x_W \in W$ and $x_{W^\perp} \in W^\perp$ in a unique manner

6. Pythagoras Theorem:

$$\|x\|^2 = \|x_W\|^2 + \|x_{W^\perp}\|^2$$

7 x_W : Orthogonal projection of x onto W

$x_{W^\perp}$: orthogonal projection of x onto $W^\perp$

Example: $\mathbb{R}^3$

$$W = \left\{ x = \begin{pmatrix} a \\ a \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$$

W is a subspace

$\boxed{\mathcal{B}_W : u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$ is a basis for W

$\boxed{\dim W = 1}$

What is $W^\perp$?

$$W^\perp = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ -\alpha-\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$\mathcal{B}_{W^\perp}$: $w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$; $w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is a basis for $W^\perp$

We see that

$$\dim W^{\perp} = 2$$

$$\boxed{\dim W + \dim W^{\perp} = 1+2 = \dim \mathbb{R}^3}$$

If we set

$$\boxed{\mathcal{B}} = \boxed{\mathcal{B}_W \cup \mathcal{B}_{W^{\perp}}}$$

$$= u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \; w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

These are l.i. (Easy to check)

They form a basis for $\mathbb{R}^3$ since any three l.i. vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$

$\mathcal{B}_W : u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ basis for W

$$\boxed{\mathcal{O}_W : \varphi_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$\mathcal{B}_{W^\perp} : w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Apply G-S orthonormalization to $\mathcal{B}_{W^\perp}$

$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \|v_1\|^2 = 2 \qquad \psi_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{(v_2, v_1)}{\|v_1\|^2} v_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} \qquad \|v_2\|^2 = \frac{1}{2^2}(1+2+\cancel{1})$$

$$\frac{v_2}{\|v_2\|} = \psi_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$\boxed{G_{W^\perp} \cdot \ \psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \psi_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}}$$

$$G = G_W \cup G_{W^\perp}$$

$$= \left.\varphi_1, \psi_1, \psi_2 \right\}$$

is an o.n.b. for $\mathbb{R}^3$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Expand in terms of the onb 6 for $\mathbb{R}^3$

The expansion is

$$(x, \varphi_1)\varphi_1 + (x, \psi_1)\psi_1 + (x, \psi_2)\psi_2 \quad \text{-- (I)}$$

$$(x, \varphi_1) = \frac{x_1 + x_2 + x_3}{\sqrt{3}}, \quad (x, \psi_1) = \frac{x_1 - x_3}{\sqrt{2}}$$

$$(x, \psi_2) = \frac{-x_1 + 2x_2 - x_3}{\sqrt{6}}$$

LHS (I) =

$$\frac{x_1+x_2+x_3}{\sqrt{3}}\,\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$+\frac{x_1-x_3}{\sqrt{2}}\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$+\frac{-x_1+2x_2-x_3}{\sqrt{6}}\,\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

$$=\underbrace{\begin{pmatrix}(x_1+x_2+x_3)/3\\(x_1+x_2+x_3)/3\\(x_1+x_2+x_3)/3\end{pmatrix}}_{x_W}+\underbrace{\begin{pmatrix}(2x_1-x_2-x_3)/3\\(-x_1+2x_2-x_3)/3\\(-x_1-x_2+2x_3)/3\end{pmatrix}}_{x_{W^\perp}}$$

$$= x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

For example if

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$$

$$x_W = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}}_{\in W} \qquad x_{W^\perp} = \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\in W^\perp}$$

