INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:

Modeling Techniques and Analysis

Lecture - 8

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Summary of the previous lecture

- Function with inequality constraints

  Minimize $f(X)$

  s.t.

  $g_j(X) \leq 0 \qquad j = 1, 2, \ldots.. m$

Kuhn – Tucker conditions:
$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots.. n$$

$$\lambda_j g_j = 0$$

$$g_j \leq 0 \qquad j = 1, 2, \ldots.. m$$

$$\lambda_j \geq 0$$

- Introduction to Linear Programming
  - Graphical method

Example – 1

Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$x_1 \le 4$$

$$2x_2 \le 12$$

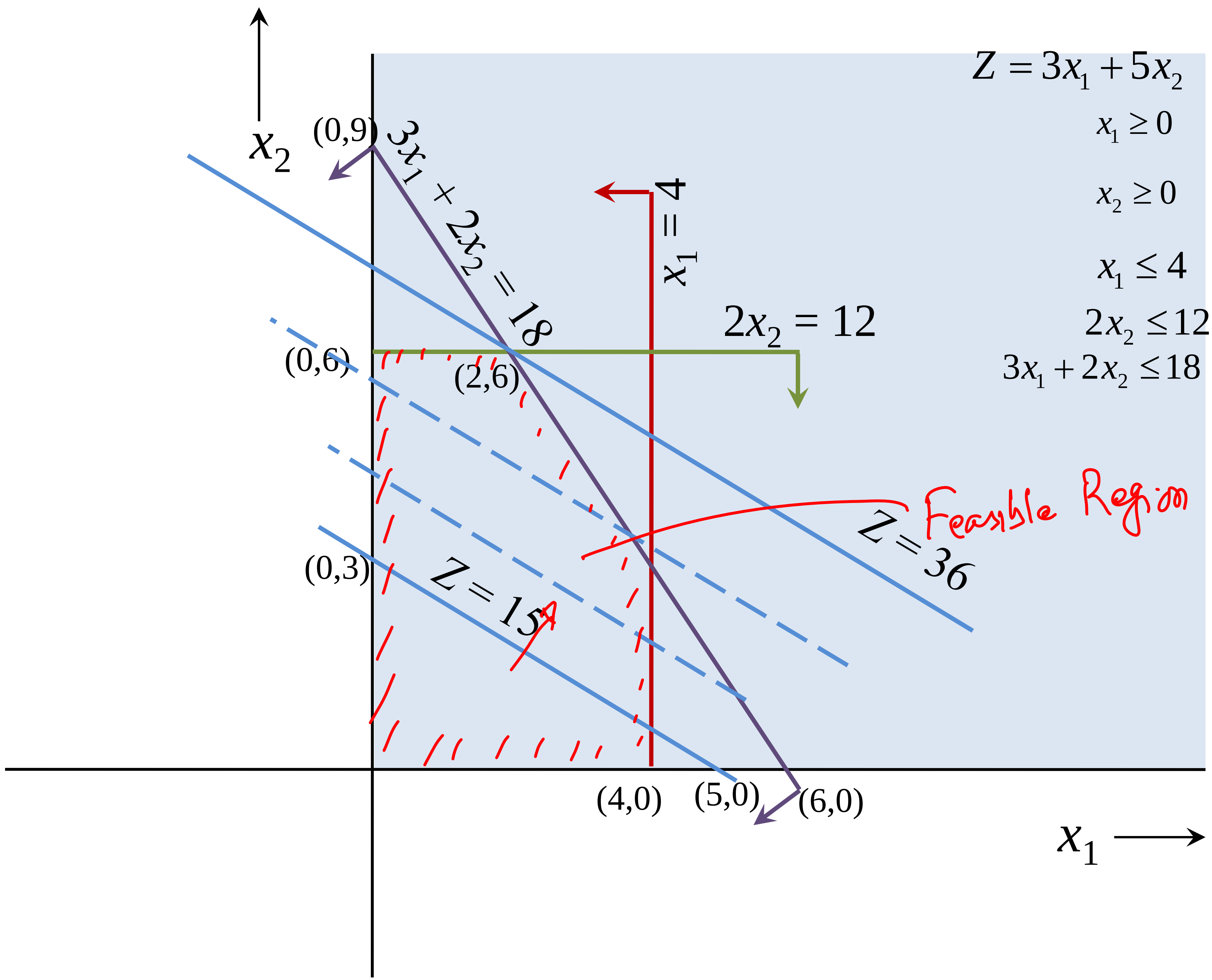
$$3x_1 + 2x_2 \le 18$$

} Constraints

$$x_1 \ge 0$$

$$x_2 \ge 0$$

} Decision variables



$Z = 3x_1 + 5x_2$
$x_1 \geq 0$
$x_2 \geq 0$
$x_1 \leq 4$
$2x_2 \leq 12$
$3x_1 + 2x_2 \leq 18$
x_2
x_1
(0,9)
(0,6)
(0,3)
(2,6)
(4,0)
(5,0)
(6,0)
$3x_1 + 2x_2 = 18$
$x_1 = 4$
$2x_2 = 12$
$Z = 36$
$Z = 15$
Feasible Region

LP – Graphical Solution

Consider

Maximize

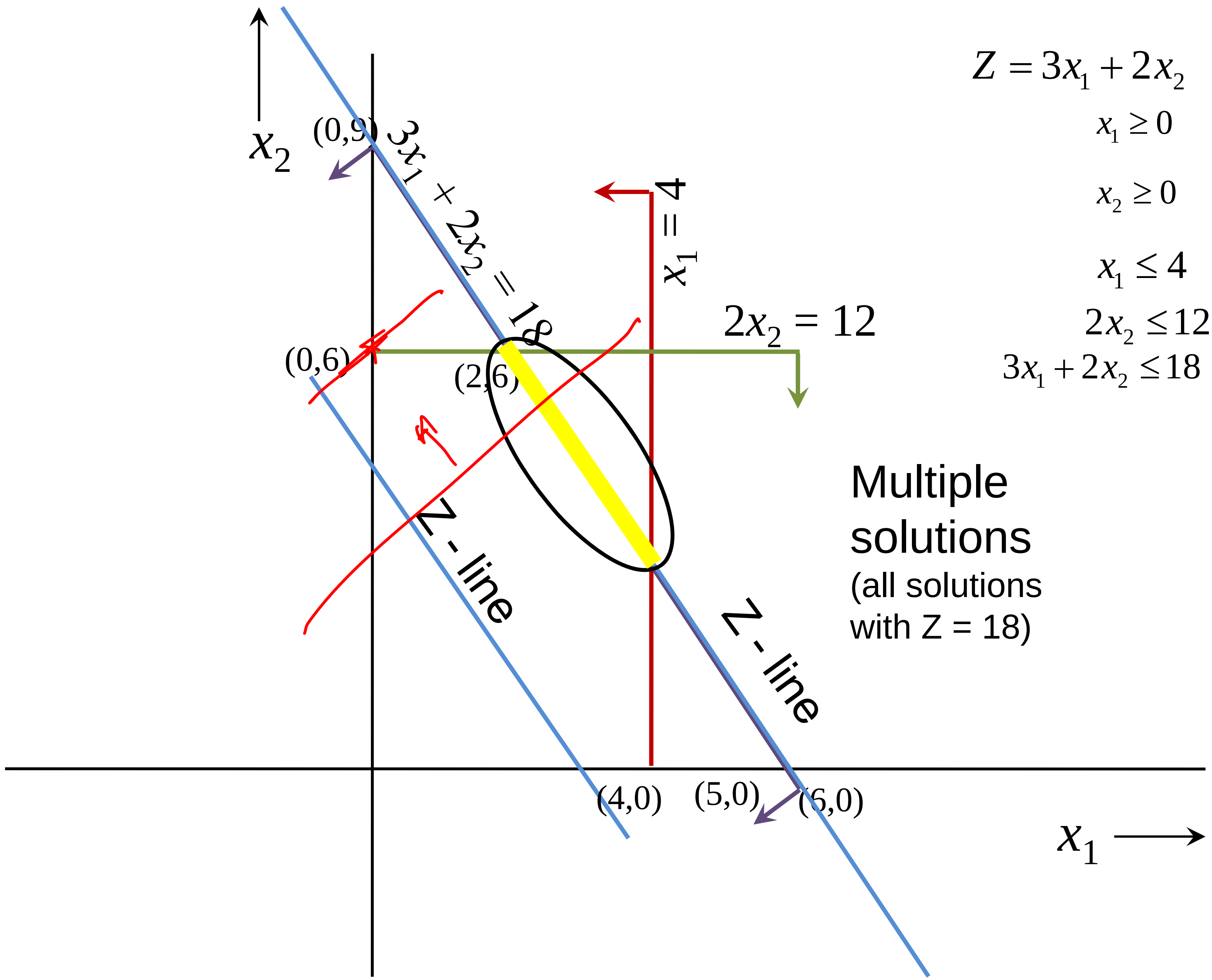
(Instead of $Z = 3x_1 + 5x_2$)

