INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 15

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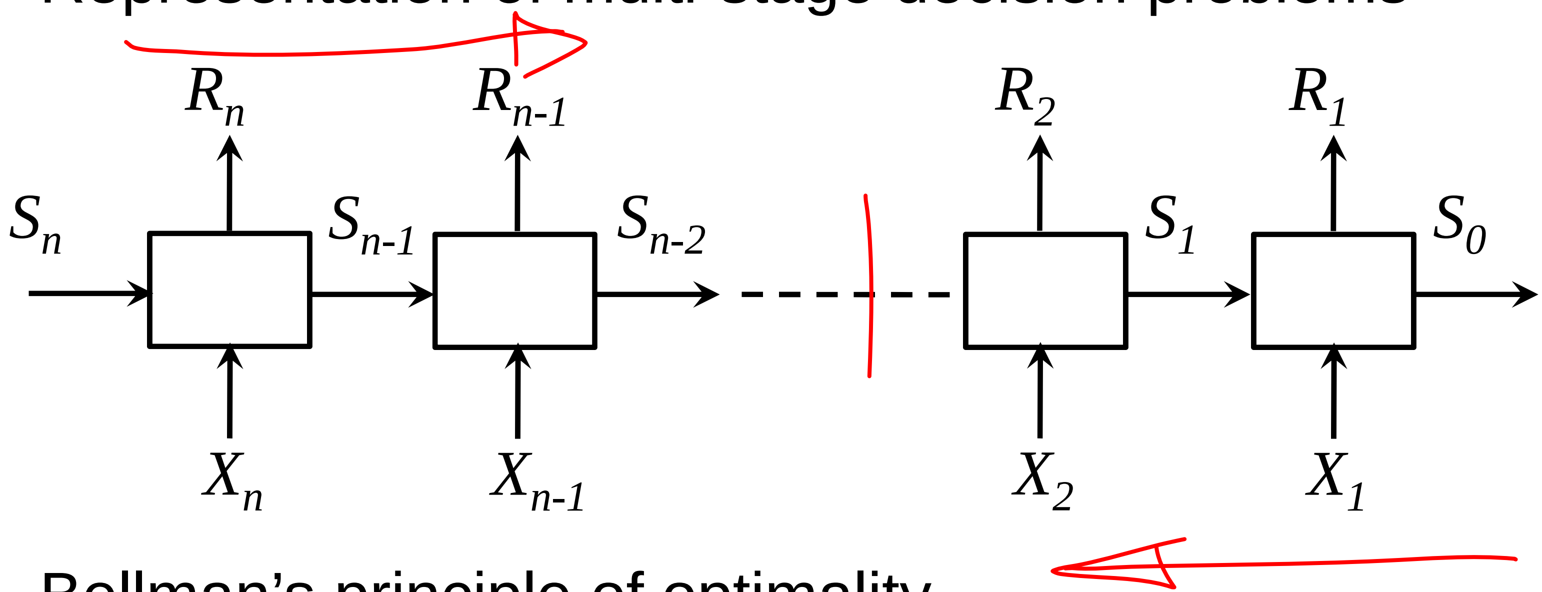
Summary of the previous lecture

- Introduction to Dynamic Programming
- Representation of multi-stage decision problems



- Bellman's principle of optimality

Dynamic Programming

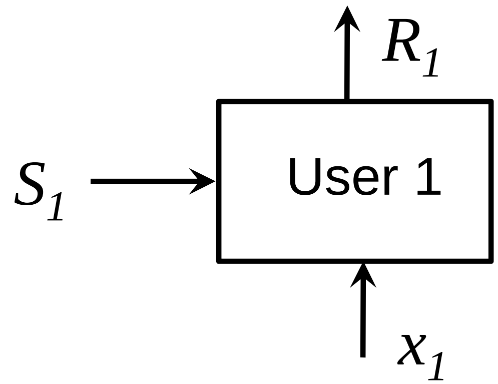
Water allocation problem:

- A total of 6 units of water is to be allocated optimally to three users, User 1, User 2 and User 3.

