INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 9

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Summary of the previous lecture

- Graphical method
- General form of LP
- Motivation for the simplex method

LP – Simplex Method

Motivation for the Simplex method

Max $$Z = 6x_1 + 8x_2$$

s.t.

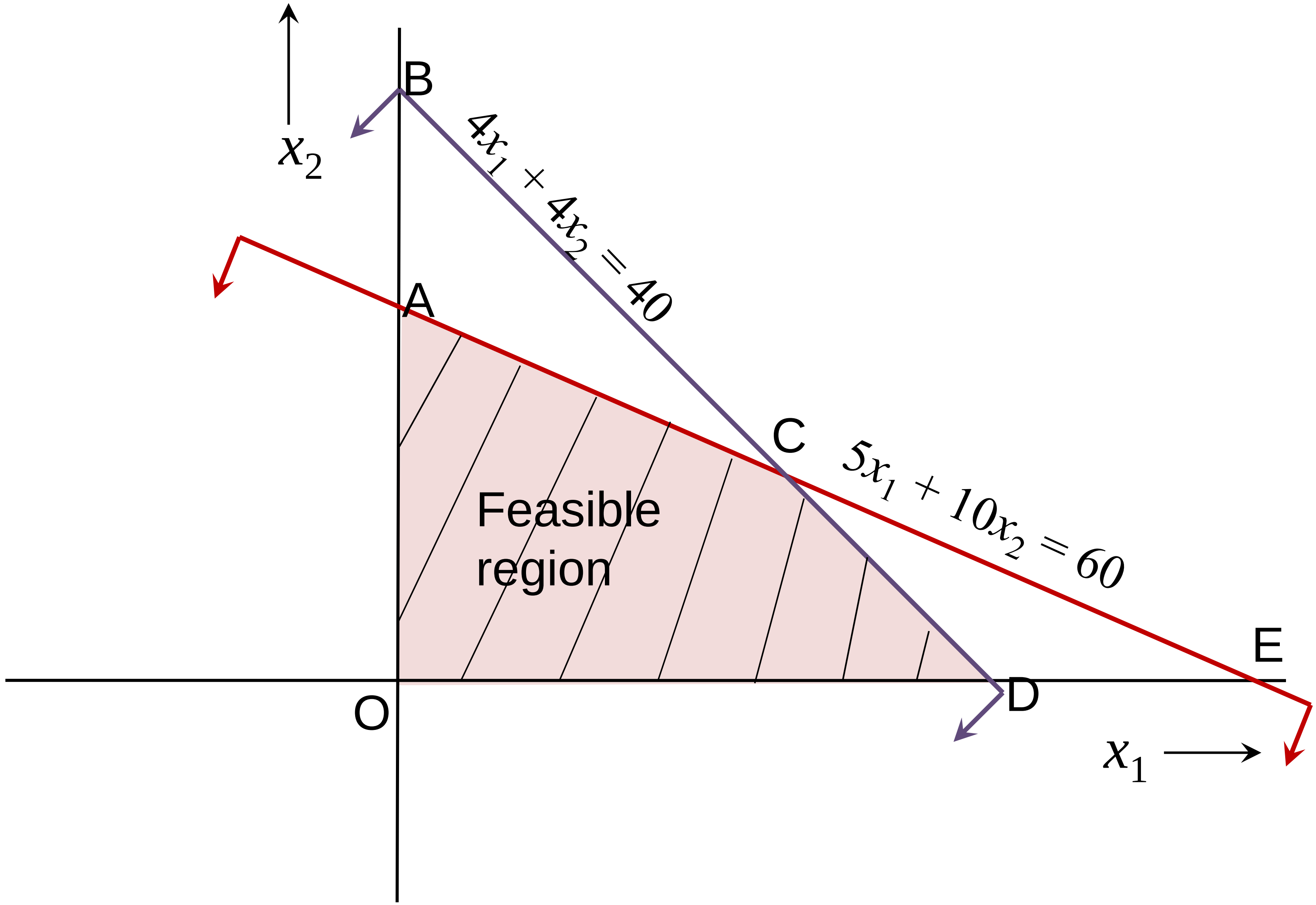
$$5x_1 + 10x_2 \leq 60$$

$$4x_1 + 4x_2 \leq 40$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

LP – Simplex Method



LP – Simplex Method

In the general form of LP, the constraints are converted as

$$5x_1 + 10x_2 \le 60 \longrightarrow 5x_1 + 10x_2 + x_3 = 60$$

$$4x_1 + 4x_2 \le 40 \longrightarrow 4x_1 + 4x_2 + x_4 = 40$$

2 equations and 4 unknowns

$$x_1 \ge 0 \qquad x_1 \ge 0; \quad x_2 \ge 0$$

$$x_2 \ge 0 \qquad x_3 \ge 0; \quad x_4 \ge 0$$

x_3, x_4 are slack variables

LP – Simplex Method