$$Z = 3x_1 + 2x_2$$

s.t.

$$x_1 \le 4; \quad 2x_2 \le 12; \quad 3x_1 + 2x_2 \le 18$$

$$x_1 \ge 0; \quad x_2 \ge 0$$



$Z = 3x_1 + 2x_2$

$x_1 \geq 0$

$x_2 \geq 0$

$x_1 \leq 4$

$2x_2 \leq 12$

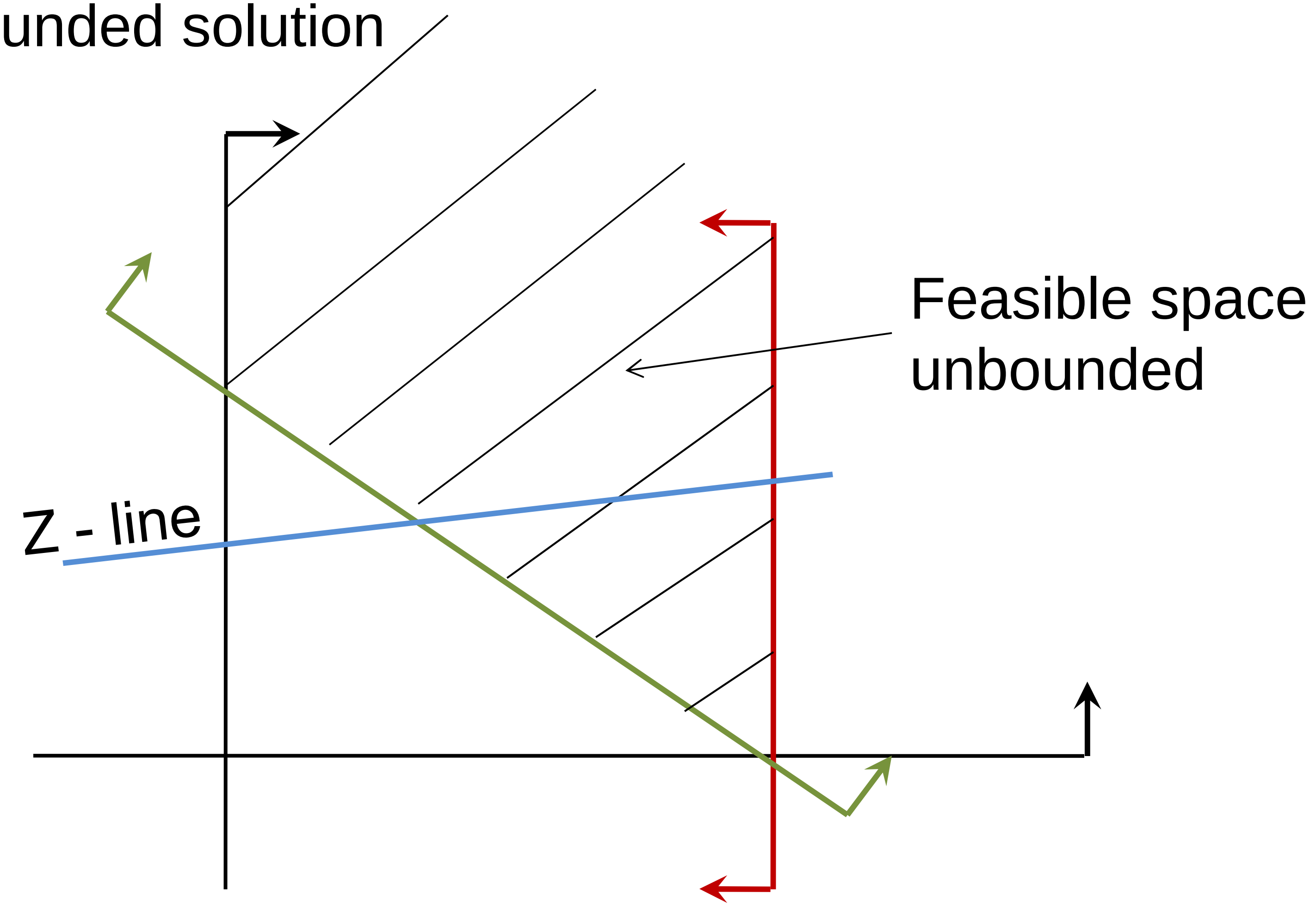
$3x_1 + 2x_2 \leq 18$

Multiple solutions

(all solutions with Z = 18)

