INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 28

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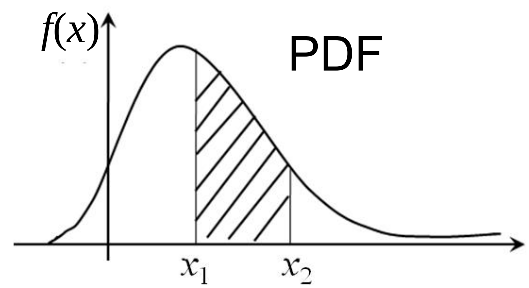
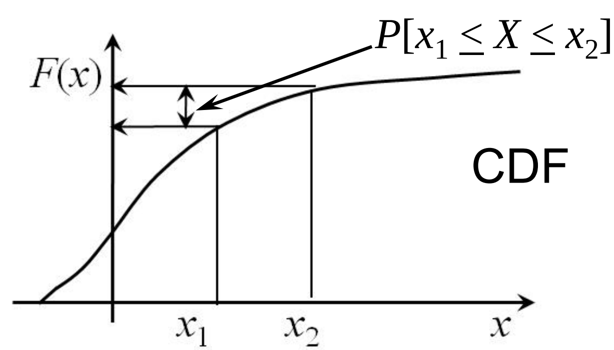
Summary of the previous lecture

- Reservoir systems – Random inflows
    - Implicit Stochastic Optimization (ISO)
    - Explicit Stochastic Optimization (ESO)
- Basic probability theory
    - Random variable
    - Discrete rv; Continuous rv
    - PMF, PDF, CDF
    - Expected value, variance, standard deviation and coefficient of variation





NPTEL

- Stochastic Hydrology

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = P[X \le x]$$

Example – 1

Consider the pdf

$$f(x) = 3x^2 \qquad 0 \leq x \leq 1$$
$$= 0 \qquad \text{elsewhere}$$

$f(x) \geq 0$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

Obtain

1. $E(X)$
2. $E(X^2)$
3. $Var(X)$

$\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

$\sigma^2 = \int_{-\infty}^{\infty} \underline{(x-\mu)^2} f(x)\,dx$

Example – 1 (Contd.)

1. $$E\left(X\right)=\int_{-\infty}^{\infty} xf(x)dx=\int_{0}^{1} x.3x^2dx$$

$$=3\left[\frac{x^4}{4}\right]_0^1=\frac{3}{4}$$ μ

2. $$E\left(X^2\right)=\int_{-\infty}^{\infty} x^2 f(x)dx=\int_{0}^{1} x^2.3x^2dx$$

$$=3\left[\frac{x^5}{5}\right]_0^1=\frac{3}{5}$$

Example – 1 (Contd.)

3. $$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \, f(x) dx$$

$$\mu_n = \int_{-\infty}^{\infty} (x-\mu)^n f(x) \, dx$$

$$= \int_0^1 \left(x - \frac{3}{4} \right)^2 3x^2 dx$$

$$= \int_0^1 \left(x^2 + \frac{9}{16} - \frac{3x}{2} \right) 3x^2 dx = \int_0^1 \left(3x^4 + \frac{27x^2}{16} - \frac{9x^3}{2} \right) dx$$

$$= \left[\frac{3x^5}{5} + \frac{27x^3}{48} - \frac{9x^4}{8} \right]_0^1 = \frac{3}{5} + \frac{27}{48} - \frac{9}{8} = \frac{3}{80}$$

Example – 2

Obtain the sample estimates of mean and standard deviation, for the following observed data of annual stream flow for 15 years.

Year	1	2	3	4	5	6	7	8	9	10
Annual stream flow (Mm^3)	150	129	160	152	165	138	149	115	97	154

Year	11	12	13	14	15
Annual stream flow (Mm^3)	168	110	108	105	125

Example – 2 (Contd.)