Amount of water allocated x	Return from		
	User 3 $R_3(x)$	User 2 $R_2(x)$	User 1 $R_1(x)$
0	0	0	0
1	5	5	7
2	8	6	12
3	9	3	15
4	8	-4	16
5	5	-15	15
6	0	-30	12

Dynamic Programming

Stage 1:



$$f_1^*(S_1) = \text{Max}\left[R_1(x_1)\right]$$

$$0 \le x_1 \le S_1$$

$$0 \le S_1 \le Q$$

S_1 : Amount of water available for allocation to User 1

x_1 : Amount of water allocated to User 1

x_1^* : Allocation to User 1, that results in $f_1^*(S_1)$

$f_1^*(S_1)$: Maximum return due to allocation of S_1

Dynamic Programming

S_1	x_1	$R_1(x_1)$	$f_1^*(S_1) = \text{Max}\left[R_1(x_1)\right]$	x_1^*
0	0	0	0	0
1	0 1	0 7	7	1
2	0 1 2	0 7 12	12	2
3	0 1 2 3	0 7 12 15	15	3

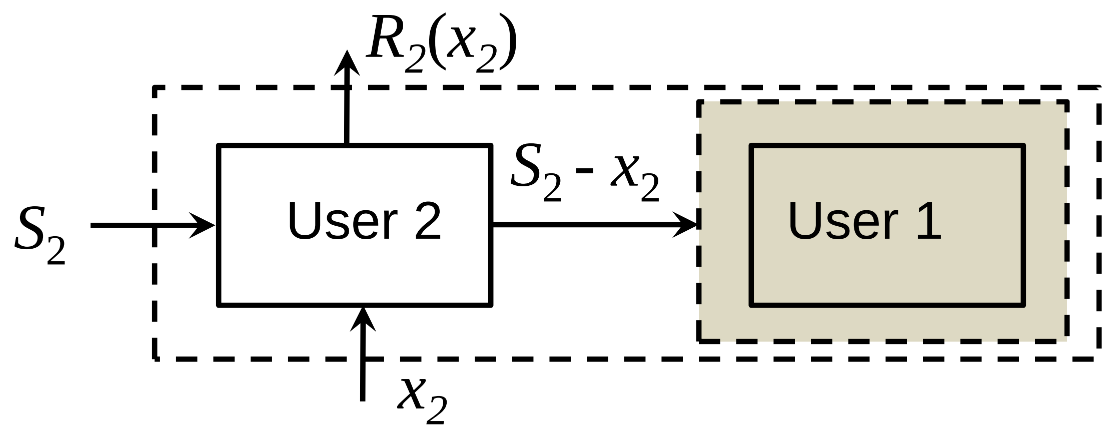
Contd.

Contd.

S_1	x_1	$R_1(x_1)$	$f_1^*(S_1) = \text{Max}\left[R_1(x_1)\right]$	x_1^*
4	0	0	16	4
	1	7		
	2	12		
	3	15		
	4	16		
5	0	0	16	4
	1	7		
	2	12		
	3	15		
	4	16		
	5	15		
6	0	0	16	4
	1	7		
	2	12		
	3	15		
	4	16		
	5	15		
	6	12		

Dynamic Programming

Stage 2:



$$f_2^*(S_2) = \text{Max}\left[R_2(x_2) + f_1^*(S_2 - x_2)\right]$$

the optimized value from previous stage.

$$0 \leq x_2 \leq S_2$$

$$0 \leq S_2 \leq Q$$

S_2 : Amount of water available for allocation to User 2 and User 1 together

x_2 : Amount of water allocated to User 2

Dynamic Programming

$S_2 - x_2$: Amount of water available for allocation at stage 1 (to User 1)

