# Linear Systems

$A$ $m \times n$ $b$ $m \times 1$

To find $x$ $n \times 1$ s.t

$$Ax = b \quad \text{———} \quad (\mathrm{I})$$

Question

What is criterion $b$ should satisfy so that (I) has a solution?

Given b we ask

Does b satisfy [C] ?

- Yes
  - Solution Exists
    - How Many
      - Unique
        - When?
          - What is THE Unique Sol
      - Infinite
        - When?
          - Find All Solution
            - CRITERION TO CHOOSE Unique Repres.
- No
  - No sol exists
    - Least Square Sol Exists
      - How Many
        - Unique
          - When
        - Inf. many
          - When
            - Find All

What is THE Unique Rep. Sol?

Find THE Unique Least Sq. Sol

CRITERION To Choose Unique Rep.

FIND THE Rep

# DIAGONALIZATION

Given $n \times n$ matrix $A$

find invertible matrix $P$ $(n \times n)$

Such that

$P^{-1}AP$ is a diagonal matrix

What is the criterion A should satisfy so that such a P exists.

[C]

Given A we ask

Does A satisfy [C]?

**Yes**

↓

Find the P s.t $P^{-1}AP$ is diagonal

**No**

↓

No such P exists

- Find a P such that $P^{-1}AP$ is nearly diag.
- Find Q & P s.t $Q^{-1}AP$ is diag

(Jordan Canonical Form)

(Singular Value Decomp

Allows for generalization to Rectangular Matrices also.

## Third Problem

$x \quad n \times 1 \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ Real

$y \quad n \times 1 \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

# INNER PRODUCT

$$(x, y) \stackrel{\text{def}}{=} y^T x$$

$$= \sum_{i=1}^{n} x_i y_i$$

— real number

Inner Product is a Number

**Example:**

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad y = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$(x, y) = -1 + 0 + 2 = 1$$

# TENSOR PRODUCT

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \; ; \; y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \text{ Real}$$

Tensor Product of $x$ with $y$ as

$$\underset{}{x \otimes y} \overset{\text{def}}{=\!=} \underset{n\times 1}{x}\,\underset{1\times n}{y^T} \rightarrow \begin{matrix} n\times n \\ \text{matrix} \end{matrix}$$

## Generalization

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} , \; y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} , \text{ Real}$$

$$x \otimes y \overset{\text{def}}{=\!=} \underset{m\times 1}{x}\ \underset{1\times n}{y^T} \rightarrow \begin{array}{c} m\times n \\ \text{matrix} \end{array}$$

$$= (x_i y_j)_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

<u>Conclusion</u>

TENSOR PRODUCT OF an $m\times 1$ matrix $x$ with an $n\times 1$ matrix $y$ is an $m\times n$ matrix

<u>Note:</u> $x \otimes y \neq y \otimes x$

Example:

$$x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad y = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$x \otimes y = x y^T = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ -2 & -3 \\ 4 & 6 \end{pmatrix}_{3 \times 2 \text{ matrix}}$$

Notice that all the rows of $x \otimes y$ are "multiples" of $y^T$

$x \otimes y$ has a simple structure

Taking Tensor Product $x \otimes y$ ($x$ $m \times 1$, $y$ $n \times 1$) is a simple way of generating an $m \times n$ matrix

__Generalization__

Consider a positive integer $k$

Look at $k$ $m \times 1$ matrices $u_1, \dots, u_k$

& $k$ $n \times 1$ matrices $v_1, v_2, \dots, v_k$

For each $i$, $1 \le i \le k$, look at $u_i \otimes v_i$

This will be an $m \times n$ matrix

Let us add all these

$$\sum_{i=1}^{k} u_i \otimes v_i$$

This is an $m \times n$ matrix

Can generate lots of $m \times n$ matrices by varying $k$, $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$

Question: Does this construction exhaust all $m \times n$ matrices?

This means: Given any $m \times n$ matrix $A$

Can we find
1) a positive integer $k$,
2) $u_1, \ldots, u_k$ $m \times 1$ matrices
3) $v_1, \ldots, v_k$ $n \times 1$ "

such that

$$A = \sum_{i=1}^{n} u_i \otimes v_i \quad ?$$

We will see that the

Answer : YES

Leads to the following question:

What is an efficient way of doing this? that is

1) What is the minimum value of $k$ required? – Say min is $p$

2) Can we choose $u_1, \dots, u_p$ $m \times 1$

$v_1, \dots, v_p$ $n \times 1$

s.t the matrix

$$A = \sum_{i=1}^{k} u_i \otimes v_i$$

is easy to analyse

– Can solve $Ax = b$ "easily"

$\downarrow$

Leads to Singular Value Decomposition

## Fourth Problem

$Ax = b$

Does $b$ satisfy [C] ?

