Easy to solve the HS

$$Bx = \theta_m$$

if B is in RRE form

- Eliminating Pivotal Variables in terms of Non Pivotal Variables and choosing Non Pivotal Variables arbitrarily over $\mathbb{F}$

Can we reduce a given $\mathbb{F}^{m\times n}$ matrix A to a Row equivalent matrix B which is in RRE form.

Then solve $Bx = \theta_m$ instead of $Ax = \theta_m$

<u>Reduction Process:</u>

<u>Step 1</u> <u>FCO</u>: $K \in \mathbb{F}^{p\times q}$

At the end of the first FCO operation:

2) Pivotal entries 1 are in "right" place

3) All entries below Pivotal entries are 0

Stage 2 Clean Up operations

Make all entries above the pivotal entries 1 as 0 by using ERO of Type 2 — End up with RRE form of $A \to$ denoted by A_R

$$A \xrightarrow{\text{ECOs}} \tilde{A} \xrightarrow[\text{up}]{\text{clean}} A_R$$

we get either $\left(\begin{array}{c|c} \bullet & \\ \hline 0 & \\ \vdots & K^{(1)} \\ 0 & \end{array}\right)$ OR $\left(\begin{array}{c|c} 0 & \\ 0 & \\ 0 & K^{(1)} \\ 2 & \\ 0 & \end{array}\right)$

We get $K^{(1)}$ having $n-1$ columns

General Reduction:

Stage 1 $A \in \mathbb{F}^{m \times n}$

$A \xrightarrow{FCO} \left(\begin{array}{c|c} 0 & A^{(1)} \end{array}\right) \xrightarrow[\text{on } A^{(1)}]{ECO}$ Submatrix $A^{(2)}$

or $\left(\begin{array}{c|c} 1 & \\ \hline & A^{(1)} \end{array}\right)$

$\xrightarrow{n \text{ steps}} \left(\quad \right)$ Its Pivotal 1 s in the way we want

1) All non zero rows will be above all zero rows

2)

$$A = \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 1 & 2 & 3 & 13 & -2 \\ -1 & -2 & -1 & -5 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{pmatrix}_{4\times 5}$$

$$\xrightarrow[\substack{R_3+R_1 \\ R_4-R_1}]{R_2-R_1} \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & -2 & -8 & 3 \end{pmatrix}$$

$$\xrightarrow[R_4+2R_2]{R_3-R_2} \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_4-R_3} \begin{pmatrix} 1 & 2 & 2 & 9 & -1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_2 + R_3]{R_1 - R_3} \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = A_R$$

The Row Reduced Echelon form of A

HS

$Ax = \theta_m$ has same set of sol as $A_R x = \theta_m$

x_1, x_3, x_5 PIVOTAL VAR

x_2, x_4 Non pivotal Variables

HS $A_R x = \theta_m$

$$x_1 + 2x_2 + x_4 = 0$$
$$x_3 + 4x_4 = 0$$
$$x_5 = 0$$

$x_1 = -2x_2 - x_4$

$x_3 = -4x_4$

$x_5 = 0$

The Non Pivotal Variables x_2 & x_4 can be chosen arbitrarily

General sol

Any sol is of the form

$$x = \begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ 0 \end{pmatrix}$$

$x_2 = \alpha, \quad x_4 = \beta$

$$x = \begin{pmatrix} -2\alpha - \beta \\ \alpha \\ -4\beta \\ \beta \\ 0 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{F}$$

$$= \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} ; \alpha, \beta \in \mathbb{F}$$

By varying α, β over $\mathbb{F}$ we get all the solutions of the HS

$$A_R x = \theta_m$$

And hence all the solutions of

$$A x = \theta_m$$

General Strategy for HS

$A \in \mathbb{F}^{m \times n}$ given

$A x = \theta_m$ HS

$$A \xrightarrow{\text{FCOs}} \tilde{A} \xrightarrow[\text{operations}]{\text{Cleanup}} A_R$$

Solve $A_R x = \theta_m$ by eliminating Pivotal variable in terms of Non Pivotal Var. And choosing Non Pivotal Variables arbitrarily over $\mathbb{F}$

These will also represent all sol of $Ax = \theta_m$

Remarks:

$A \in \mathbb{F}^{m \times n}$

$A \xrightarrow{\text{FCOs, Cleanup}} A_R$ Row Reduced Echelon Form of A

Let A_R have ρ nonzero rows

$\therefore$ It has $m-\rho$ zero rows

$$A_R = \begin{array}{c} R_1 \\ R_2 \\ \\ R_\rho \\ R_{\rho+1} \\ \vdots \\ R_m \end{array} \begin{pmatrix} \text{------------------} \\ \text{------------------} \\ \\ \text{------------------} \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ - \quad - \quad - \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \end{pmatrix}$$

Let the first nonzero entry (Pivotal entry) of R_i appear in k_i^{th} column

$$A_R = \begin{array}{c} \\ \\ \\ \\ \\ \\ R_{\rho+1} \\ \updownarrow \\ R_m \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{cccccc} & & & k_1 & k_2 & \\ & & & \downarrow & \downarrow & \\ 0 & 0 & 0 & 1 & 0 & \\ & & & 0 & 1 & \\ & & & 0 & 0 & \\ & & & 0 & 0 & \\ & & & 0 & 0 & \\ \hline & & & & & \\ & & & & & \end{array}$$

Pivotal Variables

$x_{k_1}, x_{k_2}, \ldots, x_{k_p}$ (Eliminated)

Nonpivotal Var.

x_j , $j \neq k_1, k_2, \ldots, k_p$ (arbitrarily)

Number of Pivotal Variables $= p =$ No. of Nonzero rows in A_R

– Called ROW RANK of the matrix A

Definition

Row Rank of A is the Number of Nonzero rows in the RRE form A_R of A.

Ex: In our Ex

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = A_R$$

Row Rank of $A = 3$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Row Rank $A = 0$

Row Rank A = No. of Nonzero rows

= No. of Pivotal Variables

Total No. of Variables $= n$

No. of Pivotal Variables $= p$

$\Rightarrow$ No. of Nonpivotal Variables $= n - p$

Called Nullity of the Matrix A

$$\boxed{\begin{array}{l}\text{Row Rank of } A + \text{Nullity of } A \\ = \text{Number of columns in } A\end{array}}$$

Summarize

We know how to solve Homogeneous Systems

Handling NHS

It remains to find a Particular sol. x_p of the NHS.

What was the strategy for HS used

EROs

$$A \begin{pmatrix} \rule{3cm}{0.4pt} \end{pmatrix} R_i \xleftrightarrow{\text{ERO}} a_{i1}x_1 + \cdots + a_{in}x_n = 0$$

EROs on matrix A

"Same as"

similar operation on the lhs of corr. HS

None of these ERO's affect the RHS because they are all 0 and they remain to be 0 after the EROs

$\Rightarrow$ $Ax = 0_m$ $\quad A_R x = 0_m$

will have same sol

This Will Not Work in NHS

$A \xrightarrow{\text{ERO}} A_R$ $\quad b \in F^m$

$Ax = b$ (need not be same sol) $\quad A_R x = b$

<u>Ex:</u> $A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$A \xrightarrow{R_{12}} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} = A_1$$

Look at NHS

$$Ax = b \qquad\qquad A_1 x = b$$

$$\left.\begin{aligned} 2x_1 + 3x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 1 \end{aligned}\right\} \qquad \left.\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 - x_1 &= 1 \end{aligned}\right\}$$