Mean, $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

$\sum_{i=1}^{n} x_i$ = 150+129+160+152+165+138+149+115+97+154+ 168+110+108+105+125

= 2025

Therefore mean, $\bar{x}$ = 2025/15

= 135 Mm^3

Variance, $s^2 = \dfrac{\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}{n-1}$

Example – 2 (Contd.)

Variance, $s^2 = \frac{7928}{15-1} = 566$

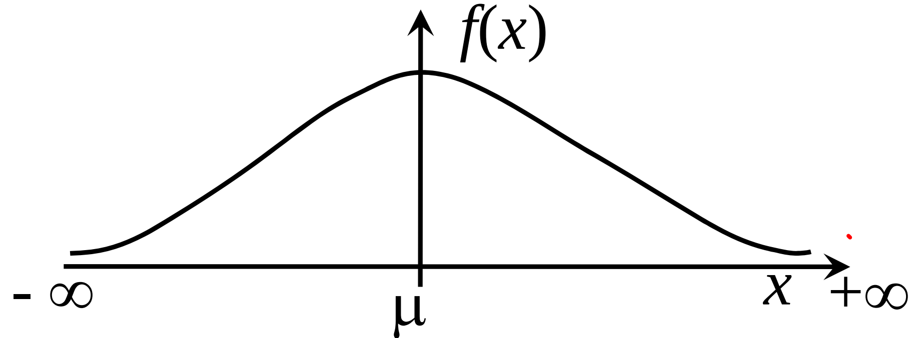
Standard deviation,
$S = +\sqrt{s^2} = 23.8$ Mm³

Coefficient of variation,
$C_v = 23.8/135 = 0.176$

$C.v = S/\bar{x}$

Year	Avg. Stream flow Mm³(x_i)	$(x_i - \overline{x})$	$(x_i - \overline{x})^2$
1	150	15	225
2	129	-6	36
3	160	25	625
4	152	17	289
5	165	30	900
6	138	3	9
7	149	14	196
8	115	-20	400
9	97	-38	1444
10	154	19	361
11	168	33	1089
12	110	-25	625
13	108	-27	729
14	105	-30	900
15	125	-10	100
Σ	**2025**	**0**	**7928**

Normal Distribution



Normal Distribution:

PDF

Symmetric about x = μ

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right\} \qquad -\infty < x < +\infty$$

Bell-shaped; Gaussian Distribution,

Two parameters, μ & σ

$X \sim N(\mu, \sigma^2)$

$f(x)$ approaches zero as $x \to \pm\infty$

e.g. $P[X < 30]$

CDF

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad -\infty < x < +\infty$$

μ: 30

$\sigma = 15$

Normal Distribution

$$Z = \frac{X-\mu}{\sigma}$$

-- Linear function

$Y = a + bX$

$Y \sim N(a+b\mu,\ b^2\sigma^2)$

$$a = \frac{-\mu}{\sigma}, b = \frac{1}{\sigma}$$

$$Z \sim N\left[\frac{-\mu}{\sigma}+\frac{\mu}{\sigma}, \frac{1}{\sigma^2}\times\sigma^2\right]$$

