INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 10

Course Instructor : Prof. P. P. MUJUMDAR

Department of Civil Engg., IISc.

Summary of the previous lecture

- Motivation for the simplex method
- Algebraic approach – Simplex algorithm

$$\text{Basis} = \left[\begin{array}{c} m \\ \text{basic} \\ \text{variables} \end{array} \right]$$

$\underline{\underline{(n-m)}}$ → Set to zero

→ non basic variables

✓ $\underline{\underline{m}}$ → basic variables

Example – 1

Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$\left.\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \end{aligned}\right\} \quad \text{Constraints}$$

$$\left.\begin{aligned} x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}\right\} \quad \text{Non-negativity of decision variables}$$

Example – 1 (Contd.)

Iteration-1

Tableau-1

$$Z - 3x_1 - 5x_2 - 0x_3 - 0x_4 - 0x_5 = 0$$ …. Row 0

$$x_1 + x_3 = 4$$ …. Row 1

$$2x_2 + x_4 = 12$$ …. Row 2

$$3x_1 + 2x_2 + x_5 = 18$$ …. Row 3

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	-3	-5	0	0	0	0
x_3	1	0	1	0	1	0	0	4
x_4	2	0	0	2	0	1	0	12
x_5	3	0	3	2	0	0	1	18

Example – 1 (Contd.)

Decisions to be made at each iteration:

- Whether the current solution is optimal?
  - The solution is optimal only if all the coefficients in the Z-row are non-negative
- If the solution is not optimal:
  - Which is the entering variable?
    - The variable with the highest negative coefficient in Z-row
  - Which is the departing variable?
    - The variable with the minimum b_i/a_{ij} value

Calculated only if both b_i and a_{ij} are +ve

i is the row, j is the entering variable

Example – 1 (Contd.)

Iteration-1 (Tableau-1)

Entering variable

Departing variable

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i	b_i/a_{ij}
Z	0	1	-3	-5	0	0	0	0	–
x_3	1	0	1	0	1	0	0	4	–
x_4	2	0	0	2	0	1	0	12	6
x_5	3	0	3	2	0	0	1	18	9

Pivotal column

Pivotal row

Pivot point

Example – 1 (Contd.)

Table for the next iteration:

- Change the basis by replacing the departing variable with the entering variable.
- Divide the pivotal row throughout by the pivot.
- Get all other numbers in the new table by

$$\widehat{E} = E - \frac{R \times C}{P}$$

$\widehat{E}$: new value
E : old value
R : number in the pivotal column in the row of E
C : number in the pivotal row in the column of E
P : pivot

Example – 1 (Contd.)

For example,

- New value for b_i in the Z-row

E : old value =0
R : number in the pivotal column in the row of E = -5
C : number in the pivotal row in the column of E = 12
P : pivot =2

$$\widehat{E} = E - \frac{R \times C}{P}$$

$$= 0 - \frac{(-5)\times 12}{2}$$

$$= 30$$

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	-3	-5	0	0	0	0
x_3	1	0	1	0	1	0	0	4
x_4	2	0	0	2 ✓	0	1	0	12 ✓
x_5	3	0	3	2 ✓	0	0	1	18

$$18 - \frac{12\times 2}{2}$$

$$= 6$$

$$-5 - \frac{(-5)\times 2}{2}$$

Example – 1 (Contd.)

Iteration-2 (Tableau-2)

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	-3	0	0	5/2	0	30
x_3	1	0	1	0	1	0	0	4
x_2	2	0	0	1	0	1/2	0	6
x_5	3	0	3	0	0	-1	1	6

Example – 1 (Contd.)

Iteration-2 (Tableau-2)

Entering variable

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i	b_i/a_{ij}
Z	0	1	-3	0	0	5/2	0	30	–
x_3	1	0	1	0	1	0	0	4	4
x_2	2	0	0	1	0	1/2	0	6	–
x_5	3	0	3	0	0	-1	1	6	2

Departing variable

Pivot point

Example – 1 (Contd.)

Iteration-3 (Tableau-3)

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	0	0	0	3/2	1	36
x_3	1	0	0	0	1	1/3	-1/3	2
x_2	2	0	0	1	0	1/2	0	6
x_1	3	0	1	0	0	-1/3	1/3	2

Example – 1 (Contd.)

Since all coefficients in the Z-row are non-negative this is the optimal solution.