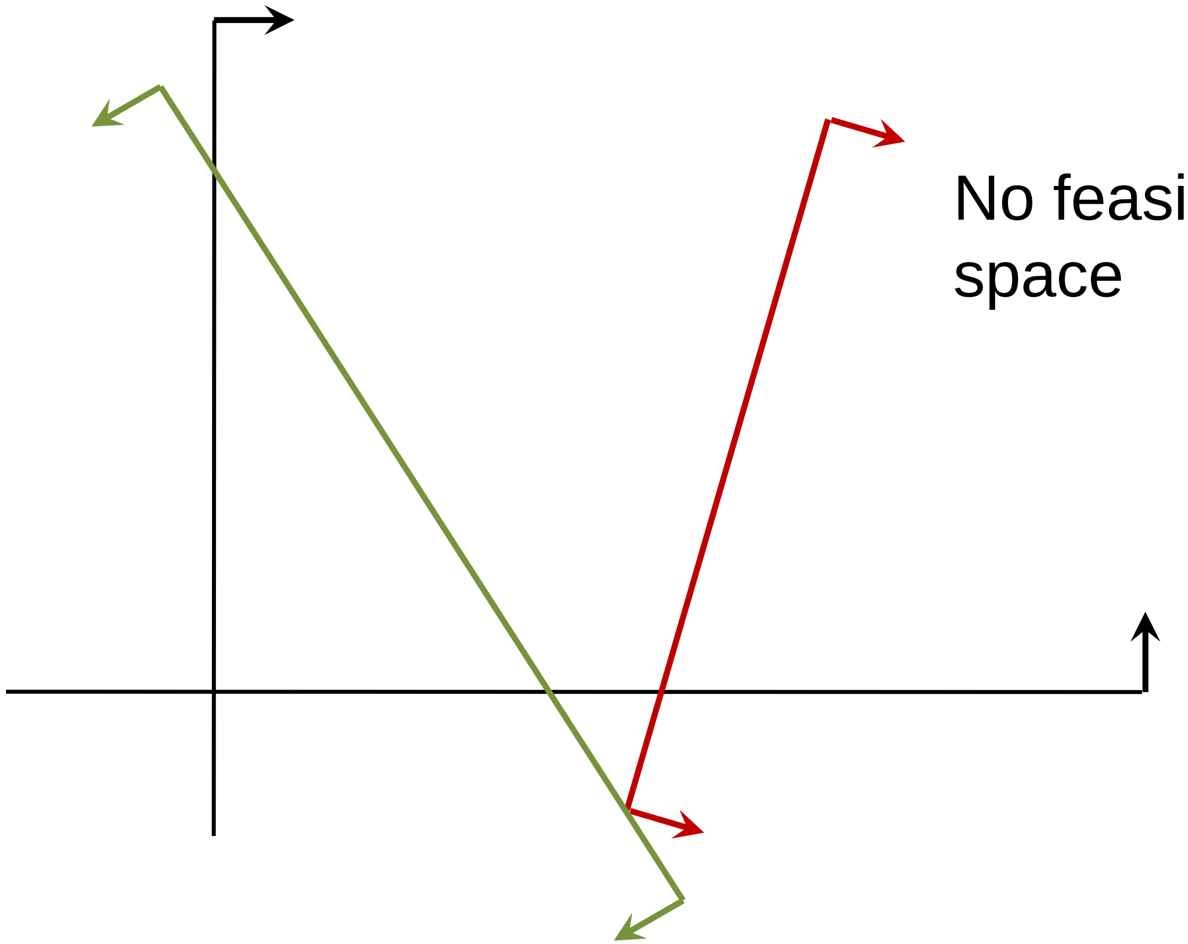
LP – Graphical Solution

Unbounded solution



LP – Graphical Solution

