Exercises for Module-III (Transform Calculus)

1) Discuss the piecewise continuity of the following functions:

a) $f(t) = \dfrac{1}{t-2}, \; t \neq 2$

b) $f(t) = \begin{cases} 2t, & t \le 1 \\ 1+t^2, & t > 1 \end{cases}$

c) $f(t) = \begin{cases} \dfrac{1-e^{-t}}{t}, & t \neq 0 \\ 0, & t = 0 \end{cases}$

d) $f(t) = \begin{cases} t \sin\left(\frac{1}{t}\right), & t \neq 0 \\ 0, & t = 0 \end{cases}$

2) Show that the function $f(t) = t\, e^{t^2} \sin\left(e^{t^2}\right)$ possesses a Laplace transform.

3) Find Laplace transform of the following functions:

a) $e^{-t} \cos^2 t, \; t > 0$

b) $f(t) = \begin{cases} 0, & 0 \le t < 2 \\ e^{at}, & t \ge 2 \end{cases}$

c) $f(t) = |\sin t|, \; t > 0$

d) $f(t) = t\, H(t-a), \; t > 0$

4) Find the inverse Laplace transform of the following functions:

a) $F(s) = \dfrac{s+3}{\left(s^2+6s+13\right)^2}$

b) $F(s) = \dfrac{4s+5}{(s-1)^2 (s+2)}$

c) $F(s) = \dfrac{e^{-\pi s}}{s^2-2}$

d) $F(s) = \dfrac{2s^2+3}{(s+1)^2 \left(s^2+1\right)^2}$

5) Solve the following initial value problems for $t > 0$ using Laplace transform method:

a) $\dfrac{d^2y}{dt^2} + y = 1; \; y(0) = y'(0) = 0$

b) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = \left(1 - H(t-1)\right); \; y(0) = 1, y'(0) = -1$

c) $\dfrac{d^2y}{dt^2} + y = f(t); \; y(0) = y'(0) = 0;$ where $f(t) = \begin{cases} \cos t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$

6) Solve the following boundary value problems using Laplace transform methods:

a) $\dfrac{d^2y}{dt^2} + y = \sin t, \quad y(0) = 1,\ y\left(\dfrac{\pi}{2}\right) = \pi,$

b) $\dfrac{d^2y}{dt^2} + 9\lambda = t, \quad y(0) = 1, \quad y'\left(\dfrac{\pi}{3}\right) = -1$

7) Solve the following differential equations using Laplace transform method:

a) $t\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} = -1;\ y(0) = 0$

b) $t\dfrac{d^2y}{dt^2} + (t+1)\dfrac{dy}{dt} + 2y = e^{-t};\ y(0) = 0$

8) Solve the following integral equations using Laplace transform method:

a) $y(t) = \sin t + 2\int_0^t y(u)\cos(t-u)\,du$

b) $\dfrac{dy(t)}{dt} + 3y(t) + 2\int_0^t y(u)\,du = t;\ y(0) = 1$

9) With the application of Laplace transform, solve the Heat equation

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t};\quad x > 0,\ t > 0$ subject to the conditions

$u(x, 0+) = 0,\quad x > 0$

$u(0, t) = f(t),\quad t > 0$

$\lim_{x \to \infty} u(x, t) = 0$

10) Using Laplace transform solve the wave equation $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2};\quad 0 < x < 1,\ t > 0$ with the initial and boundary conditions

$y(x, 0+) = \sin \pi x,\quad 0 < x < 1,\qquad y_t(x, 0+) = 0,\quad 0 < x < 1$

$y(0, t) = 0,\quad t > 0,\qquad y(1, t) = 0,\quad t > 0$

11) Expand $f(x)=e^{-x}, \quad 0<x<2\pi$ in a Fourier series.

12) Find the half range sine series of the function $f(x)=x^2, \quad 0<x<\pi$

13) Determine the Fourier integral representation of the function

$$f(x)=\begin{cases}\sin x, & -\pi<x<\pi \\ 0, & x<-\pi \text{ and } x>\pi\end{cases}.$$

14) Let the function $f(x)$ is given as

$$f(x)=\begin{cases}1-x, & 0<x<1 \\ 0, & 1<x<\infty\end{cases}.$$

Determine the Fourier sine integral of $f(x)$ and find the value of

$$\int_0^{\infty} \frac{(\alpha-\sin\alpha)\sin\left(\frac{\alpha}{2}\right)}{\alpha^2}\,\mathrm{d}\alpha.$$

15) Find the Fourier sine transform of $f(t)=te^{-t}, \quad t\geq 0.$

16) Determine the Fourier cosine transform for $f(t)=e^{-t}\cos t, \quad t\geq 0$ and show that

$$e^{-t}\cos t=\frac{2}{\pi}\int_0^{\infty}\frac{(\alpha^2+2)\cos\alpha t}{\alpha^4+4}\,\mathrm{d}\alpha, \quad t\geq 0.$$

17) Find Fourier transform of $f(t) = t\exp(-a|t|).$

18) Find $f(t)$ if $F_s(\alpha)=\sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{1+\alpha^2}\right).$

19) Solve the heat equation $\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}, \quad 0<x<\infty$ subject to the conditions

$u(x,0)=e^{-x}; \quad u_x(0,t)=0, \quad t>0$, u and u_x both tend to zero as $x\to\infty$.

