NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)
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ADDITIONAL PROBLEMS ON CALCULUS OF VARIATIONS WITH SOLUTIONS

1. Find the extremals for the functional

$$I(y) = \int_0^1 \left[(y')^2 - y^2\right] dx,$$

satisfying the boundary conditions $y(0) = 1$ and $y(1) = 1$.

Solution: Comparing the given functional to the standard form

$$I(y) = \int_0^1 F(x, y(x), y'(x))dx,$$

we have $F(x, y, y')$ =$I(y) = (y')^2 - y^2$ and the Euler equation $F_y - \frac{d}{dx}F_{y'} = 0$ implies that the extremals must satisfy the differential equation $y''+y = 0$. Thus, the extremals are given by $y(x) = A\cos x + B\sin x$. The boundary conditions imply that $A = 0$ and $B = 1/\sin 1$. Hence the function which extremizes the given functional is given by $y(x) = \sin x/\sin 1$.

2. Find the extremals for the functional

$$I(y) = \int_0^1 \left[(y')^2 + xy\right] dx,$$

satisfying the boundary conditions $y(0) = 1$ and $y(1) = 1$.

Solution: Here $F(x, y, y') = (y')^2 + xy$. The Euler equation implies that $y'' = x/2$. Integrating twice, we get the extremals as $y(x) = (x^3/12) + Ax + B$. Boundary conditions give us $B = 0$ and $A = 11/12$. Hence the extremal which extremizes the given functional is given by $y = (x^3 + 11x)/12$.

3. Show that there is no $y \in C[0, 1]$ which extremizes the functional

$$I(y) = \int_0^1 y^2 dx, \quad y(0) = 0,\ y(1) = A,$$

unless $A = 0$.

Solution: We have $F(x, y, y') = y^2$ and the Euler equation gives $y = 0$. Hence if $A \neq 0$, we have no continuous function extremizing the given functional.

4. Analyze the functional

$$I(y) = \int_0^1 [y^2 + x^4 y']dx, \quad y(0) = 0,\ y(1) = A,$$

for extremals.

Solution: We have $F = y^2 + x^4 y'$ and the Euler equation gives $y = 2x^3$. $y(0) = 0$ is satisfied but $y(1) = A$ will be satisfied only when $A = 2$. So, if $A \neq 0$ we have no extremals satisfying the boundary conditions.

5. Show that the curve of minimum length joining two points in a plane is the straight line joining these two points.

   Solution: The functional giving the length of a plane curve between two given points (x_1, y_1) and (x_2, y_2) is given by

$$l(y) = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx, \quad y(x_1) = y_1,\ y(x_2) = y_2.$$

   We have $F = \sqrt{1 + y'^2}$. Here F is independent of the variable y hence $F_y = 0$. The first integral of the Euler equation gives $F_{y'} = C$, where C is any arbitrary constant. This leads us to $y'^2 = C(1 + y'^2)$. Clearly, $C \neq 1$. Solving for y' we get $y' = D$, where D is another constant given in terms of C. Hence $y = Dx + E$, E is also an arbitrary constant. The boundary conditions can be used to determine D and D. Thus, we get the extremal as the straight line joining the given two points in the plane.

6. Formulate the functional for the lines of propagation of light in optically non-homogeneous medium in which the speed of light is $v(x, y, z)$ and hence obtain the differential equations for the same.

   Solution: According to Fermat's principle, light is propagated from a point $A(x_1, y_1, z_1)$ to another $B(x_2, y_2, z_2)$ along a curve $\Gamma(x, y(x), z(x))$, $x_1 \leq x \leq x_2$ for which the time $t(y, z)$ of passage will be the least. We have

$$t(y, z) = \int_{x_1}^{x_2} \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2 + z'^2}}{v(x, y, z)}\, dx.$$

   The system of Euler equations $F_y - \frac{d}{dx} F_{y'} = 0$ and $F_z - \frac{d}{dx} F_{z'} = 0$ gives the system of differential equations

$$v_y \left(\frac{\sqrt{1 + y'^2 + z'^2}}{v^2} \right) + \frac{d}{dx} \left(\frac{y'}{v\sqrt{1 + y'^2 + z'^2}} \right) = 0,$$

$$v_z \left(\frac{\sqrt{1 + y'^2 + z'^2}}{v^2} \right) + \frac{d}{dx} \left(\frac{z'}{v\sqrt{1 + y'^2 + z'^2}} \right) = 0.$$

7. Let S be the surface of the sphere $x^2 + y^2 + z^2 = a^2$ and let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points on S. Show that the curve joining P and Q with shortest length is a geodesic.