$f_2^*(S_2)$: Maximum return due to allocation of S_2

x_2^* : Allocation to User 2, that results in $f_2^*(S_2)$

Dynamic Programming

Stage 2

$$f_2^*(S_2) = \text{Max}\left[R_2(x_2) + f_1^*(S_2 - x_2)\right]$$

$$0 \le x_2 \le S_2 \; ; \; 0 \le S_2 \le Q$$

S_2	x_2	$R_2(x_2)$	$S_2 - x_2$	$f_1^*(S_2 - x_2)$	$R_2(x_2) + f_1^*(S_2 - x_2)$	$f_2^*(S_2)$	x_2^*
0	0	0	0	0	0	0	0
1	0	0	1	7	7	7	0
	1	5	0	0	5		
2	0	0	2	12	12	12	0, 1
	1	5	1	7	12		
	2	6	0	0	6		
3	0	0	3	15	15	17	1
	1	5	2	12	17		
	2	6	1	7	13		
	3	3	0	0	3		

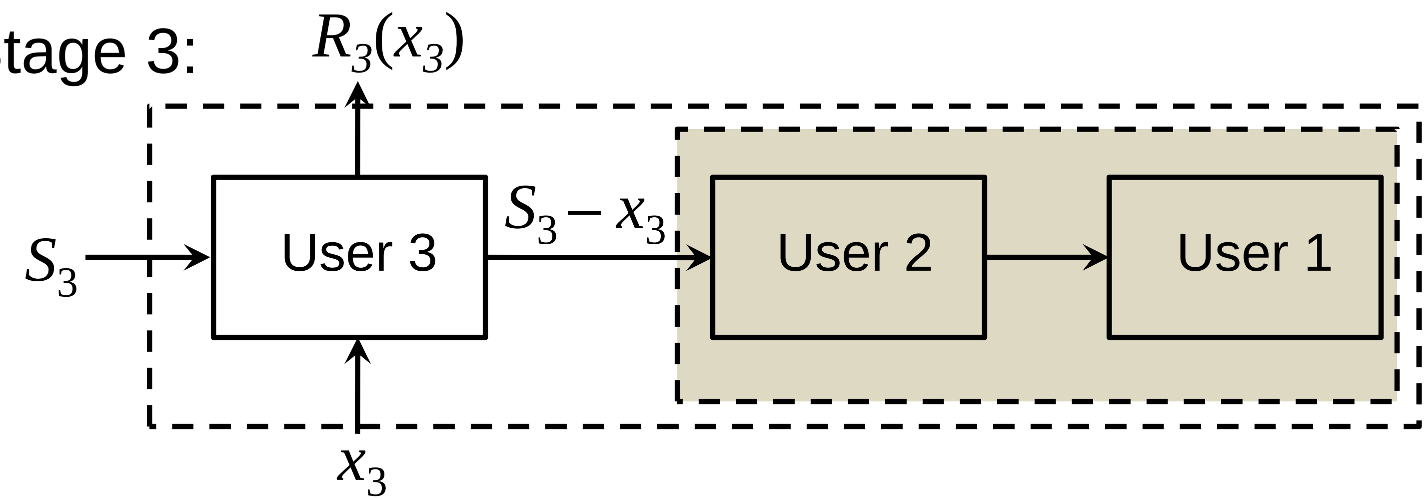
Contd.

Contd.

S_2	x_2	$R_2(x_2)$	$S_2 - x_2$	$f_1^*\left(S_2 - x_2\right)$	$R_2\left(x_2\right) + f_1^*\left(S_2 - x_2\right)$	$f_2^*\left(S_2\right)$	x_2^*
4	0	0	4	16	16	20	1
	1	5	3	15	20		
	2	6	2	12	19		
	3	3	1	7	10		
	4	-4	0	0	-4		
5	0	0	5	16	16	21	1, 2
	1	5	4	16	21		
	2	6	3	15	21		
	3	3	2	12	15		
	4	-4	1	7	3		
	5	-15	0	0	-15		
6	0	0	6	16	16	22	2
	1	5	5	16	21		
	2	6	4	16	22		
	3	3	3	15	18		
	4	-4	2	12	8		
	5	-15	1	7	-8		
	6	-30	0	0	-30		

Dynamic Programming

Stage 3:



$$f_3^*\left(S_3\right) = \text{Max}\left[R_3\left(x_3\right) + f_2^*\left(S_3 - x_3\right)\right]$$

$$0 \leq x_3 \leq S_3$$

$$S_3 = Q$$

Dynamic Programming

S_3 : Amount of water available for allocation to User 1, User 2 and User 3 together = 6 units

x_3 : Amount of water allocated to User 3

$S_3 - x_3$: Amount of water available for allocation at stage 2 (to User 1 and User 2 together)