20) Solve the following Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0; \quad y>0$ subject to the conditions u_x and $u\to 0$ as $x^2+y^2\to\infty$ and $u(x,0)=\begin{cases}1, & |x|\leq 1 \\ 0, & |x|>1\end{cases}.$

Answers/Hints:

1)

a) Function is not piecewise continuous since $\lim_{t\to 2\pm} f(t)$ do not exists.

b) Function is continuous everywhere.

c) Function has a jump discontinuity at $t=0$ and hence the function is piecewise continuous.

d) Function is continuous everywhere.

2)

Definition of Laplace transform and integration by parts give

$$L\{f(t)\}=\int_0^\infty e^{-st}te^{t^2}\sin\left(e^{t^2}\right)\mathrm{d}t=\frac{1}{2}\left[\cos(1)-sL\left\{\cos\left(e^{t^2}\right)\right\}\right]$$

Note that $L\left\{\cos\left(e^{t^2}\right)\right\}$ exists because $\cos\left(e^{t^2}\right)$ is continuous and is of exponential order. Hence $L\{f(t)\}$ exists.

3)

a) $L\left\{e^{-t}\cos^2 t\right\}=\dfrac{(s+1)^2+2}{(s+1)\left\{(s+1)^2+4\right\}}=\dfrac{s^2+2s+3}{(s+1)\left(s^2+2s+5\right)}$

b) $f(t)=e^{at}H(t-2)=e^{a(t-2)}e^{2a}H(t-2)=e^{-2(s-a)}\dfrac{1}{s-a}$

c) $L\{f(t)\}=\dfrac{1}{1-e^{-\pi s}}\displaystyle\int_0^\pi e^{-st}\sin t\ \mathrm{d}t=\dfrac{1+e^{-\pi s}}{\left(1-e^{-\pi s}\right)\left(s^2+1\right)}$

d) $L\{f(t)\}=e^{-as}\dfrac{1}{s^2}+a\dfrac{e^{-as}}{s}=\dfrac{e^{-as}}{s^2}[1+as]$

4)

a) $L^{-1}\left\{\dfrac{s+3}{\left(s^2+6s+13\right)^2}\right\}=L^{-1}\left\{\dfrac{s+3}{\left[(s+3)^2+4\right]^2}\right\}=e^{-3t}L^{-1}\left\{\dfrac{s}{\left(s^2+4\right)^2}\right\}=\dfrac{1}{4}te^{-3t}\sin 2t$

b) $\dfrac{1}{3}e^{t}+3te^{t}-\dfrac{1}{3}e^{-2t}$

c) $\dfrac{1}{\sqrt{2}}H(t-\pi)\sinh\left\{\sqrt{2}(t-\pi)\right\}$

d) $\dfrac{3}{2}e^{-t}+\dfrac{5}{4}te^{-t}-\dfrac{3}{2}\cos t+\dfrac{1}{4}\sin t-\dfrac{1}{4}t\sin t$

5)

a) $y(t) = 1 - \cos t$

b) $y = t - 1 + 2e^{-t} - H(t-1)\left(t - 2 + e^{-t+1}\right)$

c) $y = \frac{1}{2}\left[t \sin t + H(t-\pi)(t-\pi)\sin(t-\pi)\right]$

6)

a) $y = \left(-\frac{1}{2} t \cos t + \cos t + \pi \sin t\right)$

b) $y = \frac{t}{9} + \cos 3t + \frac{4}{9} \sin 3t$

7)

a) $y = t + ct^2$

b) $y = te^{-t}$

8)

a) $y(t) = te^t$

b) $y(t) = \frac{1}{2} - 2e^{-t} + \frac{5}{2} e^{-2t}$

9)

$$u(x,t) = \int_0^t \frac{x}{2\sqrt{\pi (t-u)^3}} \exp\left(-\frac{x^2}{4(t-u)}\right) f(u)\,\mathrm{d}u$$

10)

$$y(x,t) = \sin \pi x \cos \pi t$$

11)

$$f(x) \sim \frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \dots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \dots\right)\right]$$

12)

$$f(x) \sim 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x - \ldots\ ]$$

13)

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin \alpha\pi \sin \alpha x}{1-\alpha^2}\,\mathrm{d}\alpha$$

14)

$f(x) \sim \frac{2}{\pi}\int_0^{\infty} \frac{\alpha - \sin\alpha}{\alpha^2}\sin \alpha x\ \mathrm{d}\alpha$ and the value is $\frac{\pi}{4}$.

15)

$$\sqrt{\frac{2}{\pi}}\,\frac{2\alpha}{(1+\alpha^2)^2}$$

16)

$$\sqrt{\frac{2}{\pi}}\,\frac{2+\alpha^2}{(4+\alpha^4)}$$

17)

$$\sqrt{\frac{2}{\pi}}\,\frac{2ia\alpha}{(a^2+\alpha^2)^2}$$

18)

$$\mathrm{f(t)} = \exp(-t)$$

19)

$$u(x,t) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+\alpha^2} e^{-k\alpha^2 t} \cos \alpha x\ \mathrm{d}\alpha$$

20)

$$u(x,y) = \frac{y}{x}\left[\tan^{-1}\left(\frac{x-1}{y}\right) + \tan^{-1}\left(\frac{x+1}{y}\right)\right]$$