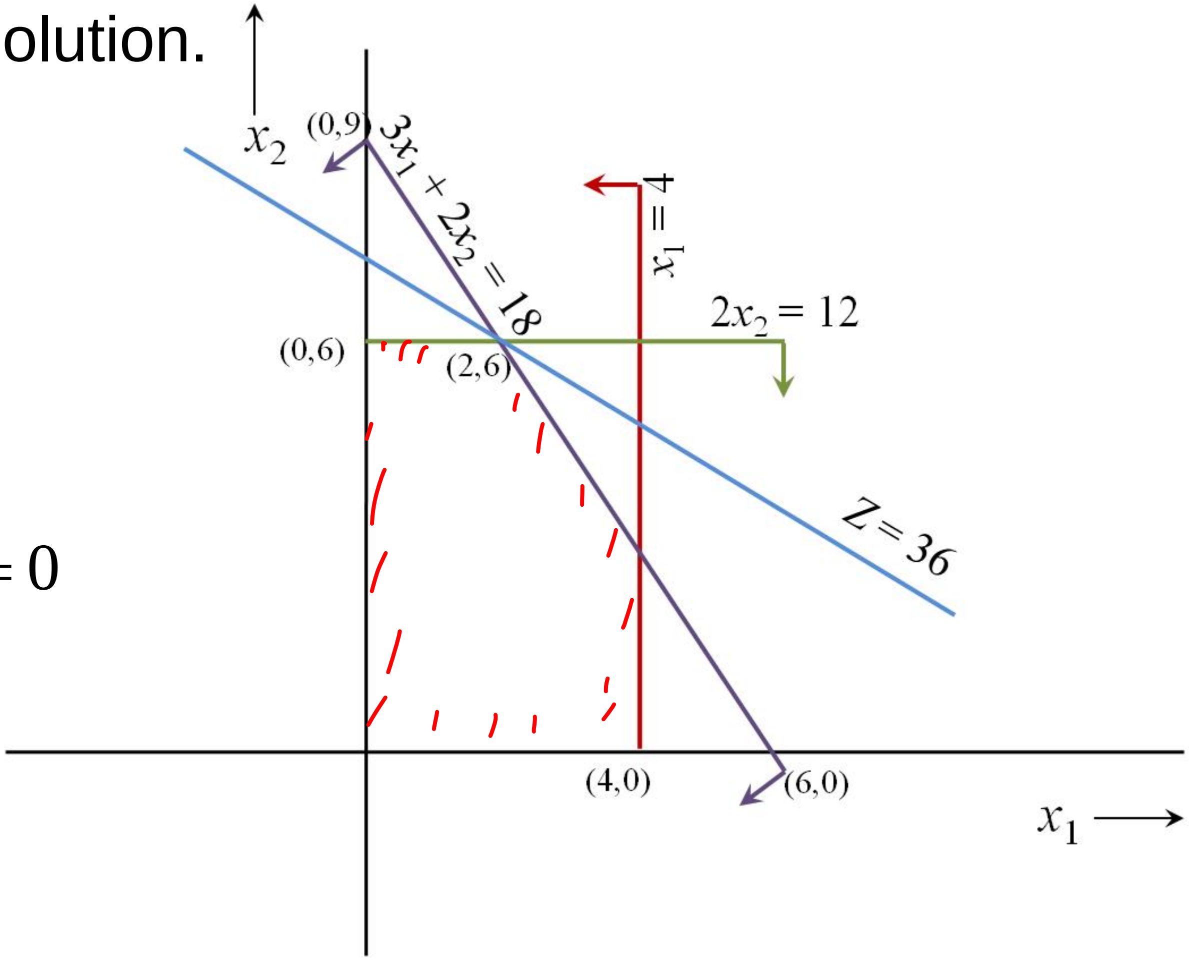
$$Z = 36$$

$$x_1 = 2$$

$$x_2 = 6$$

$$x_3 = 2;$$

$$x_4 = 0; \qquad x_5 = 0$$



LP – Simplex Method

- Tie breaking for entering variable
  - Two variables having highest negative coefficient – choose anyone arbitrarily
- Tie for departing variable
  - May lead to degeneracy
  - A degenerate solution is one in which at least one of the basic variables has zero value.
  - Degeneracy may indicate presence of redundant constraints.
  - Results in the same sequence of iterations without improving the value of objective function and without terminating the computations.

LP – Simplex Method

- In any iteration,
    - All basic variables have a coefficient of zero in the Z-row
    - Coefficients of basic variables constitute a unit matrix

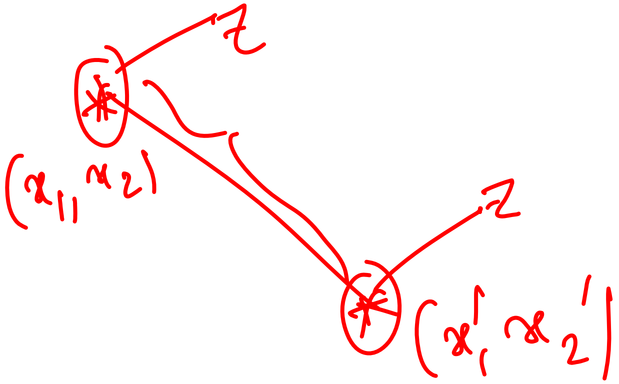
$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

LP – Multiple Solutions

Multiple solutions:

- One of the non-basic variables in the final table has a coefficient of zero in the Z-row
- When two optimal solutions X_1 and X_2 are obtained,

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} \qquad X_2 = \begin{bmatrix} x_1' \\ x_2' \\ . \\ . \\ x_n' \end{bmatrix}$$



$X^* = X_1 + (1-\alpha)X_2$ is also an optimal solution

$0 \leq \alpha \leq 1$ …. Infinite no. of solutions