$f_3^*(S_3)$: Maximum return due to allocation of S_3

x_3^* : Allocation to User 3, that results in $f_3^*(S_3)$

Dynamic Programming

Stage 3

$$f_3^*(S_3) = \text{Max}\left[R_3(x_3) + f_2^*(S_3 - x_3)\right]$$

$$0 \le x_3 \le S_3$$

$$S_3 = Q$$

S_3	x_3	$R_3(x_3)$	$S_3 - x_3$	$f_2^*(S_3 - x_3)$	$R_3(x_3) + f_2^*(S_3 - x_3)$	$f_3^*(S_3)$	x_3^*
	0	0	6	22	22		
	1	5	5	21	26		
	2	8	4	20	28	Max	
6	3	9	3	17	26	28	2
	4	8	2	12	20		
	5	5	1	5	10		
	6	0	0	0	0		

Dynamic Programming

- When the third stage is solved, all the three users are considered for allocation; thus the total maximum return is

$$f_3^*(6) = 28$$

- The allocations to individual users are traced back.

From the table for stage 3,

$$x_3^* = 2$$

From this the water available for stage 2 is obtained as,

$$S_2 = Q - x_3^* = 6 - 2 = 4$$

Dynamic Programming

From the table for stage 2, with the value of S_2 = 4 ,

$$x_2^* = 1$$

From this the amount of water available for allocation at stage 1 is obtained as,

$$S_1 = S_2 - x_2^* = 4 - 1 = 3$$

From the table for stage 1, with the value of S_1 = 3 ,

$$x_1^* = 3$$

Dynamic Programming

Thus the optimal allocations are

x_1^* = Allocation to User 1 = 3 units

x_2^* = Allocation to User 2 = 1 units

x_3^* = Allocation to User 3 = 2 units

Maximum return resulting from the allocations = 28

Dynamic Programming

Characteristics of DP problem:

- Problem is divided into stages, with a policy decision required at each stage.
- Each stage has a number of possible states associated with it.

  e.g., In the allocation problem, the amount of water available for allocation at a stage defines the state at that stage.

Dynamic Programming

- Policy decision transforms the current state into a state associated with the next stage.

State Transformation : $T(S_n , x_n)$

S_n : Current state in stage n

x_n: Decision in stage n

In allocation problem,

the transformation function is, $(S_n - x_n)$

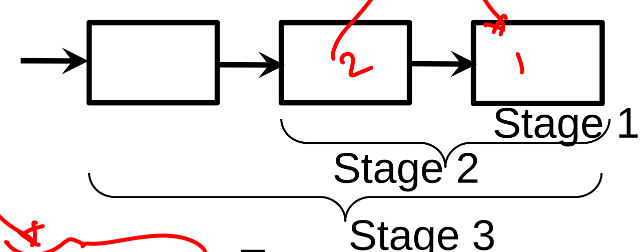
Dynamic Programming

- Recursive relationship identifies the optimal decision at stage n, for state S_n, given the optimal decisions for each state at stage $(n-1)$.

$$f_n(S_n) = \underset{\{x_n\}}{\text{Max}}\left[r_n(x_n) + f_{n-1}\{T(S_n, x_n)\}\right]$$

$r_n(x_n)$: Return for decision x_n in current state

$T(S_n, x_n)$: Transfer function to get the state S_{n-1} corresponding to S_n and x_n.



In allocation problem,

$$f_n(S_n) = \underset{\{x_n\}}{\text{Max}}\left[r_n(x_n) + f_{n-1}(S_n - x_n)\right]$$

Dynamic Programming

- Solution moves backward (or forward) stage by stage, till optimal decisions for the last stage are found.
- Optimal decisions for other stages are traced back.

Objective Function Must be Separable

Separable

$$x_1^2 + x_2^2 + \dots + x_n^2$$

Not Separable

$$x_1 x_2 + x_2 x_3 + x_3 x_4 + \dots$$