Orthonormal Basis

Expansion

$\mathbb{R}^k$

$\mathcal{B} : \varphi_1, \varphi_2, \ldots, \varphi_k$ o.n.b for $\mathbb{R}^k$

i) $x = \sum_{j=1}^{k} (x, \varphi_j)\varphi_j$ $\quad \forall x \in \mathbb{R}^k$

(Fourier exp. of x wrt $\mathcal{B}$)

ii) $(x, y) = \sum_{j=1}^{k} (x, \varphi_j)(y, \varphi_j)$ $\quad \forall x, y \in \mathbb{R}^k$

(Plancheral's formula)

iii) $\|x\|^2 = (x, x) = \sum_{j=1}^{k} (x, \varphi_j)^2$ $\quad \forall x \in \mathbb{R}^k$

Example

$\mathbb{R}^3$ $\quad \mathcal{B} : \varphi_1, \varphi_2, \varphi_3$

where $\varphi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \varphi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \varphi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

This is o.n b. for $\mathbb{R}^3$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^k$$

$$(x, \varphi_1) = \frac{1}{\sqrt{2}} (x_1 + x_2)$$

$$(x, \varphi_2) = \frac{1}{\sqrt{2}} (x_1 - x_2)$$

$$(x, \varphi_3) = x_3$$

Hence

$$x = (x, \varphi_1)\varphi_1 + (x, \varphi_2)\varphi_2 + (x, \varphi_3)\varphi_3$$

$$\underline{\text{RHS}} = \frac{1}{\sqrt{2}}(x_1 + x_2)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}(x_1 - x_2)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$+ x_3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}\frac{x_1+x_2}{2} + \frac{x_1-x_2}{2}\\ \frac{x_1+x_2}{2} - \frac{(x_1-x_2)}{2}\\ x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x = \text{LHS}$$

Fourier Exp. of x wrt $\mathcal{B}$

$$x = \frac{1}{\sqrt{2}}(x_1 + x_2)\varphi_1 + \frac{1}{\sqrt{2}}(x_1 - x_2)\varphi_2 + x_3\varphi_3$$

x, y

$$(x,\varphi_1) = \frac{1}{\sqrt{2}}(x_1+x_2), (x,\varphi_2) = \frac{1}{\sqrt{2}}(x_1-x_2), (x,\varphi_3) = x_3$$

$$(y,\varphi_1) = \frac{1}{\sqrt{2}}(y_1+y_2), (y,\varphi_2) = \frac{1}{\sqrt{2}}(y_1-y_2), (y,\varphi_3) = y_3$$

$$\underline{(x,\varphi_1)(y,\varphi_1)} + \underline{(x,\varphi_2)(y,\varphi_2)} + \underline{(x,\varphi_3)(y,\varphi_3)}$$

$$= \frac{1}{\sqrt{2}}(x_1+x_2)\frac{1}{\sqrt{2}}(y_1+y_2) + \frac{1}{\sqrt{2}}(x_1-x_2)\frac{1}{\sqrt{2}}(y_1-y_2) + x_3y_3$$

$$= x_1y_1 + x_2y_2 + x_3y_3$$

$$= \underline{(x,y)}$$ — Verifying Plancherel's Formula

$$\underline{(x,\varphi_1)}^2 + \underline{(x,\varphi_2)}^2 + \underline{(x,\varphi_3)}^2$$

$$= \frac{1}{2}(x_1+x_2)^2 + \frac{1}{2}(x_1-x_2)^2 + x_3^2$$

$= x_1^2 + x_2^2 + x_3^2$

$= \|x\|^2$ — Verifying Parseval's identity

$\mathbb{R}^k$

S l.i. set in $\mathbb{R}^k$ Either — S is a basis for $\mathbb{R}^k$ (i.e. when $r = k$)

$\shortparallel$

$\varphi_1, \dots, \varphi_r$ OR — S can be extended to a basis for $\mathbb{R}^k$

S o.n. set : $S = \varphi_1, \varphi_2, \dots, \varphi_r$

$S \Longrightarrow S$ is l.i

Either $\to$ S is a basis (if $r = k$)