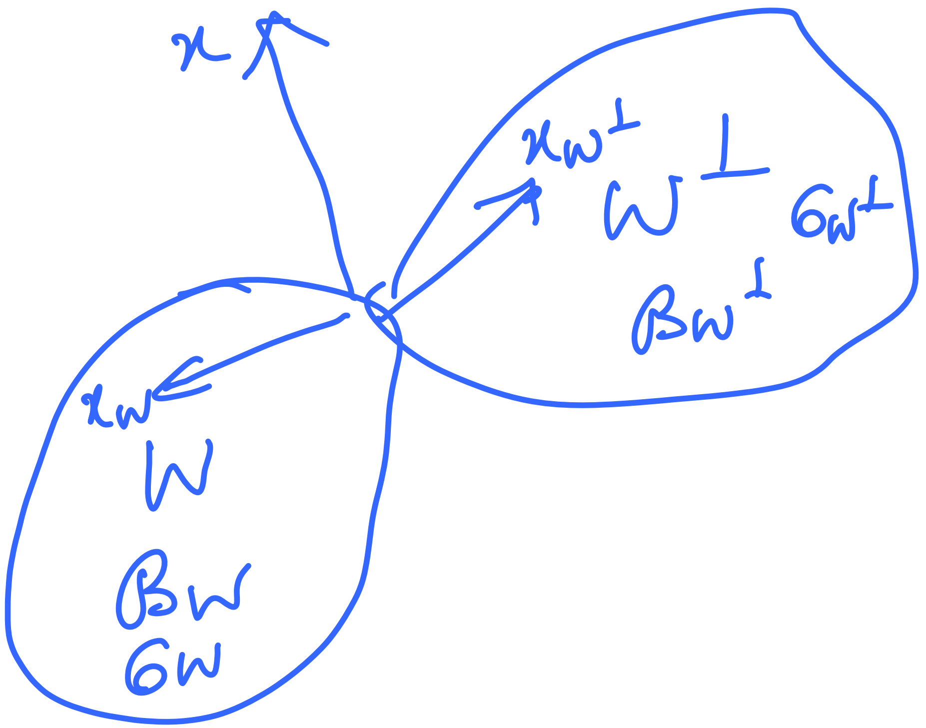
$$x = x_W + x_{W^\perp}$$

<u>Pythagoras:</u> $\|x\|^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9$
$= 14$

$$\|x_W\|^2 = 2^2 + 2^2 + 2^2 = 4+4+4 = 12$$

$$\|x_{W^\perp}\|^2 = (-1)^2 + 0^2 + 1^2 = 1+1 = 2$$

$$\|x_W\|^2 + \|x_{W^\perp}\|^2 = 12 + 2 = 14$$
$$= \|x\|^2$$



One More Simple Property of Orth Comp

$\mathbb{R}^k$, W a subspace of $\mathbb{R}^k$

$\dim W = d$

$W^\perp$: orth compl of W

$\dim W^\perp = k-d$

$X = W^\perp$ X is a subspace

$\dim X = k-d$

$X^\perp$ orth. comp. of X

$= \{x \in \mathbb{R}^k : (x, y) = 0 \ \forall\, y \in X = W^\perp\}$

Clearly $x \in W \implies x \in X^{\perp}$

$$\boxed{W \subseteq X^{\perp}}$$

$W \subseteq (W^{\perp})^{\perp}$

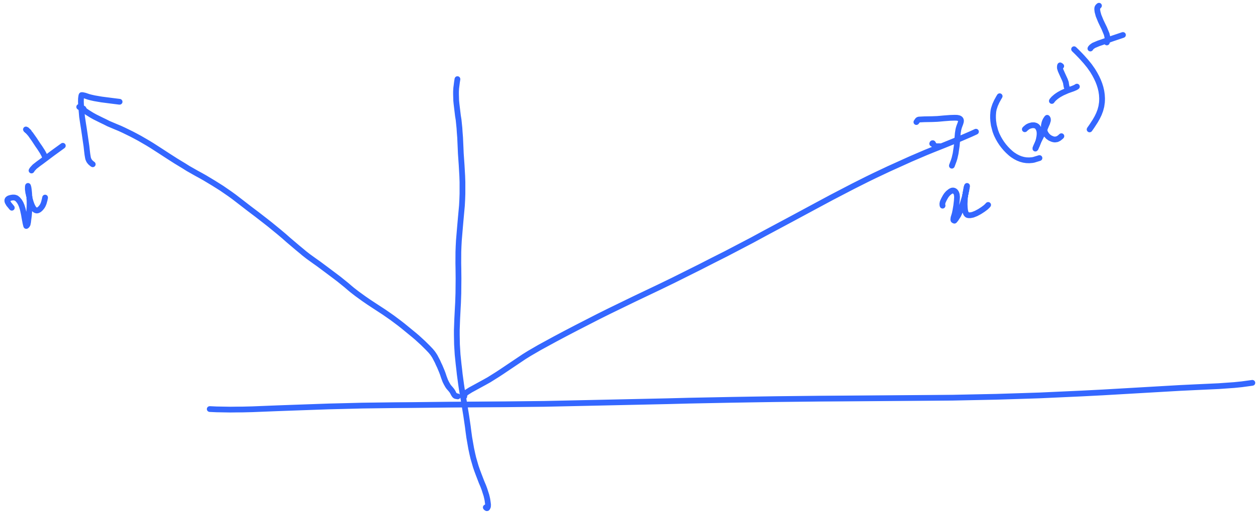
$$\dim X + \dim X^{\perp} = \dim \mathbb{R}^k$$