$$\sim N(0,1)$$

Z = Standard Normal Distribution

$X \sim N(\mu, \sigma^2)$

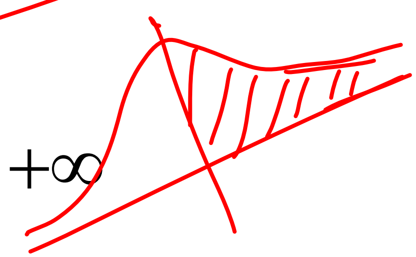
pdf of z

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < +\infty$$

cdf of z

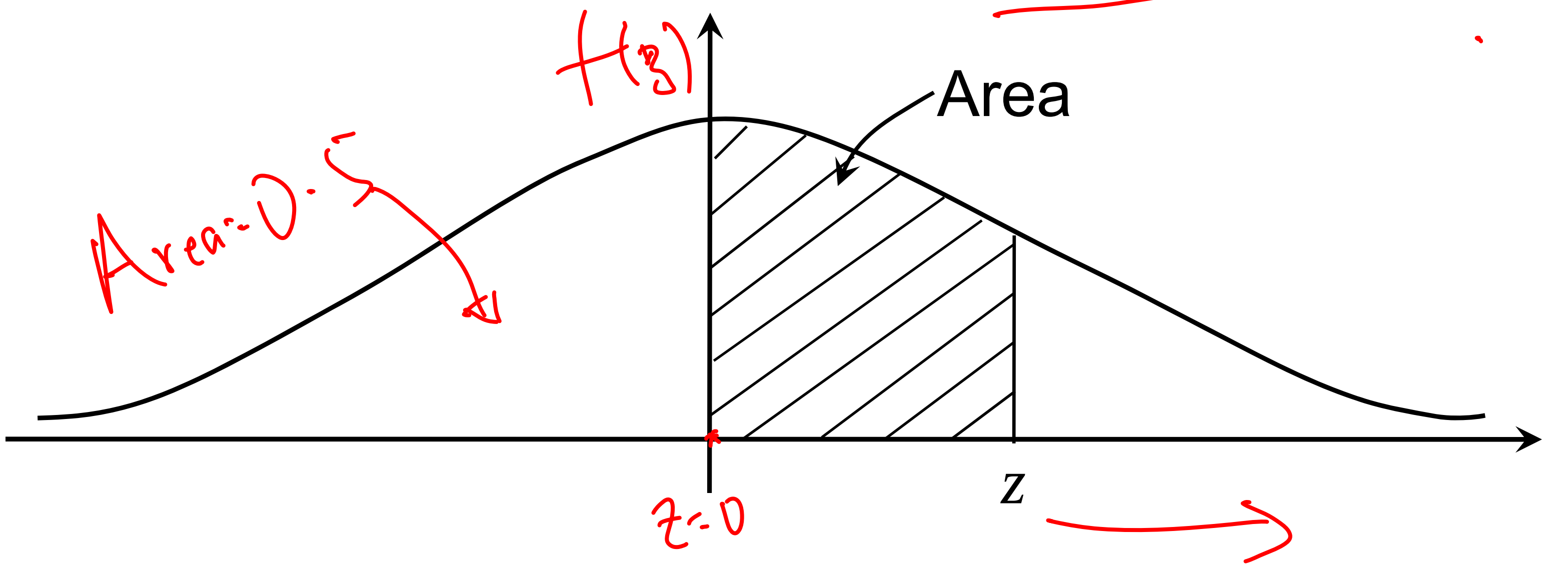
Tabulated.

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2} dz \qquad -\infty < z < +\infty$$



Normal Distribution

Obtaining standard variate 'z' using tables:



$P[Z \leq z]$ = 0.5+Area from table, for positive values of z;