- In general, m equations in n unknowns ($n > m$), including slack and surplus variables
- It is possible to solve for m variables in terms of the other ($n - m$) variables
- Basic solution: A basic solution is a solution obtained by setting ($n - m$) variables to zero
- The m variables whose solution is sought by setting the remaining ($n - m$) variables to zero are called the basic variables
- The ($n - m$) variables which are set to zero are called the non-basic variables. It may so happen that some variables may take a value of zero arising out of the solution of the m equations, but they are not non-basic variables as they are not initialized to zero.

LP – Simplex Method

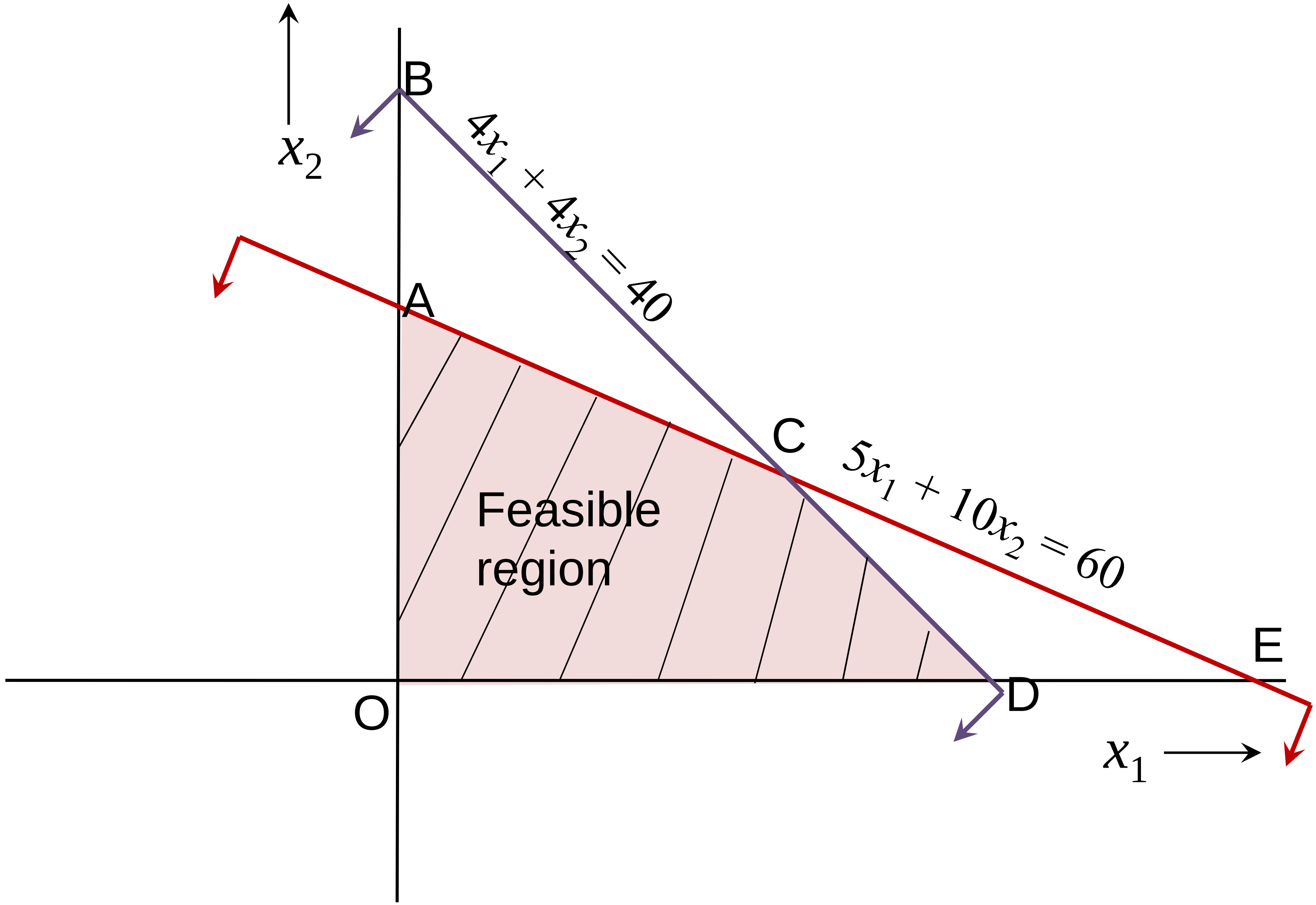
- Since out of n variables, any of the $(n - m)$ variables may be set to zero, the no. of basic solutions will correspond to the no. of ways in which m variables can be selected out of n variables i.e.,

