

Easy to solve the HS

$$Bx = 0_m$$

if B is in RRE form

- Eliminating Pivotal Variables in terms of Non Pivotal Variables and choosing Non Pivotal Variables arbitrarily over F

Can we reduce a given $F^{m \times n}$ matrix A to a Row equivalent matrix B which is in RRE form.

Then solve $Bx = 0_m$ instead of $Ax = 0_m$

Reduction Process:

Step 1 FCO: $K \in F^{p \times q}$

At the end of the first FCO operation:

2) Pivotal entries 1 are
in "right" place

3) All entries below Pivotal
entries are 0

Stage 2 Clean Up operations

Make all entries above the pivotal
entries 1 as 0 by using ERO
of Type 2 — End up with RRE
form of $A \rightarrow$ dense

$$A \xrightarrow{\text{EROs}} \tilde{A} \xrightarrow[\text{up}]{\text{Clean}} A_R$$

by A_R

$$A = \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ -1 & -2 & -1 & -5 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix}_{4 \times 5}$$

$$\begin{array}{l} \xrightarrow{R_2 - R_1} \\ \xrightarrow{R_3 + R_1} \\ \xrightarrow{R_4 - R_1} \end{array} \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & -2 & -8 & 3 \end{pmatrix}$$

$$\begin{array}{l} \xrightarrow{R_3 - R_2} \\ \xrightarrow{R_4 + 2R_2} \end{array} \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{pmatrix} \textcircled{1} & 2 & 0 & 1 & -1 \\ 0 & 0 & \textcircled{1} & 4 & -1 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} \xrightarrow{R_1 - R_3} \\ \xrightarrow{R_2 + R_3} \end{array} \begin{pmatrix} \textcircled{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 4 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = AR$$

The Row Reduced Echelon form of A

HS

$Ax = \theta_m$ has same set of sol as $ARx = \theta_m$

x_1, x_3, x_5 PIVOTAL VAR
 x_2, x_4 Non pivotal Variables

HS $ARx = \theta_m$

$$\begin{aligned} \textcircled{x_1} + 2x_2 + x_4 &= 0 \\ \textcircled{x_3} + 4x_4 &= 0 \\ \textcircled{x_5} &= 0 \end{aligned}$$

$$x_1 = -2x_2 - x_4$$

$$x_3 = -4x_4$$

$$x_5 = 0$$

The Non Pivotal Variables
 x_2 & x_4 can be chosen
arbitrarily

General sol

Any sol is of the form

$$x = \begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ 0 \end{pmatrix}$$

$$x_2 = \alpha, \quad x_4 = \beta$$

$$x = \begin{pmatrix} -2\alpha - \beta \\ \alpha \\ -4\beta \\ \beta \\ 0 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{F}$$

$$= \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}; \alpha, \beta \in \mathbb{F}$$

By Varying α, β over \mathbb{F} we get

all the solutions of the HS

$$Ax = \mathbf{0}_m$$

And hence all the solutions of

$$Ax = \mathbf{0}_m$$

General Strategy for HS

$A \in \mathbb{F}^{m \times n}$ given

$Ax = \mathbf{0}_m$ HS

$$A \xrightarrow{\text{FCOs}} \tilde{A} \xrightarrow[\text{operations}]{\text{Cleanup}} A_R$$

Solve $A_R x = \theta_m$ by eliminating
Pivotal variable in
terms of Non Pivotal Var.
And choosing Non Pivotal
Variables arbitrarily
over F

These will also represent all
sol of $Ax = \theta_m$

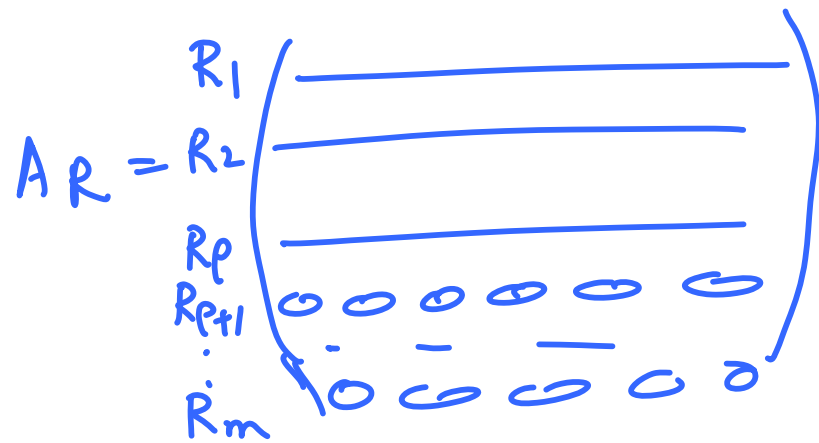
Remarks.