   Solution: Let S be parameterized spherical co-ordinates

$$x = a \sin\phi \cos\theta, \quad y = a \sin\phi \sin\theta, \quad z = a \cos\phi.$$

   Let a curve joining P and Q be given by $\theta = f(\phi)$, $\phi_1 \leq \phi \leq \phi_2$. Now, the functional for the length of a curve is

$$\int_P^Q \sqrt{dx^2 + dy^2 + dz^2} = \int_{\phi_1}^{\phi_2} \sqrt{1 + \theta'^2 \sin^2\phi}\, d\phi.$$

Here $F(\phi, \theta(\phi), \theta'(\phi)) = \sqrt{1+\theta'^2 \sin^2 \phi}$. Since $F_\theta = 0$, first integration of the Euler equation $F_\theta - \frac{d}{d\phi} F_{\theta'} = 0$ gives $F_{\theta'} = C$. Hence, we have

$$\frac{\sin^2 \phi \theta'}{\sqrt{1+\theta'^2 \sin^2 \phi}} = C.$$

Solving for θ' we get

$$\theta' = \frac{C \operatorname{cosec}^2 \phi}{\sqrt{1 - \operatorname{cosec}^2 \phi}}$$

Integrating it over (ϕ_1, ϕ_2), we get

$$\theta = \int_{\phi_1}^{\phi_2} \frac{C \operatorname{cosec}^2 \phi}{\sqrt{1 - C^2 \operatorname{cosec}^2 \phi}}\, d\phi + D = \int_{\phi_1}^{\phi_2} \frac{\operatorname{cosec}^2 \phi}{\sqrt{E - \cot^2 \phi}}\, d\phi + D,$$

where $E = \frac{1}{C^2} - 1$.

Now, we put $\cot \phi = t\sqrt{E}$ to get $\operatorname{cosec}^2 \phi \; d\phi = dt\sqrt{E}$. Thus, we get

$$\theta = \int_{t_1}^{t_2} \frac{dt}{\sqrt{1-t^2}} + D = \sin^{-1} t|_{t_1}^{t_2} + D.$$

Let t_1 be fixed and $t_2 = t$ be the movable point on the curve.

Then we have

$$\theta = \int_{t_1}^{t} \frac{dt}{\sqrt{1-t^2}} + D = \sin^{-1} t|_{t_1}^{t} + D,$$

which implies $\sin(\theta + \alpha) = t = \beta \cot \phi$, for some constants α and β. This relation leads to

$$a \sin \theta \cos \phi + b \sin \theta \sin \phi + c \cos \theta = 0$$

which is equal to $ax + by + cz = 0$ a plane passing through the origin. Thus the curve is a part of intersection of a plane passing through the origin and the sphere $x^2 + y^2 + z^2 = a^2$ which is a geodesic.

where $t_i = \cot \phi_i / \sqrt{E}$, $i = 1, 2$.

8. Show that the extremals for the functional

$$I(z) = \int\int_D \left[z_x^2 + z_y^2\right] dxdy,$$

are the solutions of the Laplace equation $z_{xx} + z_{yy} = 0$, in a bounded domain D with sufficiently smooth boundary.

Solution: In this case we have $F = z_x^2 + z_y^2$. In order $z(x, y)$ to extremize the given functional, it must satisfy

$$F_z - (F_{z_x})_x - (F_{z_y})_y = 0,$$

which leads to $-2z_{xx} - 2z_{yy} = 0$, i.e., $z_{xx} + z_{yy} = 0$.

9. Find the extremals for the functional

$$I(y,z) = \int_0^{x_1} \left[y'^2 + z'^2 + 2yz \right] dx, \quad y(0) = 0 = z(0),$$

and the point $(x_1, y(x_1), z(x_1))$ moves on the plane $x = x_1$.