Use symmetry, for negative values of z

Normal Distribution Tables

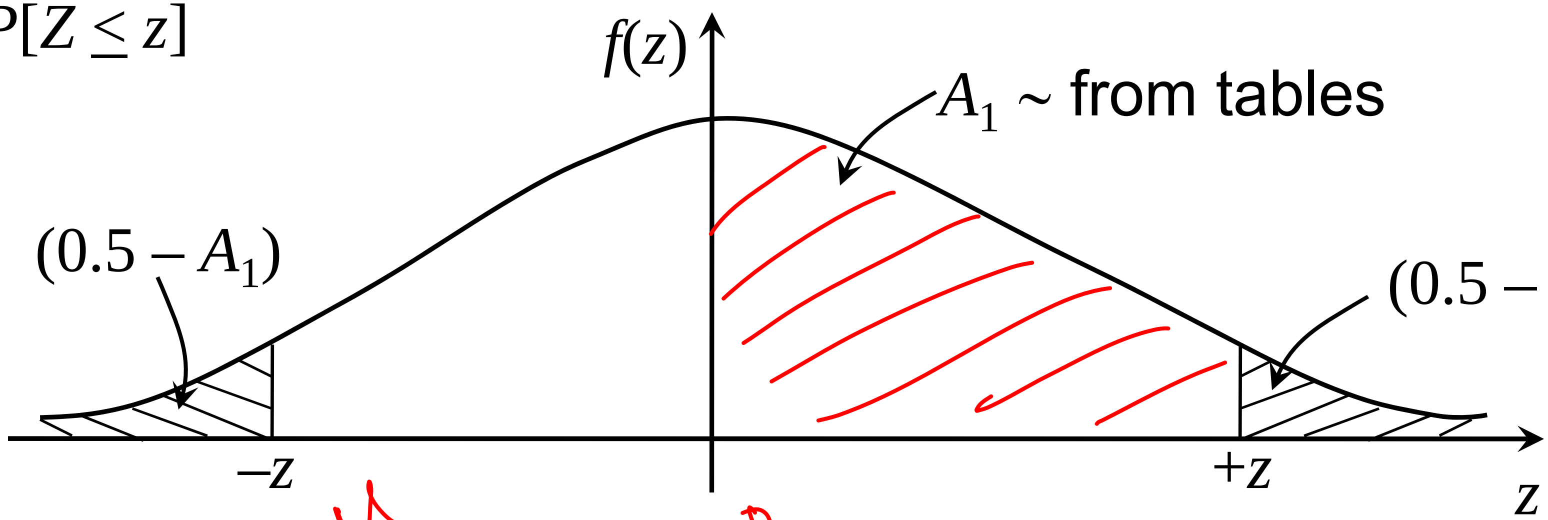
z	0	2	4	6	8
0	0	0.008	0.016	0.0239	0.0319
0.1	0.0398	0.0478	0.0557	0.0636	0.0714
0.2	0.0793	0.0871	0.0948	0.1026	0.1103
0.3	0.1179	0.1255	0.1331	0.1406	0.148
0.4	0.1554	0.1628	0.17	0.1772	0.1844
0.5	0.1915	0.1985	0.2054	0.2123	0.219
0.6	0.2257	0.2324	0.2389	0.2454	0.2517
0.7	0.258	0.2642	0.2704	0.2764	0.2823
0.8	0.2881	0.2939	0.2995	0.3051	0.3106
0.9	0.3159	0.3212	0.3264	0.3315	0.3365
1	0.3413	0.3461	0.3508	0.3554	0.3599

Normal Distribution Tables

z	0	2	4	6	8
3.1	0.499	0.4991	0.4992	0.4992	0.4993
3.2	0.4993	0.4994	0.4994	0.4994	0.4995
3.3	0.4995	0.4995	0.4996	0.4996	0.4996
3.4	0.4997	0.4997	0.4997	0.4997	0.4997
3.5	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5	0.5	0.5	0.5	0.5

Normal Distribution

$P[Z \leq z]$



$z = \frac{x - \mu}{\sigma}$ $\quad z = 0$

$z = \frac{x - \mu}{\sigma}$

e.g., $P[Z \leq -0.7] = 0.5 - 0.258$ ← from table

$= 0.242$

z	0
0.5	0.1915
0.6	0.2257
0.7	0.258

Example – 4

The monthly streamflow at a reservoir follows normal distribution with mean of 300 Mm^3 and standard deviation of 150 Mm^3.

Obtain,

1. The probability of monthly streamflow being greater than or equal to 450 Mm^3.
2. The probability of monthly streamflow being less than or equal to 200 Mm^3.
3. Monthly streamflow which will be exceeded with a probability of 0.9.

Let monthly streamflow be a rv 'X'

Example – 4 (Contd.)