$$\dim W^{\perp} + \dim (W^{\perp})^{\perp} = \dim \mathbb{R}^k$$

Also have
$$\dim W^{\perp} + \dim W = \dim \mathbb{R}^k$$

$$\implies \boxed{\dim W = (\dim W^{\perp})^{\perp}}$$

$$\implies W = (W^{\perp})^{\perp}$$



Context of Matrices

$A \in \mathbb{R}^{m \times n}$

The Four Fundamental Bases

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

Two Subspacs

R_{A^T} and N_A

Two Subspacs

R_A and N_{A^T}

$$N_A = \{x \in \mathbb{R}^n : Ax = \theta_m\}$$

$$N_{A^T} = \{x \in \mathbb{R}^m : A^T x = \theta_n\}$$

$R_{A^T} = \{x \in \mathbb{R}^n : x = A^T b \text{ for } b \in \mathbb{R}^m\}$

$R_A = \{b \in \mathbb{R}^m \; b = Ax \text{ for } x \in \mathbb{R}^n\}$

$\boxed{\text{Look at } N_A, R_{A^T}}$

$$x \in N_A \iff Ax = \theta_m$$
$$\iff (Ax, b) = 0 \quad \forall b \in \mathbb{R}^m$$
$$\iff b^T(Ax) = 0 \quad \forall b \in \mathbb{R}^m$$
$$\iff (b^T A)x = 0 \quad \forall b \in \mathbb{R}^m$$

$\iff (A^T b)^T x = 0 \quad \forall b \in \mathbb{R}^m$

$\iff (x, A^T b) = 0 \quad \forall b \in \mathbb{R}^m$

$\iff$ x orth to all vectors of the form $A^T b$, $b \in \mathbb{R}^m$

$\iff$ x is orth to all vect in R_{A^T}

$\iff x \in R_{A^T}^{\perp}$

$N_A = R_{A^T}^{\perp}$	$N_{A^T} = R_A^{\perp}$
$N_A^{\perp} = R_{A^T}$	$N_{A^T}^{\perp} = R_A$
$\mathbb{R}^n$	$\mathbb{R}^m$

$$\dim R_{A^T} + \dim R_{A^T}^{\perp} = n$$

$$\rho_{A^T} + \nu_A = n$$

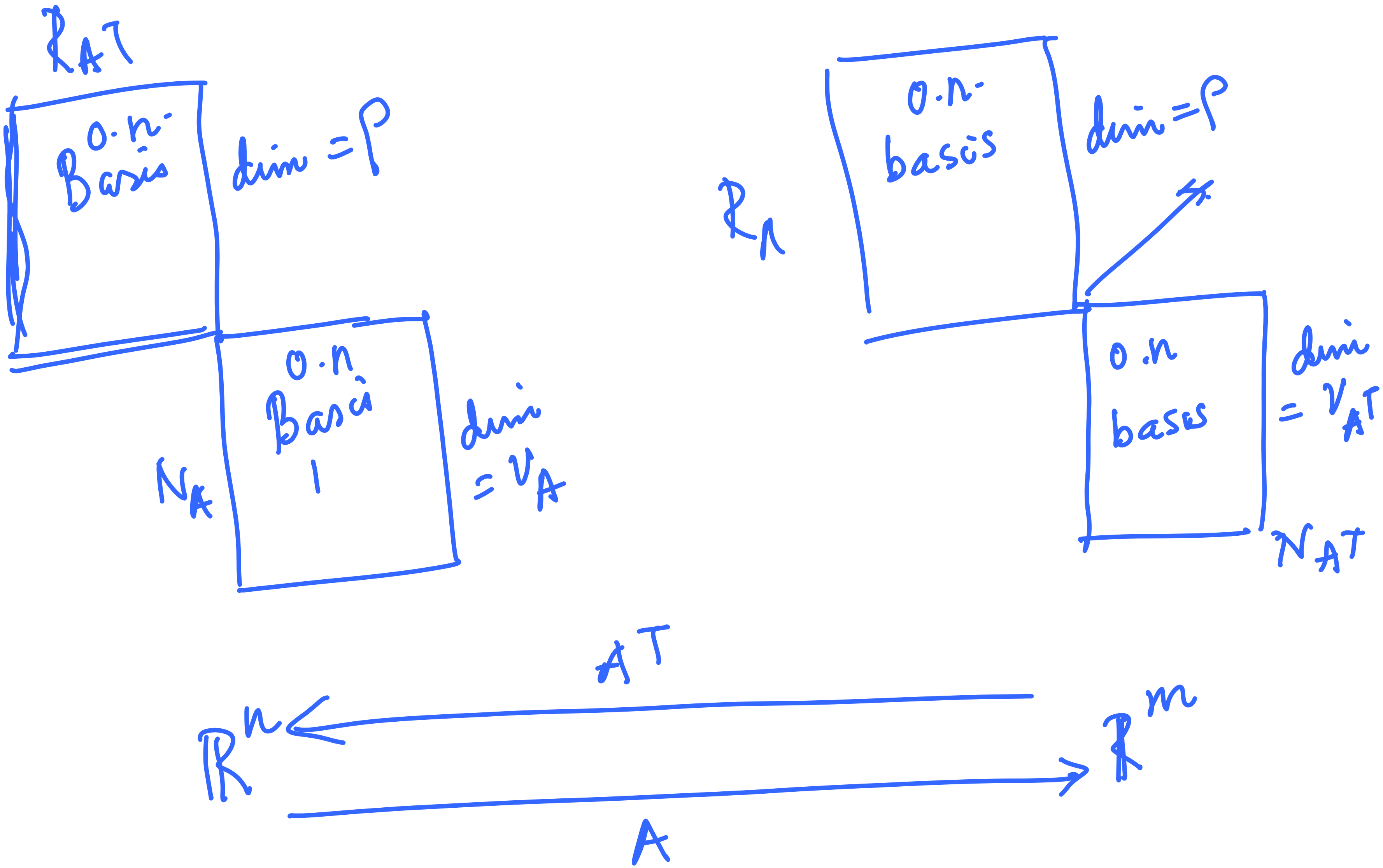
On the other hand Rank-Nullity Thm gives

