Inner Product

$x, y \in \mathbb{R}^k$

$$(x,y) = \underbrace{\sum_{j=1}^{k} x_j y_j} = y^T x \ (\text{or } x^T y)$$

Orthogonality

$x, y \in \mathbb{R}^3$

$x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3$

$x \perp y$ iff $x \cdot y = 0$

Generalize:

If $x, y \in \mathbb{R}^k$ we say that x is ORTHOGONAL to y if $(x, y) = 0$

$$\left(\text{i.e. } y^T x = 0,\ x^T y = 0 \quad \sum_{j=1}^{k} x_j y_j = 0 \right)$$

x is orthogonal to y

iff y is orthogonal to x

i.e. x and y are orthogonal to each other

ORTHONORMAL SETS

is the generalization of the

$\vec{i}$, $\vec{j}$, $\vec{k}$ vectors in $\mathbb{R}^3$

<u>We generalize this</u>

If $S: u_1, u_2, \ldots, u_r$ is a set of vectors in $\mathbb{R}^k$ we say S is an <u>orthonormal</u> set in $\mathbb{R}^k$ if

$$(u_i, u_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\longrightarrow$ (orthogonality)

$\longrightarrow$ (normality)

$$u_j^T u_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Examples

(1) $\mathbb{R}^3$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

is an orthonormal set in $\mathbb{R}^k$

(2) S. $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

is an orthogonal set in $\mathbb{R}^3$ but not an orthonormal set since

$$\langle v_1, v_1 \rangle = 2 \neq 1$$
$$\langle v_2, v_2 \rangle = 2 \neq 1$$

If we take

$$w_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

is an orthonormal set in $\mathbb{R}^3$

3) $\mathbb{R}^4$

$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

$(u_1, u_2) = 1 + (-1) + 1 + (-1) = 0$

$(u_1, u_3) = 0$ $(= u_2, u_3)$

$\therefore$ This set is an orthogonal set

But $(u_1, u_1) = 1^2 + 1^2 + 1^2 + 1^2 = 4 \neq 1$

$(u_1, u_1) = 1 + 1 + 1 + 1 = 4 \neq 1$

$(u_2, u_2) = 1 + 1 + 1 + 1 = 4 \neq 1$

$(u_3, u_3) = 1 + 0 + 1 + 0 = 2 \neq 1$

Hence this is <u>not</u> an orthonormal set

$$S_1: \; v_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \; v_2 = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}; \; v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$$

is an orthonormal set in $\mathbb{R}^4$

A very Important Property of o.n. sets

We have

$$S: u_1, u_2, \ldots, u_r$$

an o.n. set in $\mathbb{R}^k$.

Is S l.i.?

$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r = \theta_k$$

$$\Rightarrow (\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r, \underline{x}) = (\theta_k, x) = 0 \quad \forall x \in \mathbb{R}^k$$

In particular if we let $x = u_1$

$$(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r, u_1) = 0$$

$$\Rightarrow \alpha_1 (u_1, u_1) + \alpha_2 \underbrace{(u_2, u_1)}_{=0} + \cdots + \alpha_r \underbrace{(u_r, u_1)}_{=0} = 0$$

$$\Rightarrow \alpha_1 = 0$$

$|||^{ly}$ if we successively let $x = u_2, u_3, \ldots, u_r$

we get $\alpha_2 = \alpha_3 = \cdots = \alpha_r = 0$

Then $\alpha_1 u_1 + \cdots + \alpha_r u_r = \theta_k$

$$\Longrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$$

$$\Longrightarrow S: u_1, u_2, \ldots, u_r \ \text{l.i}$$

Conclusion

Every orthonormal set in $\mathbb{R}^k$ is l.i

Orthonormal Basis for $\mathbb{R}^k$

If a set S of vectors in $\mathbb{R}^k$ s.t it is

i) orthonormal, and

ii) $L[S] = \mathbb{R}^k$

is called an o.n. basis

Remark: If W is a subspace of $\mathbb{R}^k$ then a subset S of W is called an o.n. basis for W if

i) S is orthonormal, and

ii) $L[S] = W$

Examples

1) $\mathbb{R}^3$ Clearly

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is an orthonormal basis for $\mathbb{R}^3$