1. The probability of monthly streamflow being greater than or equal to 450 Mm^3.

$$Z = \frac{X-\mu}{\sigma}$$

$$= \frac{450-300}{150} = 1$$

$$P[X \geq 450] = P[Z \geq 1]$$

$$= 1 - P[Z \leq 1]$$

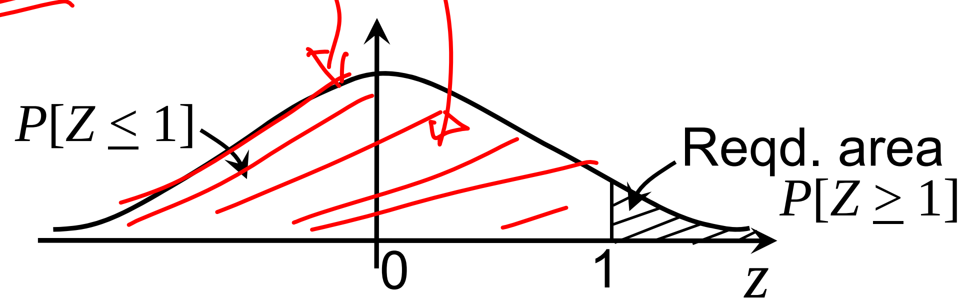
$$= 1 - (0.5 + 0.3413)$$

$$= 0.1587$$

$Z = \frac{x-\mu}{\sigma}$

$P[X \geq x] = 1 - P[X \leq x]$

$F(x)$



Example – 4 (Contd.)

2. The probability of monthly streamflow being less than or equal to 200 Mm^3.

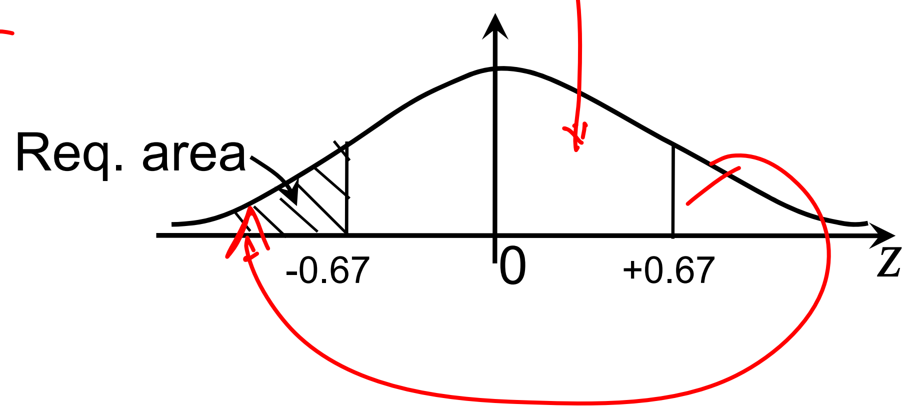
$$Z = \frac{X - \mu}{\sigma}$$

$$= \frac{200 - 300}{150} = -0.67$$

$$P[X \le 200] = P[Z \le -0.67]$$

$$= 0.5 - 0.2486$$

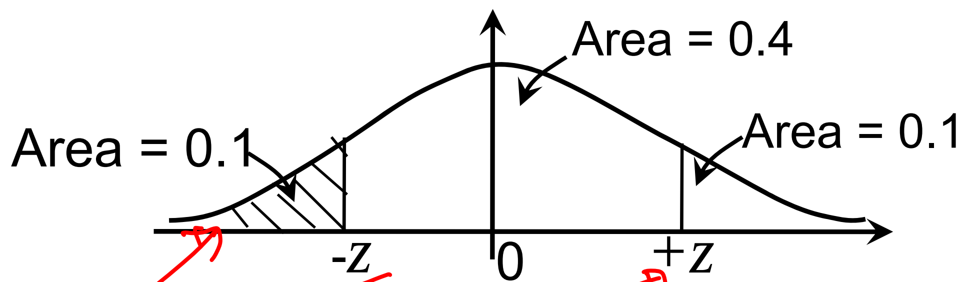
$$= 0.2514$$



Example – 4 (Contd.)

