Linear Algebra problems

1. Show that the set $F = (\{1, 0\}, +, .)$ is a field where + and . are defined as 1+1=0, 0+0=0, 0+1=1+0=1, 0.0=0.1=1.0=0, 1.1=1.

2. Let X be a non-empty set and F be any field. Let X_F be the set of all functions from X to F. Show that X_F is a vector space over F under the operations: $(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$.

3. Let X be any set and $P(X)$ be the power set of X. Prove that $P(X)$ is a vector space over $F = \{0, 1\}$ under the operation: $A + B = A\triangle B$ (symmetric difference) and $\alpha A = A$ if $\alpha = 1$ and is equal to ϕ if $\alpha = 0$.

4. Find which of the axioms will be violated if addition of vectors in problem 3 is changed to $A+B = A\bigcup B$.

5. In the following, find out whether S froms a subspace of V?

   (a) $V = R^3,\ S = \{(x_1, x_2, x_3) : x_1 + 5x_2 + 3x_3 = 0\}$

   (b) $V = R^3,\ S = \{(x_1, x_2, x_3) : x_1 + 5x_2 + 3x_3 = 1\}$

   (c) $V = R^3,\ S = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$

   (d) $V = P(R)$, the set of all polynomials over reals and $S = \{p(x) \in P(R) : P(5) = 0\}$

   (e) $V = R^n, S = \{(x_1, x_2, ..., x_n) : x_1 = x_2\}$

   (f) $V = R^3, S = \{(x_1, x_2, ..., x_n) : x_1^2 = x_2^2\}$.

6. Prove or disprove:

   (a) Union of two subspaces of V is a subspace of V.

   (b) Intersection of any number of subspaces is a subspace.

7. Check whether the following set of vectors are linearly dependent or independent.

   (a) $S = \{(1, 2, -2, -1), (2, 1, -1, 4), (-3, 0, 3, -2)\}$

   (b) $S = \{(1, 3, -2, 5, 4), (1, 4, 1, 3, 5), (1, 4, 2, 4, 3), (2, 7, -3, 6, 13)\}$

8. Determine whether or not the following form a basis for $\mathbb{R}^3$?

   (a) $\{(1, 1, 1), (1, -1, 5)\}$

(b) $\{(1,1,1),(1,2,3),(2,-1,1)\}$

(c) $\{(1,2,3),(1,0,-1),(3,-1,0),(2,1,-2)\}$

(d) $\{(1,1,2),(1,2,5),(5,3,4)\}$

9. Let W be a subspace of $\mathbb{R}^5$ generated by the vectors in 7(b). Find dimension and a basis for it.

10. Applying Gauss Jordan elimination method find inverse of the matrix
$$A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

11. Whether f is a linear transformation in each of the following? If yes then whether it is as isomorphism?

(a) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + x_2, x_1 x_2)$.

(b) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_2, x_1, 0)$.

(c) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1, x_3, x_1)$.

(d) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1, x_2, x_3) = (x_1 - 2, x_2 - 4, x_3)$.

12. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$. Find the matrix of T with respect to the basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ and $\{u_1', u_2'\}$ of $\mathbb{R}^2$ respectively, where $u_1 = (1, -1, 0)$, $u_2 = (2, 0, 1)$, $u_3 = (1, 2, 1)$, $u_1' = (-1, 0)$ and $u_2' = (0, 1)$.

13. Whether the system below is consistent? Justify.

$$\begin{aligned} x + 2y - 3z &= 1 \\ 3x - y + 2z &= 5 \\ 5x + 3y - 4z &= 2 \end{aligned}$$

14. Solve

$$\begin{aligned} x + 2y - 3z + 2w &= 2 \\ 2x + 5y - 8z + 6w &= 5 \\ 3x + 4y - 5z + 2w &= 4 \end{aligned}$$

15. For the system

$$\begin{aligned} x + 2y - z &= 0 \\ 2x + 5y + 2z &= 0 \\ x + 4y + 7z &= 0 \\ x + 3y + 3z &= 0 \end{aligned}$$

find the solution space as well as its dimension.

16. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$
For this find all eigenvalues and a basis for each eigenspace. Is A diagonalizable?

17. Find Jordan canonical form of $A = \begin{pmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{pmatrix}$.

18. Prove or disprove: $< x, y >= 2x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ is an inner product in $\mathbb{R}^2$ where $x = (x_1, y_1)$ and $y = (y_1, y_2)$.

19. Applying Gram-Schmidt Orthogonalization process construct an orthogonal set of vectors from the linearly independent set $\{x_1, x_2, x_3\}$, where $x_1 = (1, 1, 0)$, $x_2 = (0, 1, 1)$ and $x_3 = (1, 0, 1)$.

20. Check whether the matrices below are positive definite or positive semi-definite?

(i) $\begin{pmatrix} 10 & 2 & 0 \\ 2 & 4 & 6 \\ 0 & 6 & 10 \end{pmatrix}$.

(ii) $\begin{pmatrix} 8 & 2 & -2 \\ 2 & 8 & -2 \\ -2 & -2 & 11 \end{pmatrix}$.