2) Similarly for $\mathbb{R}^k$,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \ldots, e_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

is an o.n. basis

3) $\mathbb{R}^3$ length $\sqrt{2}$ $u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

is l.i. They are 3 of them

& $\dim \mathbb{R}^3 = 3$

$\therefore$ These form a basis for $\mathbb{R}^3$

" an orthogonal basis for $\mathbb{R}^3$

Do NOT " an orthonormal basis for $\mathbb{R}^3$

because these vectors do not have length 1

But $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

is an o.n. basis for $\mathbb{R}^3$

4) $\mathbb{R}^3$ Consider the subspace

$$W = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

Clearly $\underline{\underline{u_1}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\underline{\underline{u_2}} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

is a basis for W

This is not an orthogonal basis for W

$\because (u_1, u_2) = 0 + 0 + 1 = 1 \neq 0$

$\therefore$ Not an orthonormal basis

If we take

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

i) $v_1, v_2 \in W$

ii) $(v_1, v_2) = 0$

iii) $(v_1, v_1) = 1$, $(v_2, v_2) = 1$

$\Rightarrow$ v_1, v_2 an o.n. set in W

$\therefore$ l.i. set in W

$\Rightarrow$ (Since $\dim W = 2$) v_1, v_2 is an o.n. basis for W

<u>Expansion in terms of o.n. basis</u>

Let us consider $\mathbb{R}^k$

$\mathcal{B} \cdot \varphi_1, \varphi_2, \ldots, \varphi_k$ o.n. basis for $\mathbb{R}^k$

$x \in \mathbb{R}^k \implies \exists\, \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ s.t $x = \alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \cdots + \alpha_k \varphi_k$

$$\Rightarrow (x, \varphi_1) = (\alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_k\varphi_k, \varphi_1)$$

$$\Rightarrow (x, \varphi_1) = \alpha_1$$

& similarly $(x, \varphi_j) = \alpha_j$ for $j = 1, 2, \ldots, k$

and hence

Every $x \in \mathbb{R}^k$ can be expanded in terms of the o.n.b. as

$$x = \underline{(x, \varphi_1)}\, \underline{\underline{\varphi_1}} + \underline{(x, \varphi_2)}\, \underline{\varphi_2} + \cdots + (x, \varphi_k)\varphi_k$$

(Fourier Expansion of x w.r.t the o.n. basis $\mathcal{B} = \varphi_1, \ldots, \varphi_k$)

(x, φ_j) is called the j^{th} Fourier coeff of

x wrt $\mathcal{B}$.

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}$ $\quad S: e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \ldots \quad e_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Fourier exp. of $x\,y$ wrt S

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_k e_k \qquad x_j = (x, e_j)$$

$$y = y_1 e_1 + y_2 e_2 + \cdots + y_k e_k \qquad y_j = (y, e_j)$$

$$(x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_k y_k$$

Fourier Exp in Terms of o.n.b

$$\mathcal{B} = \varphi_1, \ldots, \varphi_k$$

$$x = \underline{(x,\varphi_1)\varphi_1 + (x,\varphi_2)\varphi_2 + \cdots + (x,\varphi_k)\varphi_k}$$

$$y = \underline{(y,\varphi_1)\varphi_1 + (y,\varphi_2)\varphi_2 + \cdots + (y,\varphi_k)\varphi_k}$$

$$\Rightarrow (x,y) = \underline{(x,\varphi_1)(y,\varphi_1)} + \underline{(x,\varphi_2)(y,\varphi_2)} + \cdots + \underline{(x,\varphi_k)(y,\varphi_k)}$$

$$(x,y) = \sum_{j=1}^{k} (x,\varphi_j)(y,\varphi_j) \quad \forall\, x,y \in \mathbb{R}^k$$

(Plancheral's formula)

$$\underline{\underline{\|x\|^2}} = (x,x) = \sum_{j=1}^{k} \underline{\underline{(x,\varphi_j)^2}} \quad \forall\, x \in \mathbb{R}^k.$$

<u>Parseval's identity</u>

Note: $x = \theta_k \implies (x, \varphi_j) = (\theta_k, \varphi_j) = 0$

$\implies \|x\|^2 = 0$

$$x = \theta_k \iff (x, \varphi_j) = 0 \quad \forall j = 1, 2, \dots, k$$

Remark: Let W be a subspace of $\mathbb{R}^k$

$\dim W = d$

$B_W : \varphi_1, \varphi_2, \dots, \varphi_d$ o.n. basis for W

Then

1) $$x = \sum_{j=1}^{d} (x, \varphi_j)\varphi_j \quad \forall x \in W$$

2) $$(x, y) = \sum_{j=1}^{d} (x, \varphi_j)(y, \varphi_j) \quad \forall x, y \in W$$

3) $$\|x\|^2 = \sum_{j=1}^{d} (x, \varphi_j)^2 \quad \forall x \in W$$

Question:- $\mathbb{R}^k$

S l.i. set Either → S is basis for $\mathbb{R}^k$ (if S has k vectors)

OR → Can extend S to a basis for $\mathbb{R}^k$

S o.n. set $\Rightarrow$ S l.i Either → S is basis (& $\therefore$ o.n. basis

OR → Extend it to a basis

Question Can we extend to an o.n. basis?

GRAM-Schmidt orthonormalization