Solution: The extremals are given by the system $F_y - \frac{d}{dx}F_{y'} = 0$ and $F_z - \frac{d}{dx}F_{z'} = 0$, where $F = y'^2 + z'^2 + 2yz$. Hence y and z must satisfy $z'' - y = 0$ and $y'' - z = 0$. Differentiating the first equation twice, we get $z^{(4)} - z = 0$ which has the general solution $z(x) = Ae^x + Be^{-x} + C\cos x + D\sin x$. Now $z(0) = 0$ implies $A + B + C = 0$. $y(0) = z''(0) = 0$ implies that $A + B - C = 0$. Thus $C = 0$ and $B = -A$. Hence $z = A_1 \sinh x + B_1 \sin x$. The condition at the moving point is

$$[F - y'F_{y'} - z'F_{z'}]|_{x=x_1} \delta x_1 + F_{y'}|_{x=x_1}\delta y_1 + F_{z'}|_{x=x_1}\delta z_1 = 0.$$

Since the point $(x_1, y(x_1), z(x_1))$ is moving on $x = x_1$, we have $\delta x_1 = 0$. The variations δy_1 and δ_1 are arbitrary, we have

$$F_{y'}|_{x=x_1} = 0, \quad F_{z'}|_{x=x_1} = 0.$$

These conditions imply that $y'(x_1) = 0 = z'''(x_1)$ and $z'(x_1) = 0$. Thus

$$A_1 \cosh x_1 + B_1 \cos x_1 = 0, \quad A_1 \cosh x_1 - B_1 \cos x_1 = 0.$$

If $\cos x_1 \neq 0$ then $A_1 = B_1 = 0$. Then $y = z = 0$. If $\cos x_1 = 0$ then $x_1 = (2n+1)\pi/2$ where $n \in \mathbb{Z}$, and $A_1 = 0$. In this case $y = B_1 \sin x$ and $z = -B_1 \sin x$. The value of $I(y,z) = 0$ for these functions.

10. Test the functional

$$\int_{x_1}^{x_2} [6y'^2 - y'^4 + yy']dx, \quad y(x_1) = 0, \ y(x_2) = \alpha, \ x_2 > x_1 > 0, \ \alpha > 0,$$

for an extremum with extremals $y \in C^1[x_1, x_2]$.

Solution: We have $F = 6y'^2 - y'^4 + yy'$ and the Euler equation imply

$$y' - 12y'' + 12y'^2 y'' - y' = 0.$$

Thus,

$$(1 - y'^2)y'' = 0.$$

So, either $y'' = 0$ which gives $y = Ax + B$ or $y' = \pm 1$ which give $y = \pm x + D$. Hence extremals are straight lines. $y(x_1) = 0$ implies $0 = Ax_1 + B$ hence $A = -B/x_1$. The condition $y(x_2) = \alpha$ implies $\alpha = -B[(x_2/x_1) - 1]$. Thus, $B = -\alpha x_1/(x_2 - x_1)$. Putting these values of the constants A and B we get the extremal as

$$y = \alpha \frac{x - x_1}{x_2 - x_1}.$$

This is a part of the pencil of extremals $y = C(x - x_1)$ that form a central field at $(x_1, 0)$.

Now we construct the Weierstrass function $E(x, y, y', p) = F(x, y, y') - F(x, y, p) - (y' - p)F_p(x, y, p)$ for the given functional. Here we have $F(x, y, y') = 6y'^2 - y'^4 + yy'$ and $F(x, y, p) = 6p^2 - p^4 + yp$. Thus, we have

$$\begin{aligned} E(x, y, y', p) &= 6y'^2 - y'^4 + yy' - 6p^2 + p^4 - yp - (y' - p)(12p - 4p^3 + y) \\ &= (y' - p)[6(y' - p) - 3(y'^3 - p^3) + y'(y'^2 - p^2) + y'^2(y' - p)] \\ &= -(y' - p)^2[-6 + 3(y'^2 + y'p + p^2) - y'(y' + p) - y'^2] \\ &= -(y' - p)^2[y'^2 + 2y'p + (3p^2 - 6)]. \end{aligned}$$

