EXERCISE(*Complex Variable*)

1. Show that the following limits does not exit.

(a) $\lim\limits_{z\to 0}\dfrac{Re\, z^2}{|z|^2}$ (b) $\lim\limits_{z\to 0}\left[\dfrac{1}{1-e^{1/y}}+ix^2\right]$

2. Examine the continuity of the function

$$f(z)=\begin{cases}\dfrac{\text{Img } z^3}{|z|^2} & , \ z\neq 0\\ 0 & , \ z=0\end{cases}$$

3. Show that for the function

$$f(z)=\begin{cases}\dfrac{x^3-3xy^2}{x^2+y^2}+i\left(\dfrac{y^3-3x^2y}{x^2+y^2}\right) & , \ (x,y)\neq 0\\ 0 & , \ (x,y)=0\end{cases}$$

Cauchy-Riemann Equations are satisfied at the origin.
Does $f'(0)$ exist ?

4. Find an analytic function $f(z)=u+iv$ such that Img $f'(z)=1+y$ and $f(1+i)=0$.

5. Find the values of the constants $a,b,c\,\&\,d$ such that the function $f(z)=(y^2+axy+bx^2)+i(cy^2+dxy+x^2)$ is analytic.

6. Let $f(z)=u+iv$ and $g(z)=v+iu$ be analytic functions for all z. Show that u and v are constant. If $f(0)=2i$ and $g(0)=5$, then find the value of $h(z)=\dfrac{f(z)}{g(z)}$.

7. If $f'(z)=-f(z)$ for all z, then find $f(z)$.

8. Show that if $f(z)=u+iv$ and $g(z)=A+iB$ are differentiable at a point z,

Then $|f'(z).g'(z)|^2=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$

9. If $f(z)=u+iv$ is an analytic function, show that $|f(z)|$ is not Harmonic.

10. If $f(z)=u+iv$ is an analytic function, show that $\log|f(z)|$ is a Harmonic.

11. Prove that $|f(z)|=|z|^4$ is differentiable but not analyticat $z=0$.

12. If C is the curve $y = x^4 + x^2 - x + 1$ joining points $(0,1)$ and $(1,2)$, then find the value of $\int_C (z^2 - iz)\,dz$.

13. Evaluate $\oint_C \frac{\sin z + \cos z}{\left(z - \frac{\pi}{2}\right)(z - \pi)}$ where C is the circle $|z| = 4$.

14. Evaluate $\oint_C \frac{\cos z}{(z - \pi i)\left(z - \frac{\pi i}{2}\right)}$ if C is the ellipse $|z-1| + |z+1| = 4$.

15. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{\left(z^2 + 4\right)^2}\,dz$ if $t > 0$ and C is the circle $|z| = 3$.

16. Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give region of convergence of each series.

(a) $(z-3)\cos\frac{1}{z+4}$, $z = -4$

(b) $\frac{1}{(z+3)^2 z^2}$, $z = -3$

17. Expand the function

$f(z) = \frac{1}{(z+2)(z+4)}$ in the Laurent series valid for

(a) $2 < |z| < 4$ (b) $|z| > 4$

(c) $0 < |z+2| < 2$ (d) $|z| < 2$

18. Determine the poles and Residues at the poles for $f(z) = \operatorname{sec}h\ z$.

19. Evaluate $\oint_C \frac{2\cos\frac{\pi}{2}z}{z(z-2)^2}\,dz$.

20. Evaluate $\int_0^\infty \frac{dx}{\left(x^2+1\right)(x^2+4)^2}$

Hint & Answers

Ex -1. (a) Use paths $x \to 0$, $y \to 0$ and $y \to 0, x \to 0$

(b) Use paths $x \to 0, y \to 0+$ and $x \to 0, y \to 0-$

2. Continuous. Use r, θ definitions

3. At (0,0) we have $\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 1$, but $\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ does not exist (e.g. choose path y=mx)

4. Ans. $f(z) = \frac{1}{2}(z^2 + 2iz + 2 - 4i)$

5. $a = -2, b = -1, c = -1, d = -2$ by showing partial derivatives are continuousand then calculate by assuming $C - R$ are satisfied

6. $h(z) = \frac{2i}{5}$

7. $f(z) = ke^{-z}$

Use $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ where $f(z) = u + iv$ then apply $C - R$ equations

8. Use $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

9. $w = |f(z)| = \sqrt{u^2 + v^2}$ does not satisfy Laplace equation.

