$A \in \mathbb{F}^{m \times n}$ $b \in \mathbb{F}^{m \times 1}$

NHS $Ax = b$

HS $Ax = \theta_m$

HS plays imp role in analyses NHS

1) A consistent NHS has unique sol $\Longleftrightarrow$ HS has only trivial sol

2) Gen sol of a consistent NHS is of the form

$$x = x_p + x_H$$

x_p Particular sol of NHS

x_H sol of HS

Finding Sol of NHS has two parts

1) Find x_H sol HS

2) Find x_p a Particular sol of NHS

Sol of the HS

Main Tool : EROs

Three Types

1) R_{ij} Interchange of Rows

2) $R_j + \alpha R_i$, $\alpha \in \mathbb{F}$

3) αR_i ; $\alpha \in \mathbb{F}, \alpha \neq 0$

Properties of EROs

1) ERO's are invertible & of the same type inverse

2) ERO's do not alter the set of sol of the HS

3) ERO's can be effected by premultiplication of A by an elementary matrix

$$I_m \xrightarrow[\text{}]{\text{one ERO}} E \to \begin{array}{c} m \times m \\ \text{elementary} \\ \text{matrix} \end{array}$$

Row Equivalence

$A, B \in \mathbb{F}^{m \times n}$

There are EROs $O_1, O_2, \ldots, O_k$ (finite number)

s.t.

$$A \xrightarrow{O_1} A_1 \xrightarrow{O_2} A_2 \longrightarrow \cdots \xrightarrow{O_k} A_k = B$$

Then we say $A \overset{R}{\sim} B$

Properties

1) $A \overset{R}{\sim} A$ Reflexive

2) $A \overset{R}{\sim} B \iff B \overset{R}{\sim} A$ (Symmetry)

3) $A \overset{R}{\sim} B$, & $B \overset{R}{\sim} C$
$\implies A \overset{R}{\sim} C$ Transitivity

Row Equivalence is an equivalence relation on $\mathbb{F}^{m\times n}$

$A \overset{R}{\sim} B$

$\Longrightarrow$ $Ax = \theta_m$ HS corr to A

$Bx = \theta_m$ HS " " B

have the same set of sol.

<u>STRATEGY FOR HS</u>

Given $A \in \mathbb{F}^{m\times n}$

Find a $B \in \mathbb{F}^{m\times n}$ s.t

1) $A \overset{R}{\sim} B$, and

2) $Bx = \theta_m$ is EASY to solve

What are some "easy" B ?
How to reduce A by ERO's
to 'easy' B ?

EXAMPLE:

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 5} \in \mathbb{F}^{4\times 5}$$

Basic Features

(1) Has Rows in which all entries are 0 & such rows are called ZERO ROWS.

NONZERO ROW: is a row which has nonzero entries

R_1, R_2, R_3 Nonzero Rows

R_4 zero Row

The Zero Rows All Appear below all NonZero Rows

2) The first nonzero entry is each nonzero row is 1 This is called the PIVOTAL entry of that row

3) If R_i & R_j are nonzero rows and the pivotal entry of R_i appears in k_i^{th} column and pivotal entry of R_j row appears in k_j^{th} column, then

$i < j \implies k_i < k_j$

4) If a column contains a pivotal entry then all other entries in that column are zero

A matrix which has these properties is said to be in Row Reduced Echelon form (RRE)

$B \in \mathbb{F}^{m \times n}$ RRE form

1) $R_1, R_2, \ldots, R_p$ are the nonzero rows

$R_{p+1}, R_{p+2}, \ldots, R_m$ Zero Rows

2) The first (from the left) nonzero entry in each R_i is 1 for $1 \le i \le p$

Suppose R_i has its first nonzero entry in column k_i

$$\begin{array}{cc} & \begin{matrix} & & & k_i \end{matrix} \\ i \rightarrow & \begin{pmatrix} 0 & \cdots & 0 & 1 & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