The sign of E will depend on the sign of $Q = y'^2 + 2y'p + (3p^2 - 6)$. That is, $E \geq 0$ if and only if $Q \leq 0$ and $E \leq 0$ if and only if $Q \geq 0$. Q changes sign when y' passes through the value

$$y' = -p \pm \sqrt{6 - 3p^2}.$$

For Large positive value of p and y' close to p, $Q > 0$ and hence if $6 - 3p^2 < 0$ then we have no real value of y' for which Q will vanish and hence it remains positive for $6 - 3p^2 \leq 0$. For $6 - 3p^2 > 0$, Q changes sign. For $p = 1$, we have $Q = y'^2 + 2y' - 3$ and $Q = 0$ for $y' = 1$. Hence for $p > 1$ and y' close to p, i.e., $y' > 1$ we have $Q > 0$. Similarly, for $p < 1$ and $y' < 1$, we have $Q < 0$. Thus, we have, for the slop of the extremal $p = \alpha/(x_2 - x_1) > 1$ and the slop of neighboring extremals y' close to p, $E < 0$, i.e., we have weak maximum. and for the case $p = \alpha/(x_2 - x_1) < 1$ and y' close to p, we have $E > 0$, i.e., we have weak minimum.

ADDITIONAL PROBLEMS ON INTEGRAL EQUATIONS WITH SOLUTIONS

1. Show that $u(x) = \cosh x$ is a solution of the integral equation $u(x) = 2\cosh x - x \sinh x - 1 + \int_0^x tu(t)dt$.

   Solution: $\int_0^x t \cosh t dt = x \sinh x - \cosh x + 1$, hence the result follows.

2. Convert the following initial value problem to an equivalent integral equation,

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0, \; y(0) = 2, \; y'(0) = 0, \; y''(0) = 2.$$

   Solution: Let $y'''(x) = u(x)$, then $y''(x) = 2 + \int_0^x u(t)dt$, $y'(x) = 2x + \int_0^x (x-t)u(t)dt$, $y(x) = 2 + x^2 + \frac{1}{2}\int_0^x (x - t)^2 u(t)dt$. Substituting into the given equation we find the required integral equation

$$u(x) = 2x - x^2 + \int_0^x \left[1 + (x - t) - \frac{1}{2}(x - t)^2\right] u(t)dt.$$

3. Solve the following Volterra integral equation by the successive approximations method,

$$u(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x - t)u(t)dt.$$

   Solution: We assume the first approximation as $u_0(x) = 1$. Then we can find successively, $u_1(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x-t)u_0(t)dt = 1 - x$, $u_2(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x-t)u_1(t)dt = 1 - x - \frac{x^3}{6}$, $u_3(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x - t)u_2(t)dt = 1 - x - \frac{x^3}{6} - \frac{x^5}{120}$ and so on. Finally we can verify that $u(x) = 1 - \sinh x$.

4. Solve the following Volterra integral equation by the series solution method,

$$u(x) \;=\; x\cos x + \int_0^x tu(t)dt.$$

Solution: Substituting $u(x) = \sum_{n=0}^{\infty} a_n x^n$ on both sides of the given equation and then integrating we get,

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \left(x - \frac{x^3}{2!} + \frac{x^5}{4!} - \cdots \right) + \left(a_0 \frac{x^2}{2} + a_1 \frac{x^3}{3} + a_2 \frac{x^4}{4} \cdots \right).$$

Equating like powers of x from both sides we get, $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = -\frac{1}{6}$, $a_4 = 0$, $a_5 = \frac{1}{5!}$. Hence the required solution is $u(x) = \sin x$.

5. Use Adomian decomposition method to solve the following integral equation,

$$u(x) \;=\; 6x - x^3 + \frac{1}{2}\int_0^x tu(t)dt.$$

Solution: $u_0(x) = 6x - x^3$, $u_1(x) = x^3 - \frac{x^5}{10}$, $u_3(x) = \frac{x^5}{10} - \frac{x^7}{140}$ and hence the required solution is $u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots = 6x$.

6. Use the modified Adomian decomposition method to solve the following integral equation,

$$u(x) \;=\; \sec x \tan x + (e - e^{\sec x}) + \int_0^x e^{\sec t} u(t)dt, \;\; x \;<\; \pi/2.$$

Solution: According to the modified Adomian decomposition method we assume $f_1(x) = \sec x \tan x$ and $f_2(x) = (e - e^{\sec x})$. Then $u_0(x) = f_1(x)$, $u_2(x) = f_2(x) + \int_0^x e^{\sec t} u_0(t)dt = 0$ and so on. Hence the required solution is $u(x) = \sec x \tan x$.

