NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)
IIT KANPUR

ADDITIONAL PROBLEMS ON CALCULUS OF VARIATIONS

1. Find the extremals for the functional

$$I(y) = \int_0^1 \left[(y')^2 - y^2\right] dx,$$

satisfying the boundary conditions $y(0) = 1$ and $y(1) = 1$.

2. Find the extremals for the functional

$$I(y) = \int_0^1 \left[(y')^2 + xy\right] dx,$$

satisfying the boundary conditions $y(0) = 1$ and $y(1) = 1$.

3. Show that there is no $y \in C[0, 1]$ which extremizes the functional

$$I(y) = \int_0^1 y^2 dx, \quad y(0) = 0, \ y(1) = A,$$

unless $A = 0$.

4. Analyze the functional

$$I(y) = \int_0^1 [y^2 + x^4 y'] dx, \quad y(0) = 0, \ y(1) = A,$$

for extremals.

5. Show that the curve of minimum length joining two points in a plane is the straight line joining these two points.

6. Formulate the functional for the lines of propagation of light in optically non-homogeneous medium in which the speed of light is $v(x, y, z)$ and hence obtain the differential equations for the same.

7. Let S be the surface of the sphere $x^2+y^2+z^2 = a^2$ and let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points on S. Show that the curve joining P and Q with shortest length is a geodesic.

8. Show that the extremals for the functional

$$I(z) = \iint_D \left[z_x^2 + z_y^2\right] dxdy,$$

are the solutions of the Laplace equation $z_{xx} + z_{yy} = 0$, in a bounded domain D with sufficiently smooth boundary.

9. Find the extremals for the functional

$$I(y, z) = \int_0^{x_1} \left[y'^2 + z'^2 + 2yz \right] dx, \quad y(0) = 0 = z(0),$$

and the point $(x_1, y(x_1), z(x_1))$ moves on the plane $x = x_1$.

10. Test the functional

$$\int_{x_1}^{x_2} [6y'^2 - y'^4 + yy'] dx, \quad y(x_1) = 0,\ y(x_2) = \alpha,\ x_2 > x_1 > 0,\ \alpha > 0,$$

for an extremum with extremals $y \in C^1[x_1, x_2]$.

ADDITIONAL PROBLEMS ON INTEGRAL EQUATIONS

1. Show that $u(x) = \cosh x$ is a solution of the integral equation $u(x) = 2\cosh x - x\sinh x - 1 + \int_0^x tu(t)dt$.

2. Convert the following initial value problem to an equivalent integral equation,

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0, \; y(0) = 2, \; y^{'}(0) = 0, \; y^{''}(0) = 2.$$

3. Solve the following Volterra integral equation by the successive approximations method,

$$u(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x-t)u(t)dt.$$

4. Solve the following Volterra integral equation by the series solution method,

$$u(x) = x\cos x + \int_0^x tu(t)dt.$$

5. Use Adomian decomposition method to solve the following integral equation,

$$u(x) = 6x - x^3 + \frac{1}{2}\int_0^x tu(t)dt.$$

6. Use the modified Adomian decomposition method to solve the following integral equation,

$$u(x) = \sec x \tan x + (e - e^{\sec x}) + \int_0^x e^{\sec t}u(t)dt, \; x < \pi/2.$$

7. Solve the integral equation $u(x) = 1 + \lambda\int_0^1 (1-3xt)u(t)dt$ by using the resolvent kernel method.

8. Solve the following Fredholm integral equation by using successive substitution,

$$u(x) = \sin x + \frac{1}{2}\int_0^{\pi/2} \cos x u(t)dt.$$

9. Use the method of degenerate kernel to solve the integral equation,

$$u(x) = e^x + \lambda\int_0^1 2e^x e^t u(t)dt.$$

10. Solve the following singular integral equation by using the Laplace transform method,

$$\int_0^x \frac{u(t)}{\sqrt{x-t}}dt = 1 + x + x^2.$$