$\therefore S$ is an o.n basis for $\mathbb{R}^k$

OR

S can be extended to a basis for $\mathbb{R}^k$

— Can we extend S to an o.n. basis for $\mathbb{R}^k$

Gram-Schmidt Orthonormalization

Goal Given

$S = u_1, \dots, u_r$ a l.i. set in $\mathbb{R}^k$

To Produce

$O_S = \varphi_1, \varphi_2, \dots, \varphi_r$ an o.n. set in $\mathbb{R}^k$

such that

(i) $L[S] = L[O_S]$

Actually the procedure does even more

It does the following :

$$S_j = u_1, u_2, \ldots, u_j \quad ; \quad (1 \le j \le r)$$

$$O_{S_j} = \varphi_1, \varphi_2, \ldots, \varphi_j \quad (\text{o.n})$$

s.t

$$L[S_j] = L[O_{S_j}]$$

(when $j = r$ we get $S_j = S$, $O_{S_j} = O_S$

and $\quad L[S] = L[O_S]$

The GS procedure

FIRST STEP To get orthogonalization

$S : u_1, u_2, \dots, u_r$

Find $S_1 : v_1, v_2, \dots, v_r$

s.t $(v_i, v_j) = 0$ if $i \neq j$

and $L[u_1, \dots, u_j] = L[v_1, \dots, v_j]$

for $j = 1, 2, \dots, r$

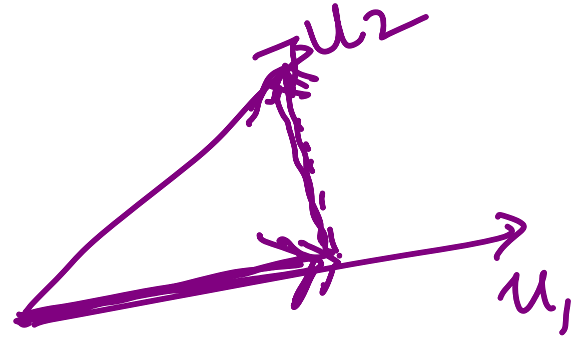
We get $v_1, v_2, \dots, v_r$ as follows:

$v_1 = u_1$ $\qquad\qquad \|v_1\|^2 = (v_1, v_1) = (u_1, u_1)$

$$v_2 = u_2 - (u_2, v_1)\frac{v_1}{\|v_1\|^2}$$

(Note $(v_2, v_1) = 0$)

$\|v_2\|^2$



v_1, v_2 orthogonal

v_1, v_2 can be written as a lc of u_1, u_2

& u_1, u_2 " " " v_1, v_2

$$\therefore L[u_1, u_2] = L[v_1, v_2]$$

Suppose we have defined $v_1, v_2, \ldots, v_{j-1}$

then we define v_j as follows:

$$v_j = u_j - \sum_{i=1}^{j-1} (u_j, v_i)\frac{v_i}{\|v_i\|^2}$$

$$\begin{cases} v_1 = u_1 \\ v_j = u_j - \sum_{i=1}^{j-1} (u_j, v_i) \dfrac{v_i}{\|v_i\|^2} \quad \text{for } j \geq 2 \end{cases}$$

Then $(v_i, v_j) = 0$ for $i \neq j$

& $L[u_1, \ldots, u_j] = L\{v_1, \ldots, v_j\}$ $\quad 1 \leq j \leq r$

& in particular if $j = r$ we get

$L[u_1, \ldots, u_r] = L\{v_1, \ldots, v_r\}$

<u>Second Step</u> Normalization

Let $\quad \varphi_j = \dfrac{v_j}{\|v_j\|}$

Then $O_S = \varphi_1, \varphi_2, \dots, \varphi_r$

is the orthonormal set we are looking for

i.e. $L[u_1, \dots, u_j] = L[\varphi_1, \dots, \varphi_j]$

for $1 \le j \le r$

And in particular

$$L[S] = L[O_S]$$

Examples

(1) $\mathbb{R}^4$ $\quad S = u_1, u_2, u_3$

where $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

Easy to verify that S is l.i.

Apply G-S to this

The first step of G-S is orthogonalization.

But we note that u_1, u_2, u_3 are already orthogonal to each other

The G-S will give

$$v_1 = u_1, \quad v_2 = u_2, \quad v_3 = u_3$$

Second Step

$$\varphi_1 = \frac{v_1}{\|v_1\|}, \quad \varphi_2 = \frac{v_2}{\|v_2\|}, \quad \varphi_3 = \frac{v_3}{\|v_3\|}$$

$$\varphi_1 = \frac{v_1}{2}, \quad \varphi_2 = \frac{v_2}{2}, \quad \varphi_3 = \frac{v_3}{\sqrt{2}}$$

$$\varphi_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \varphi_2 = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \quad \varphi_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

$$O_S = \varphi_1, \varphi_2, \varphi_3.$$

Example: $\mathbb{R}^4$ $\quad S : u_1, u_2, u_3$

where

$$\underline{u}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \underline{u}_2 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \quad \underline{u}_3 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