7. Solve the integral equation $u(x) \;=\; 1 + \lambda \int_0^1 (1 - 3xt)u(t)dt$ by using the resolvent kernel method.

Solution: $K_1(x,\xi) = K(x,\xi) = (1 - 3x\xi)$, $K_2(x,\xi) = 1 - \frac{3}{2}(x+\xi) + 3x\xi$, $K_3(x,\xi) = \frac{1}{4}K_1(x,\xi) = \frac{1}{4}(1 - 3x\xi)$, $K_4(x,\xi) = \frac{1}{4}K_2(x,\xi)$, $K_5(x,\xi) = \left(\frac{1}{4}\right)^2 K_1(x,\xi)$. Hence,

$$R(x,\xi;\lambda) = [K_1(x,\xi) + \lambda^2 K_3(x,\xi) + \lambda^4 K_5(x,\xi) + \cdots] + [\lambda K_2(x,\xi) + \lambda^3 K_4(x,\xi) + \lambda^5 K_6(x,\xi) + \cdots]$$

$$= (1 - 3x\xi)\left[1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{4^2} + \cdots\right] + \lambda\left(1 - \frac{3}{2}(x+\xi) + 3x\xi\right)\left[1 + \frac{\lambda^2}{4} + \frac{\lambda^4}{4^2} + \cdots\right]$$

$$= \frac{4}{4-\lambda^2}\left[1 + \lambda - \frac{3}{2}\lambda x - 3\xi\left(x + \frac{\lambda}{2} - \lambda x\right)\right], \;\; |\lambda| \;<\; 2.$$

Hence the required solution is

$$u(x) = 1 + \lambda \int_0^1 R(x,t;\lambda)dt = \frac{4 + 2\lambda(2 - 3x)}{4 - \lambda^2}, \;\; |\lambda| \;<\; 2.$$

8. Solve the following Fredholm integral equation by using successive substitution,

$$u(x) = \sin x + \frac{1}{2}\int_0^{\pi/2} \cos x u(t) dt.$$

Solution: Using successive substitution method we find,

$$u(x) = \sin x + \frac{1}{2}\int_0^{\pi/2} \cos x \sin t dt + +\frac{1}{4}\int_0^{\pi/2} \cos x \left(\int_0^{\pi/2} \cos t \sin s ds\right) dt + \cdots .$$

Evaluating the successive integrals,

$$u(x) = \sin x + \frac{1}{2}\cos x + \frac{1}{4}\cos x + \frac{1}{8}\cos x + \cdots = \sin x + \cos x.$$

9. Use the method of degenerate kernel to solve the integral equation,

$$u(x) = e^x + \lambda \int_0^1 2e^x e^t u(t) dt.$$

Solution: Let $c = \int 0^1 2e^t u(t)dt$, then from the given equation, $u(x) = e^x + 2\lambda e^x c$. Substituting in the given equation and then solving for c we find $c = \frac{e^2-1}{2[1-\lambda(e^2-1)]}$. Hence the required solution is $u(x) = \frac{e^x}{1-\lambda(e^2-1)}$, $\lambda \neq \frac{1}{e^2-1}$.

10. Solve the following singular integral equation by using the Laplace transform method,

$$\int_0^x \frac{u(t)}{\sqrt{x-t}} dt = 1 + x + x^2.$$

Solution: Taking Laplace transform of the given equation we find,

$$\mathcal{L}[u(x)]\mathcal{L}\left[\frac{1}{\sqrt{x}}\right] = \mathcal{L}[1] + \mathcal{L}[x] + \mathcal{L}[x^2] \Rightarrow \mathcal{L}[u(x)] = \frac{1}{\sqrt{\pi}}\left[\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{p^3}} + \frac{2}{\sqrt{p^5}}\right].$$

Taking inverse Laplace transform, we get

$$u(x) = \frac{1}{\pi}\left[\frac{1}{\sqrt{x}} + 2\sqrt{x} + \frac{8}{3}\sqrt{x^3}\right].$$