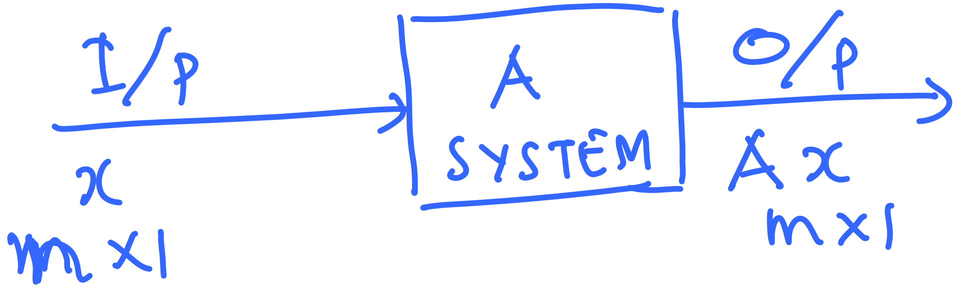
- Yes
  - THE Unique Sol
  - THE Unique Rep. Sol
- No
  Least Sq Sol
  - The Unique least Sol
  - The Unique Rep Least Sq Sol

## Situation

Given $b$

To find "The" Sol

$$x_0(b)$$

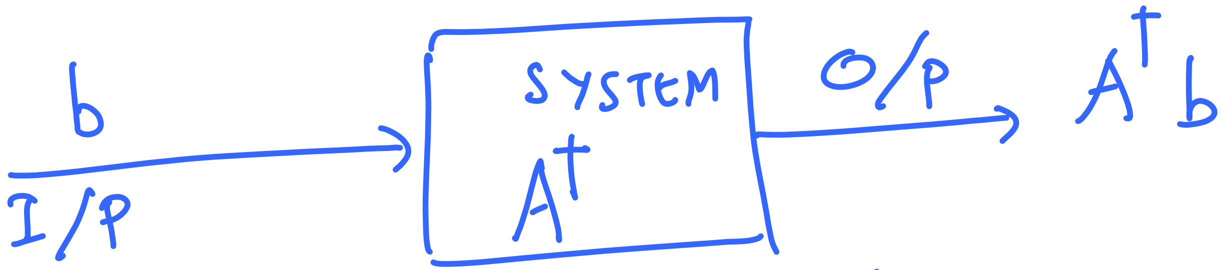



A : Transfer function

Solve $Ax = b$

Looking for the input $x$ which will produce the output $b$

<u>Answer</u>: $x_0(b)$




Want to construct $A^\dagger$ such that

$$\underset{n\times 1}{x_0(b)} = \underset{n\times m}{A^{+}}\ \underset{m\times 1}{b}$$

## MAIN PROBLEM

Given an $m \times n$ matrix $A$

Construct an $n \times m$ matrix $A^{+}$

s.t $x_0(b) = A^{+} b$

$A^{+}$ : $\longrightarrow$ $\left\{ \begin{array}{l} \text{(1) Inverse of } A \text{ whenever} \\ A \text{ is a square matrix} \\ \text{and has an inverse} \\ \text{(2) Pseudo inverse} \end{array} \right.$

## FOUR PROBLEMS

1) $Ax = b$

2) DIAGONALIZATION
   - NEAR DIAGONALIZATION

3) DECOMPOSITION OF A MATRIX AS SUM OF TENSOR PRODUCTS

4) PSEUDO INVERSE

Major Role — Decisive Role in 2), 3) & 4) will be played by the notion of eigenvalues and eigenvectors

## Problem (3)

Decomposing a matrix as sum of Tensor Products

Two Important

1) A square real symmetric matrix (Hermitian complex)
2) A square Normal Matrix

— SPECTRAL DECOMPOSITION

GOAL FOR THE COARSE

1) Develop the Mathematical Framework for analysing these problems,

2) Find the Answers to the various questions raised in the above four problems

3) Look for generalizations and abstractions of these ideas

# LINEAR SYSTEMS

**NOTATIONS** : $\mathbb{F}$ : (denote $\mathbb{R}$ or $\mathbb{C}$)

$$\mathbb{F}^k = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}_{k\times 1} : x_i \in \mathbb{F} \right\}$$

$$\theta_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{k\times 1}$$

$$\mathbb{F}^{m\times n} = \left\{ A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}} : a_{ij} \in \mathbb{F} \right\}$$

$0_{m \times n}$ : zero $m \times n$ matrix

$\mathbb{F}^{n^2}$ : $\{ A = (a_{ij})_{n \times n} : a_{ij} \in \mathbb{F} \}$

$I_n$ : $n \times n$ identity matrix

$O_n$ : $n \times n$ zero matrix