3. Monthly streamflow which will be exceeded with a probability of 0.9.

$$P[X \geq x] = 0.9$$



$$P[Z \geq z] = 0.9$$

$$1 - P[Z \leq z] = 0.9$$

z	8	9
1.1	0.381	0.383
1.2	0.3997	0.4015
1.3	0.4162	0.4177

$$P[Z \leq z] = 0.1$$

area between 0 to $-z = 0.5 - 0.1 = 0.4$

From the table, corresponding to area of 0.4, $-z = 1.28$

$z = -1.28$

Example – 4 (Contd.)

$$z = \frac{x - \mu}{\sigma}$$

$$-1.28 = \frac{x - 300}{150}$$

$$x = 108\ Mm^3$$

Lognormal Distribution

Lognormal Distribution:

'X' is said to be log-normally distributed if $Y = \ln X$ is normally distributed

PDF

$$f(x) = \frac{1}{\sqrt{2\pi}\, x\sigma_y} e^{-(\ln x - \mu_y)^2 / 2\sigma_y^2} \qquad 0 < x < \infty, 0 < \mu_y < \infty, \sigma_y > 0$$

The parameters of $Y = \ln X$ may be estimated from

$y = \ln X$

$$\left.\begin{aligned} \mu_y &= \frac{1}{2} \ln\left[\frac{\bar{x}^2}{1 + C_v^2}\right] \\ \sigma_y^2 &= \ln\left[1 + C_v^2\right] \end{aligned}\right\} \quad where \quad C_v = \frac{S_x}{\bar{x}}$$

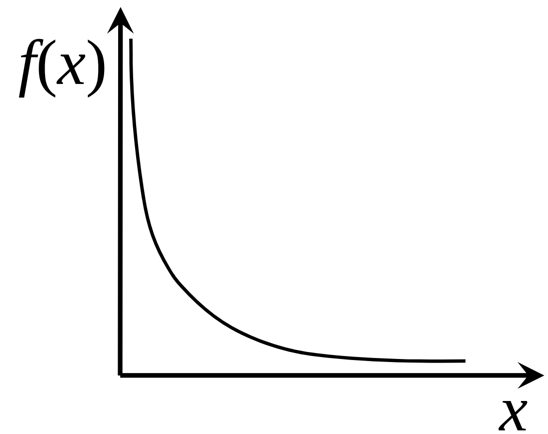
Chow et al (1988) "Applied Hydrology"

Exponential Distribution

Exponential Distribution:

PDF

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0, \lambda > 0$$



CDF

$$F(x) = \int_0^x f(x)\,dx = 1 - e^{-\lambda x} \qquad x > 0, \lambda > 0$$

$F(x) = P[X \le x]$

$E[X] = 1/\lambda$

$\lambda = 1/\mu$

$Var(X) = 1/\lambda^2$

Example – 5

The annual peak flow at a location is assumed to follow exponential distribution with mean 1000 Mm^3. Obtain the peak flow which has an exceedence probability of 0.8.

$\mu = 1000$

Let annual peak flow be the rv 'X'

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0, \lambda > 0$$

$$F(x) = 1 - e^{-\lambda x} \qquad x > 0, \lambda > 0$$

$P[X \le x]$

$$\lambda = \frac{1}{\mu} = \frac{1}{1000}$$

Example – 5 (Contd.)

$$P[X \geq x] = 0.8$$

$$\lambda = \frac{1}{\mu} = \frac{1}{1000}$$

$$1 - P[X \leq x] = 0.8$$

$$1 - F(x) = 0.8$$

$$1 - \left(1 - e^{-\lambda x}\right) = 0.8$$

$$e^{-\lambda x} = 0.8$$

$$-\lambda x = \ln(0.8) = -0.223$$

$$x = \frac{0.223}{\lambda} = 0.223 \times 1000 = 223 \;\; Mm^3$$