$$\rho_A + \nu_A = n$$

$$\Longrightarrow$$

$$\rho_A = \rho_{A^T}$$

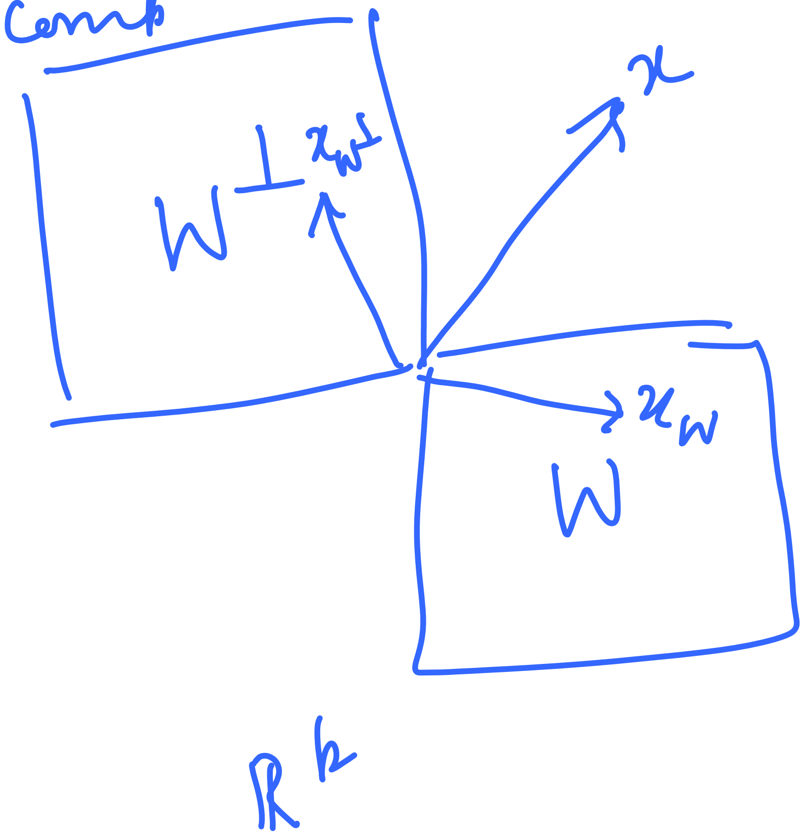
For any matrix (over reals) its rank = rank of its transpose



R_{A^T}
o.n. Basis
dim = ρ
N_A
o.n. Basis
dim = ν_A
R_A
o.n. basis
dim = ρ
o.n basis
dim = ν_{A^T}
N_{A^T}
A^T
R^n
R^m
A