$$A \in F^{m \times n}$$

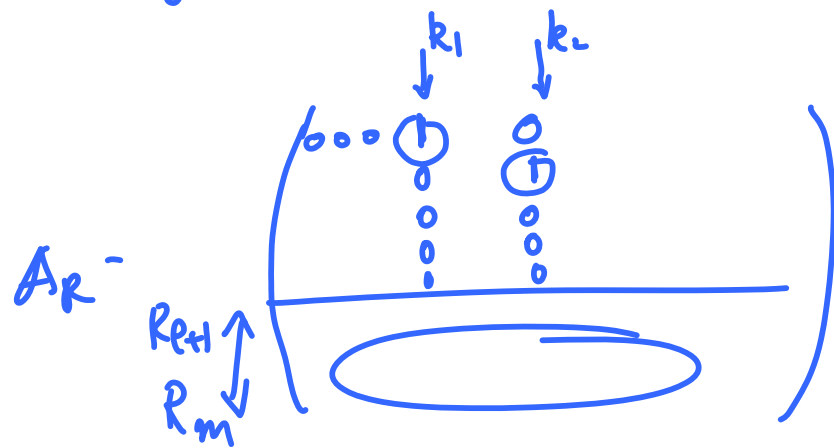
$$A \xrightarrow{\text{FCOs, Cleanup}} A_R \quad \text{Row Reduced
Echelon Form
of } A$$

Let A_R have p nonzero rows

\therefore It has $m-p$ zero rows



Let the first nonzero entry (Pivotal entry) of R_i appear in k_i^{th} column



Pivotal Variables

$x_{k_1}, x_{k_2}, \dots, x_{k_p}$ (Eliminated)

Nonpivotal Var.

$x_j, j \neq k_1, k_2, \dots, k_p$ (arbitrarily)

Number of Pivotal Variables
 $= p =$ No. of Nonzero rows
in AR

— Called ROW RANK of the
matrix A

Definition

Row Rank of A is the

Number of Nonzero
rows in the RRE form AR
of A .

Ex: In our Ex

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = A_R$$

Row Rank of $A = 3$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Row Rank $A = 0$

Row Rank $A = \text{No. of Nonzero rows}$

$= \text{No. of Pivotal Variables}$

Total No. of Variables = n

No. of Pivotal Variables = p

\Rightarrow No. of Nonpivotal Variables = $n - p$
Called Nullity of the Matrix A

Row Rank of A + Nullity of A
= Number of columns in A

Summarize

We know' how to solve
Homogeneous Systems

Handling NHS

It remains to find
a Particular sol. x_p of
the NHS.

What was the strategy for HS
used
EROs

$$A \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) R_{ii} \xrightarrow{\text{ERO}} a_{i1}x_1 + \dots + a_{in}x_n = 0$$

EROs on matrix A

"Same as"

similar operation on
the lhs of Corr. HS

None of these ERO's affect the RHS because they are all 0 and they remain to be 0 after the ERO's

$\Rightarrow Ax = 0_m \quad A_R x = 0_m$
will have same sol

This will not work in \mathbb{NHS}
 $A \xrightarrow{\text{ERO}} A_R \quad b \in \mathbb{F}^m$

$Ax = b$ (need not be same sol)
 $A_R x = b$

Ex: $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$A \xrightarrow{R_{12}} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} = A_1$$

Look at NHS

$$Ax = b$$

$$A_1 x = b$$

$$2x_1 + 3x_2 - x_3 = 2$$

$$x_1 - x_2 + x_3 = 1$$

$$x_1 - x_2 + x_3 = 2$$

$$2x_1 + 3x_2 - x_3 = 1$$