$$nC_m = \frac{n!}{m!(n-m)!}$$

For the problem being considered, with n = 4 and m = 2, the no. of basic solutions will be

$$\frac{4!}{2!(4-2)!} = 6$$

LP – Simplex Method



LP – Simplex Method

$$5x_1 + 10x_2 + x_3 = 60$$
$$4x_1 + 4x_2 + x_4 = 40$$

- In the example, the basic solutions are

x_1	x_2	x_3	x_4	Point on graph	Feasible
0	0	60	40	O	Y
0	6	0	16	A	Y
0	10	-40	0	B	N
8	2	0	0	C	Y
10	0	10	0	D	Y
12	0	0	-8	E	N

LP – Simplex Method

- All the basic solutions need not (and, in general, will not) be feasible.

- A basic solution which is also feasible is called as the Basic Feasible Solution.

- All the corner points of the feasible space are basic feasible solutions.

LP – Simplex Method

- As the size of a LP problem increases, the no. of basic solutions also increases.
- The possible no. of basic feasible solutions can be too large to be enumerated completely.
- The goal would be, starting with an initial basic feasible solution, to generate better and better basic feasible solutions until the optimal basic feasible solution is obtained.

LP – Simplex Method

Algebraic approach (Simplex algorithm):

Maximize $Z = 6x_1 + 8x_2 + 0x_3 + 0x_4$

s.t.

$$5x_1 + 10x_2 + x_3 = 60$$

$$4x_1 + 4x_2 + x_4 = 40$$

$$x_1 \geq 0; \quad x_2 \geq 0$$

$$x_3 \geq 0; \quad x_4 \geq 0$$

- To solve the problem algebraically, we must be able to sequentially generate a set of basic feasible solutions

LP – Simplex Method

1. Determine an initial basic feasible solution:

- The best way to obtain an initial basic feasible solution would be to put all the decision variables ($n - m$ in no., if we have one slack/surplus variable associated with each constraint) to zero.
- In the example, choose the slack variables x_3 and x_4 to be basic and x_1 and x_2 to be non-basic

  (i.e., set $x_1 = 0$ and $x_2 = 0$)

Basis : $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$

$$5x_1 + 10x_2 + x_3 = 60$$
$$4x_1 + 4x_2 + x_4 = 40$$

LP – Simplex Method

- Solve the m (=2) equations for m basic variables

$$5x_1 + 10x_2 + x_3 = 60 \qquad \longrightarrow \qquad x_3 = 60$$
$$4x_1 + 4x_2 + x_4 = 40 \qquad\qquad\qquad x_4 = 40$$