10. Use Laplace equation for $w = \frac{1}{2}\log(u^2 + v^2)$

11. $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} \le \lim_{r \to \infty} 2r^3 = 0$ using $x = r\cos\theta, y = r\sin\theta$ But C.R. are not satisfied at (0,0) for $f(z) = |z|^4 = (x^2 + y^2)^2, z \neq 0$

12. Ans. $= \frac{1}{6} + \frac{5i}{2}$. As $f(z) = z^2 - iz$ is analytic, hence Integral is Independent of path. So choose path $c = c_1 \cup c_2$ when c_1: straight line joining (0, 1) to (1, 1) & c_2: straight line joining (1, 1) (1, 2)

$\int_{c_1} (z^2 - iz)dz = \frac{1}{3} + \frac{i}{2}$ while $\int_{c_2} (z^2 - iz)dz = 2i - \frac{1}{6}$

13. Ans. $-8i$

Singular points are $\frac{-\pi}{2}$ & π which lie inside |z|=4. Apply Cauchy Integral formula after breaking the

Integral into $\frac{2}{\pi}\left[\int_{|z|=4}\frac{\sin z+\cos z}{z-\pi}dz-\int\frac{\sin z+\cos z}{z-\frac{\pi}{2}}dz\right]$

14. Ans. $-4\cosh\frac{\pi}{2}$

Singular points $z=\pi i$ (lies outside) while $\frac{\pi}{2}i$ (lies inside). Break up the integral into two parts & apply Cauchy Theorem and Cauchy Integral Formula.

15. Ans. $\frac{1}{8}\left[-t\cos 2t+\sin 2t\right]$

Singularities are $z=\pm 2i$, both are of order 2 lying inside |z|=3.

$$\underset{z=+2i}{\text{Residue}}\frac{e^{zt}}{\left(z^2+4\right)^2}=\frac{e^{2it}}{4^4}\left[-16t-8i\right] \text{ \& } \underset{z=-2i}{\text{Residue}}\frac{e^{zt}}{\left(z^2+4\right)^2}=\frac{e^{-2it}}{4^4}\left[-16t+8i\right]$$

Apply Residue Theorem

16. Ans. (a) $z-3-\frac{1}{2(z+4)}+\frac{7}{2(z+4)^2}$............; $z=-4$ is essential singularity. Put $z+4=4$ & use the expansion of $\cos\frac{1}{u}$ around 0. The series converges for all values of $z\neq -4$

(b) $\frac{1}{9(z+3)^2}\left[1+\frac{2}{3}(z+4)+\frac{1}{3}(z+3)^2+......\right]$ series converges $0<|z+3|<3, z=-3$ is double pole

17. Ans. (a) $......-\frac{4}{z^2}+\frac{2}{z^3}-\frac{1}{z^2}+\frac{1}{2z}-\frac{1}{8}+\frac{z}{32}-\frac{z^2}{128}+......$

(b) $\frac{1}{z^2}-\frac{6}{z^3}+.....$

(c) $\frac{1}{2(z+2)}-\frac{1}{4}+\frac{1}{8}(z+2)+......$

(d) $\frac{1}{8}-\frac{3}{32}z+.......$

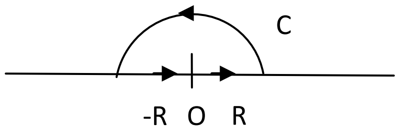
Hint use $\frac{1}{(z+2)(z+4)} = \frac{1}{2}\left(\frac{1}{z+2} - \frac{1}{z+4}\right)$. If |z|>2, $\frac{1}{(z+2)} = \frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}$ expand

Assuming $\frac{2}{|z|} < 1$ similarly others

18. Ans. $z = \frac{1}{2}(2n+1)\pi i$ for $n = 0, \pm 1, ...$ poles; $\underset{z=(2n+1)\frac{\pi}{2}}{\text{Residue}} = (-1)^{k+1} i$, k=0,1,....

19. Ans. $2\pi i$, Apply Residue theorem. Poles are z=0(order 1), z = 2 (order 2).

20. Ans . $\frac{5\pi}{288}$ .Use $\oint_c \frac{dz}{(z^2+1)(z^2+4)}$ where C:



Poles z = i of order 1& z = 2i of order 2 lies inside c

$$\oint_c \frac{dz}{(z^2+1)(z^2+4)} = 2\pi i\left(\frac{1}{18i} - \frac{11}{288i}\right) = \frac{5\pi}{288}.$$