(iii) $\begin{pmatrix} 3 & 10 & -2 \\ 10 & 6 & 8 \\ -2 & 8 & 12 \end{pmatrix}$.

Answer and Hints

1. Verify all the axioms of a field taking 0 as additive identity and 1 as multiplicative identity.

2. Verify all the axioms of a vector space taking zero function as zero vector.
3. Take ϕ set as zero vector and additive inverse of a set itself.
4. Additive inverse does not exist.

5. (a) yes, (b) neither closed under addition nor under scalar multiplication, (c) not closed under scalar multiplication, (d) yes, (e) yes, (f) not closed under addition.

6. (a) No, Counter Example: $V = \mathbb{R}^2,\ S_1 = \{(x_1, x_2) : x_1 = x_2\},\ S_2 = \{(x_1, x_2) : x_1 + 2x_2 = 0\}$.
$(1, 1) \in S_1,\ (-2, 1) \in S_2$ but $(1, 1) + (-2, 1) = (-1, 2) \in S_1, S_2$.

(b) Yes. Let $S_i (i = 1, 2, \cdots)$ be subspaces of V and $S = \cap_{i=1}^{\infty} S_i$. $x, y \in S \Rightarrow x, y \in S_i\ \forall i$. Then $x + y \in S_i\ \forall i$ and so $x + y \in S$. Similarly S is closed under scalar multiplication. So (b) is true.

7. (a) Echelon form of the corresponding matrix
$\begin{pmatrix} 1 & 2 & -2 & -1 \\ 2 & 1 & -1 & 4 \\ -3 & 0 & 3 & -2 \end{pmatrix}$ is $\begin{pmatrix} 1 & 2 & -2 & -1 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & -3 & -7 \end{pmatrix}$. So the given set of vectors are linearly independent.

(b) Echelon form of the corresponding matrix
$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 3 \end{pmatrix}$ is $\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 0 & -1 & -3 & 2 & -1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. So the given set of vectors are linearly dependent.

8. (a) No, because $dim \mathbb{R}^3 = 3$.

(b) Yes, because the set is linearly independent.

(c) No, because it contains more than 3 vectors.

(d) No, because it is linearly dependent.

9. First 3 rows of the echelon form in 7(b) forms a basis for W. Therefore $dim\ W = 3$.

10. Consider $(A|I) = \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$.
Apply each of these elementary row operations in the updated matrix $R_1 \to \frac{1}{2}R_1,\ R_2 \to R_2 - 5R_1,\ R_3 \to R_3 + R_2,\ R_1 \to R_1 + R_3,\ R_2 \to -R_2,\ R_2 \to R_2 - 5R_3,\ R_3 \to 2R_3$ and get
$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right)$. So, $A^{-1} = \begin{pmatrix} 3 & -1 & -1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$.

11. (a) No. (b) Yes; not an isomorphism. (c)Yes; an isomorphism. (d) No

12. $T(u_1) = (2, 1) = -2(-1, 0) + 1(0, 1)$
$T(u_2) = (2, 3) = -2(-1, 0) + 3(0, 1)$
$T(u_3) = (-1, 2) = 1(-1, 0) + 2(0, 1)$. So, answer is $\begin{pmatrix} -2 & -2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$.

13. Not consistent. Check that rank of the co-efficient matrix is 2 where as that of the augmented matrix is 3.

14. Check that the system is consistent, where the rank of both the co-efficient matrix and augmented matrix is 2. In echelon form the system is

$$\begin{aligned} x + 2y - 3z + 2w &= 2 \\ y - 2z + 2w &= 1 \end{aligned}$$

Taking z and w as free variables, i.e., $z = \alpha$, $w = \beta$, we get the set of all solutions is $\{(-\alpha + 2\beta, 1 + 2\alpha - 2\beta, \alpha, \beta) : \alpha, \beta \in \mathbb{R}\}$.

15. The system in echelon form is

$$\begin{aligned} x + 2y - z &= 0 \\ y + 4z &= 0 \end{aligned}$$

The solution space is $\{(9\alpha, -4\alpha, \alpha) : \alpha \in \mathbb{R}\}$. It's dimension is 1.

16. Eigenvalues are $2, 2, 3$. Basis for eigenspace corresponding to 2 and 3 are $\{(1,0,0)\}$ and $\{(1,1,-2)\}$ respectively. The matrix is not diagonalizable beacuse sum of dimension of eigenspaces is not equal to 3.

17. $\lambda = 2$ with $m = 3$, $p_1 = 2$, $\rho_1 = 2$, $\rho_2 = 2$.
$X_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $Y_1 = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, $Y_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
$M = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{pmatrix}$, $J = M^{-1}AM = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

18. Yes it is an inner product. One can check that last axiom as
$< x, x >= 2x_1^2 - 2x_1x_2 + 3x_2^2 = 2(x_1 - \frac{x_2}{2} + \frac{5}{2}x_2^2)^2$. This gives that $< x, x >= 0$ iff $x = 0$.

19. Answer is $\{(1,1,0), (-\frac{1}{2}, \frac{1}{2}, 1), (\frac{2}{3}, -\frac{2}{3}, \frac{2}{3})\}$.

20. (i) Positive semi-definite.
(ii) Positive definite. (iii) Neither of them.