W subsp of $\mathbb{R}^n$

$W^\perp$ orth comp



$x \in \mathbb{R}^k$

$\Rightarrow$

$x = x_W + x_{W^\perp}$

Given x find a vector $w_0 \in W$ s.t

$$\|x - w_0\|^2 < \|x - w\|^2 \quad \forall\, w \neq w_0 \quad w \in W$$

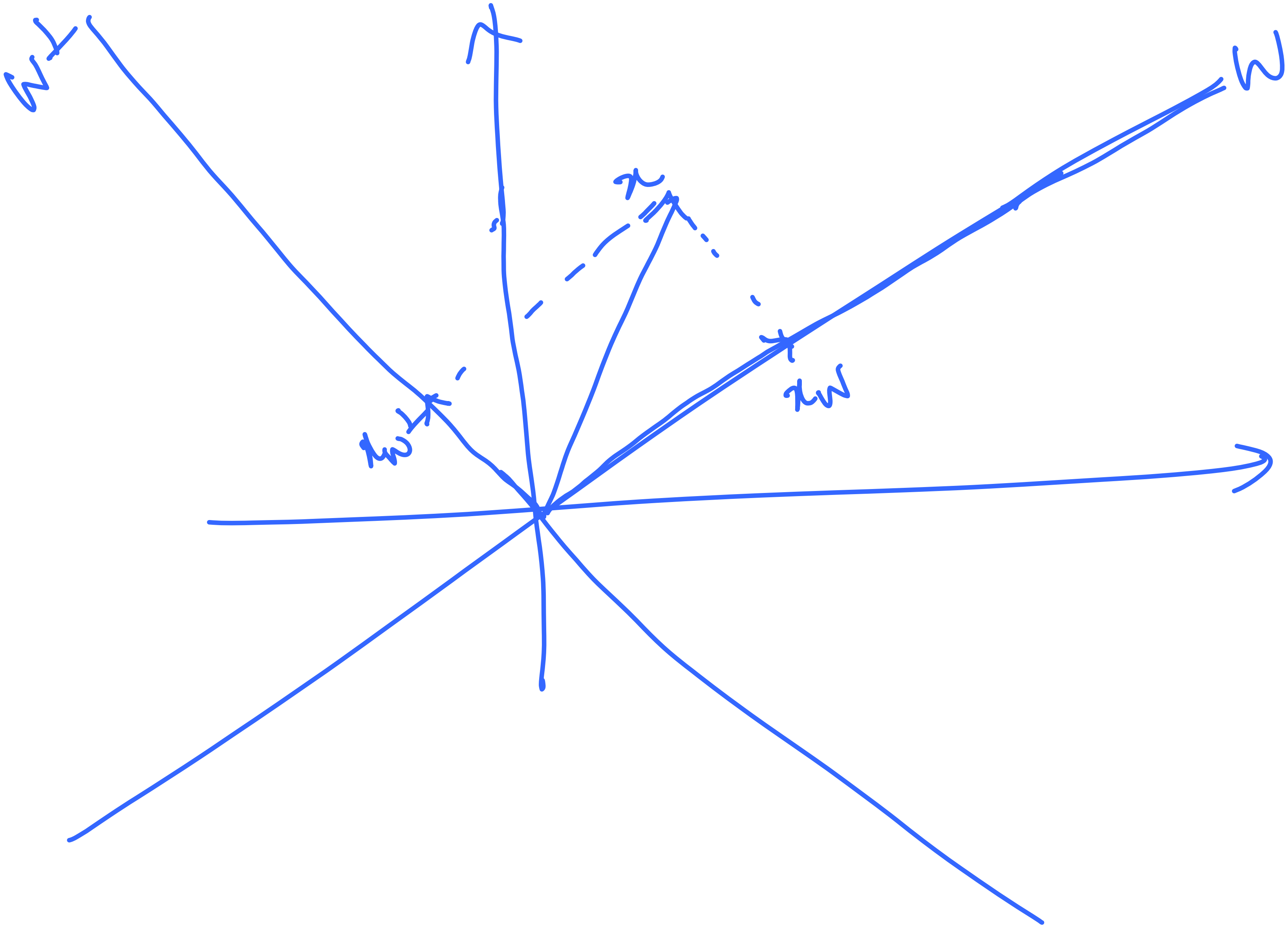
Best Approximation Problem

$\mathbb{R}^k$; W subspace of $\mathbb{R}^k$

Given $x \in \mathbb{R}^k$ find $w_0 \in W$

s.t $\|x - w_0\|^2 < \|x - w\|^2 \ \forall w \in W \ w \neq w_0$

The vector w_0 which gives this least error for x is precisely x_W — the orthogonal proj of x onto W



x_W : The orth proj of x onto W gives the best app. of x from W