Check that S is l-i

Apply G-S

Step I Orthogonalization

$$\underline{\underline{v_1}} = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \|v_1\|^2 = 4$$

$$\underline{\underline{v_2}} = u_2 - (u_2, v_1) \frac{v_1}{\|v_1\|^2}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{(3)}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ -3/4 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$$

$$\|v_2\|^2 = \frac{12}{16} = \underline{\frac{3}{4}}$$

$$\underline{\underline{v_3}} = u_3 - (u_3, v_1)\frac{v_1}{\|v_1\|^2} - (u_3, v_2)\frac{v_2}{\|v_2\|^2}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{(2/4)}{(3/4)}\frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 1/2 - 1/6 \\ 1 - 1/2 - 1/6 \\ -1/2 - 1/6 \\ -1/2 + 1/2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\|v_3\|^2 = \frac{4}{9}$$

Second Step Normalization

$$\varphi_1 = \frac{v_1}{\|v_1\|} \qquad \varphi_2 = \frac{v_2}{\|v_2\|} \qquad \varphi_3 = \frac{v_3}{\|v_3\|}$$

$$\varphi_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \varphi_2 = \frac{1}{2\sqrt{3}}\begin{pmatrix}1\\1\\1\\-3\end{pmatrix}, \quad \varphi_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\\0\end{pmatrix}.$$

$$L[\varphi_1] = L\{u_1\}$$

$$L[\varphi_1, \varphi_2] = L\{u_1, u_2\}$$

$$L[\varphi_1, \varphi_2, \varphi_3] = L[u_1, u_2, u_3]$$

$\varphi_1, \varphi_2, \varphi_3$ is an o.n.b. for $L[S]$

Consequence

$\mathbb{R}^k$

S o.n. set

$\|$

$\varphi_1, \dots, \varphi_r$

Either: S is a basis ($r=k$) & $\therefore$ o.n. basis for $\mathbb{R}^k$

OR: It can be extended to a basis for $\mathbb{R}^k$

$\mathcal{B} = \varphi_1, \varphi_2, \dots, \varphi_r, \psi_1, \psi_2, \dots, \psi_{k-r}$

where $\underbrace{\psi_1, \dots, \psi_{k-r}}_{l-i} \in \mathbb{R}^k \setminus L[S]$

$\Big\downarrow$ Apply GS to $\mathcal{B}$

$\mathcal{G}_{\mathcal{B}} : \varphi_1, \varphi_2, \dots, \varphi_r, \underline{\varphi_{r+1}, \dots, \varphi_k}$

s.t. $L[\mathcal{G}_{\mathcal{B}}] = L[\mathcal{B}] = \mathbb{R}^k$

$\therefore$ $\mathcal{G}_{\mathcal{B}}$ is an o.n basis for $\mathbb{R}$

— it is an extension of S

Conclusion:

Any o.n. set in $\mathbb{R}^k$ is either a basis (o.n. basis) or can be extended to an o.n. basis

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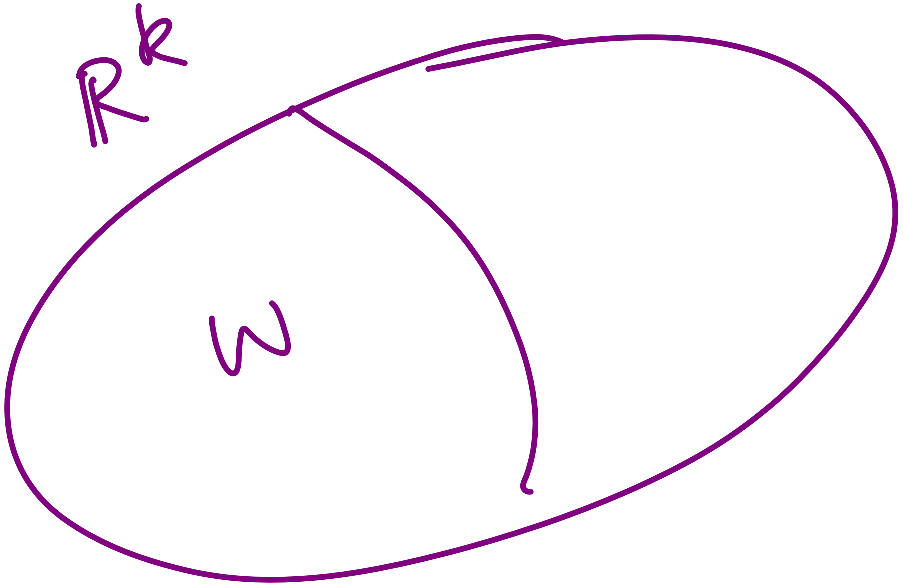
ORTHOGONAL COMPLEMENT

$\mathbb{R}^k$

Let W be a subspace of $\mathbb{R}^k$



Let $W^\perp$ denote the collection of all vectors in R^k which are orthogonal to all the vectors in W

$$W^\perp = \{x \in R^k : (x, w) = 0 \ \forall w \in W\}$$

— Orthogonal complement of W

Is $W^\perp$ a subspace of R^k?