Point ‘O’ on the graph

Initial basic feasible solution is $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 40 \end{bmatrix}$

Z = 0

LP – Simplex Method

- Now we have to generate another basic feasible solution from the initial basic feasible solution such that there is an improvement in the objective function value, Z.
- The new solution is obtained as follows
  - ONE presently non-basic variable (x_1 or x_2) must be selected to enter the current basis, thus becoming basic.
  - ONE presently basic variable (x_3 or x_4) must be selected to leave the current basis, thus becoming non-basic.

LP – Simplex Method

Current basis : $\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$

First operation:

$$Z = 6x_1 + 8x_2 + 0x_3 + 0x_4$$

- Either x_1 or x_2 can enter the basis.
- The question now is, out of x_1 and x_2 which variable should enter the basis?
- Since the problem is to maximize the objective function (OF), we must look for the variable which will increase the OF value the fastest.
- Because the coefficient of x_2 in the OF is higher than that of x_1, x_2 increases the OF value faster than x_1 does. Hence x_2 should enter the basis.

LP – Simplex Method

Second operation:

- As the value of newly selected basic variable x_2 is increased (x_1 still being zero), the other two variables, x_3 and x_4 (which are currently in the basis) keep reducing.
- One of these will reach its lower limit (zero) earlier than the other

$$5x_1 + 10x_2 + x_3 = 60$$

$$4x_1 + 4x_2 + x_4 = 40$$

$$x_3 = 60 - 10x_2$$

$$x_4 = 40 - 4x_2$$

- $x_3 = 0$ when $10x_2 = 60$ or $x_2 = 6$
- $x_4 = 0$ when $4x_2 = 40$ or $x_2 = 10$

LP – Simplex Method

- To increase the OF value, we would be interested in increasing x_2 to the greatest extent possible.
- As soon as x_2 reaches 6, x_3 becomes zero (x_4 is still non-zero)
- Therefore for x_2 entering the basis, x_3 must leave the basis.
- Therefore new basis is $\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$
- The new basic solution is obtained by Guass Jordan elimination method.
- The process continues until the maximum value of the function is arrived at.

Example – 1

Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$\left.\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \end{aligned}\right\} \quad \text{Constraints}$$

$$\left.\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}\right\} \quad \text{Decision variables}$$

Example – 1 (Contd.)

The problem is converted into standard LP form

Maximize $Z = 3x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5$

s.t.

$$x_1 \le 4 \longrightarrow x_1 + x_3 = 4$$

$$2x_2 \le 12 \longrightarrow 2x_2 + x_4 = 12$$

$$3x_1 + 2x_2 \le 18 \longrightarrow 3x_1 + 2x_2 + x_5 = 18$$

$$x_1 \ge 0 \qquad x_1 \ge 0; \; x_2 \ge 0$$

$$x_2 \ge 0 \qquad x_3 \ge 0; \; x_4 \ge 0; \; x_5 \ge 0$$

n = no. of variables = 5; m = no. of constraints =3

Example – 1 (Contd.)

Seek solution from m variables by setting $(n - m)$ variables to zero

x_1 and x_2 are non basic variables (whose values are set to zero)

x_3, x_4 and x_5 are basic variables (whose solution is sought)

The solution is obtained by creating a table with coefficients of variables.

Example – 1 (Contd.)

$x_1 + x_3 = 4$ — (1)

$2x_2 + x_4 = 12$ — (2)

$3x_1 + 2x_2 + x_5 = 18$ — (3)

Iteration-1

$$Z - 3x_1 - 5x_2 - 0x_3 - 0x_4 - 0x_5 = 0$$ (0)

Basis	Row	Z	x_1	x_2	x_3	x_4	x_5	b_i
Z	0	1	-3	-5	0	0	0	0
x_3	1	0	1	0	1	0	0	4
x_4	2	0	0	2	0	1	0	12
x_5	3	0	3	2	0	0	1	18