$b_{ij} = 0, \; j = 1, 2, \ldots, k_i - 1$

$b_{ik_i} = 1$

If $i < j$

$$\Longrightarrow k_i < k_j$$

4)

$$\begin{array}{cc} & \downarrow k_i \\ i \longrightarrow & \begin{pmatrix} & 0 & \\ & 0 & \\ & \vdots & \\ & 1 & \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix} \end{array}$$

$$b_{j k_i} \begin{cases} = 0 & \text{if } j \neq k_i \\ = 1 & \text{if } j = k_i \end{cases}$$

It is easy to solve the HS

$$Bx = \theta_m$$

if B is in RRE form

Example:

$$B = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{F_{4\times 5}}$$

$$\begin{aligned} x_1 + 2x_2 \quad + x_4 \quad &= 0 \\ x_3 + 4x_4 \quad &= 0 \\ x_5 &= 0 \end{aligned}$$

The variable corresp. to the columns in which the pivotal 1 appear are called PIVOTAL VARIABLES

x_1, x_3, x_5 are the Pivotal Variables

x_2, x_4 NonPivotal Variable

No. of Pivotal Var. = No. of nonzero rows

" NonPivotal Var = n − No. of Nonzero rows

The i^{th} equation eliminates the k i^{th} Pivotal Var. in terms of nonpivotal variables.

$$x_1 = -2x_2 - x_4$$
$$x_3 = -4x_4$$
$$x_5 = 0$$

Any sol is of the form

$$x = \begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ 0 \end{pmatrix}$$

By varying x_2 & x_4 we get all solutions

Moral

If a matrix B is in RRE form

$R_1, \dots, R_\ell$ Nonzero rows

$R_{\ell+1}, \dots, R_m$ zero rows

Let the first nonzero entry (i.e. the Pivotal 1) in row R_i appear in k_i^{th} col. for $1 \le i \le \ell$

The i^{th} pivotal variable x_{k_i}

PIVOTAL VAR: $x_{k_1}, x_{k_2}, \dots, x_{k_\ell}$

NONPIVOTAL VAR: x_j, $j \neq k_1, \dots, k_\ell$

$x_{k_1}, x_{k_2}, \dots, x_{k_\ell}$ Pivotal Variable can be eliminated in terms of the Nonpivotal Variables

The Nonpivotal Variables

can be chosen arbitrarily in $\mathbb{F}$

The Question

Given $A \in \mathbb{F}^{m \times n}$

Can we find a matrix

i) $B \in \mathbb{F}^{m \times n}$,

ii) $A \overset{R}{\sim} B$, and

iii) B is in RRE form

Then the system (Homog)

$Bx = \theta_m$ has the

same set of sol as

$Ax = \theta_m$

and $Bx = \theta_m$ is easy to solve

because B is in RRE form

thereby getting the sol of

$$Ax = \theta_m$$

The answer is Yes

How to find such a B?

Obviously since we want

$$A \overset{R}{\sim} B$$

we should be able to find

a finite seq. of ERO's

say $E_1, E_2, \ldots, E_k$ s.t.

$A \xrightarrow{E_1} A_1 \xrightarrow{E_2} A_2 \rightarrow - \; -$

$A_{k-1} \xrightarrow{E_k} A_k = B$; which is in RREF

How do we do this?

Reduction Process

The first Basic Operation:

FIRST COLUMN OPERATION (FCO)

(Elementary Row operations Applied to the first column of any matrix)

FCO

$K \in \mathbb{F}^{p \times q}$

Does the first Column have a Nonzero Entry?

YES

↓

Bring a Nonzero entry to the leading position (if necessary using ERO of Type 1)

↓

If necessary use ERO of type 3 to make this top nonzero entry as 1

↓

Make all entries below this 1 as zero by ERO

NO

↓

Do nothing

↓

$$\left(\begin{array}{c|c} 0 & \\ 0 & K^{(1)} \\ 0 & \\ \vdots & \\ 0 & \end{array}\right)$$

type 3

$$\left(\begin{array}{c|cccc} 1 & \times & \times & \times & r \\ \hline 0 & & & & \\ 0 & & \underline{\underline{K^{(1)}}} & & \\ \vdots & & & & \\ 0 & & & & \end{array}\right)$$