We observe the following:

i) θ_k is orth. to all vect in R^k

$\therefore \theta_k$ is orth to all vect in W

$\therefore \ \theta_R \in W^{\perp}$
($\because$ $W^{\perp}$ is nonempty)

ii) $x, y \in W^{\perp} \Rightarrow (x, w) = 0$, and $(y, w) = 0$
$\forall w \in W$

$\Rightarrow (x, w) + (y, w) = 0$

$\Rightarrow (x+y, w) = 0$

$\Rightarrow x+y \in W^{\perp}$

$W^{\perp}$ is closed under addition

iii) $x \in W^{\perp}, \alpha \in \mathbb{R} \Rightarrow (x, w) = 0 \quad \forall w \in W$
& $\alpha \in \mathbb{R}$

$\Rightarrow \alpha(x, w) = 0 \quad \forall w \in W$

$\Rightarrow (\alpha x, w) = \alpha(x, w) = 0 \quad \forall w \in W$

$\Rightarrow \alpha x \in W^{\perp}$

$W^\perp$ is closed under scalar mult

(i), (ii), (iii) $\Longrightarrow$

$W^\perp$ is a nonempty subset of $\mathbb{R}^k$

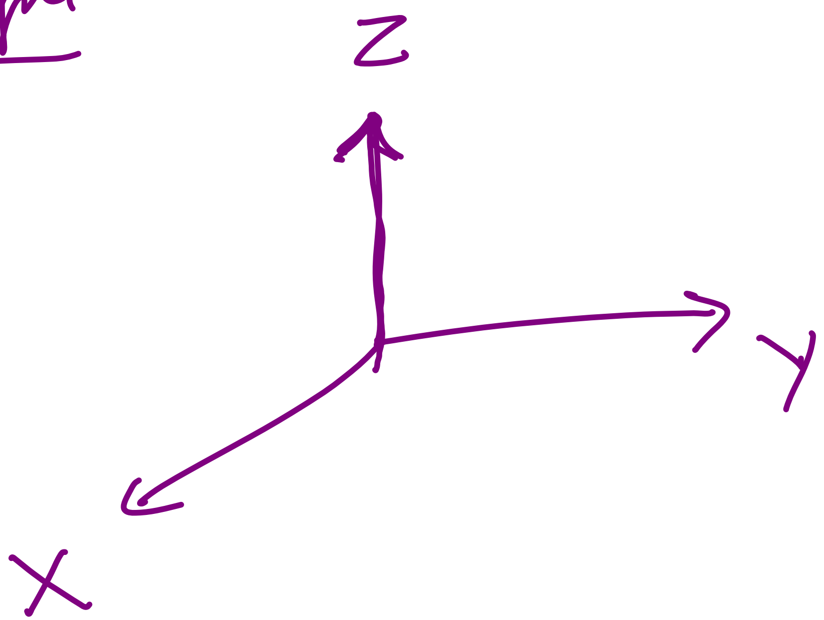
which is closed under addition

and scalar multiplication

$\therefore$ $W^\perp$ is also a subspace of $\mathbb{R}^k$

Example

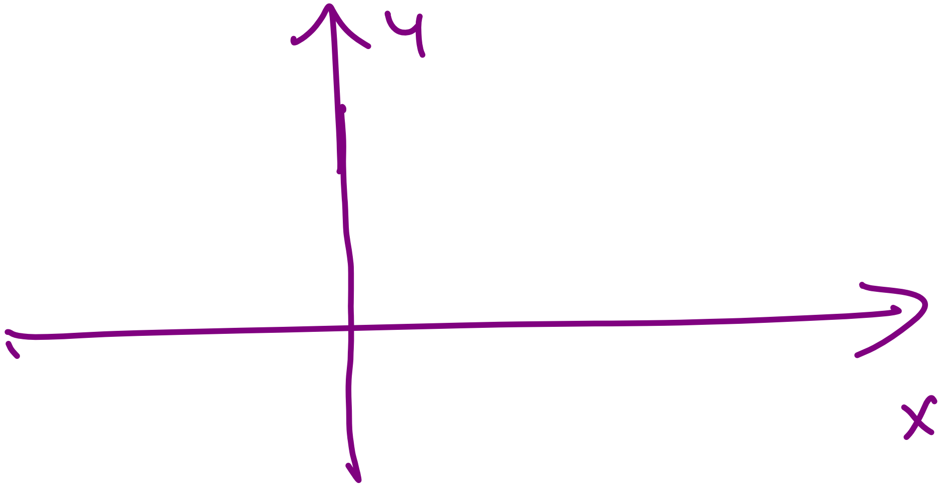
$\mathbb{R}^3$



W : XY pla

$W^\perp$: Z Axis

$\mathbb{R}^2$



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