Infeasible solution

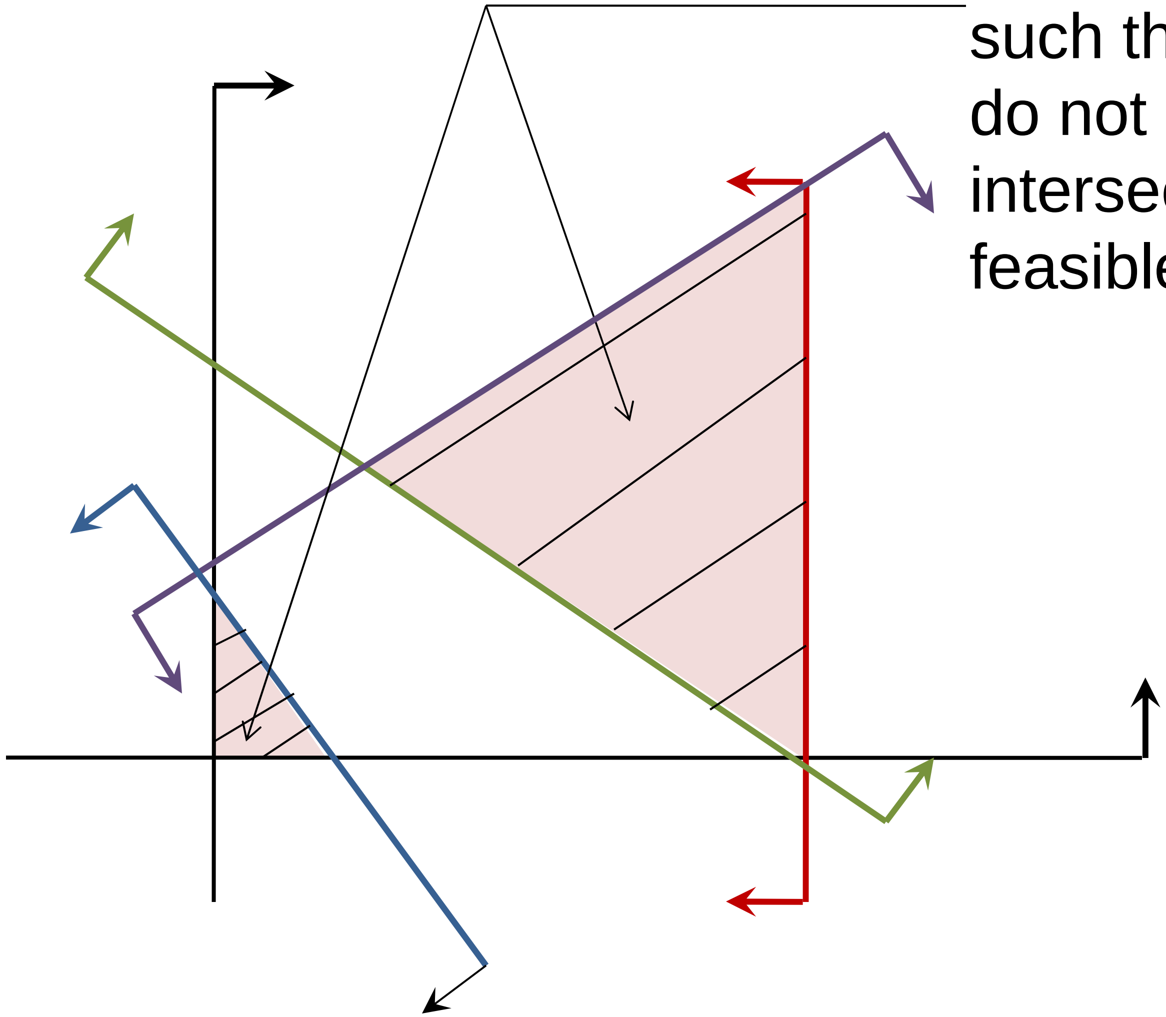


No feasible
space

LP – Graphical Solution

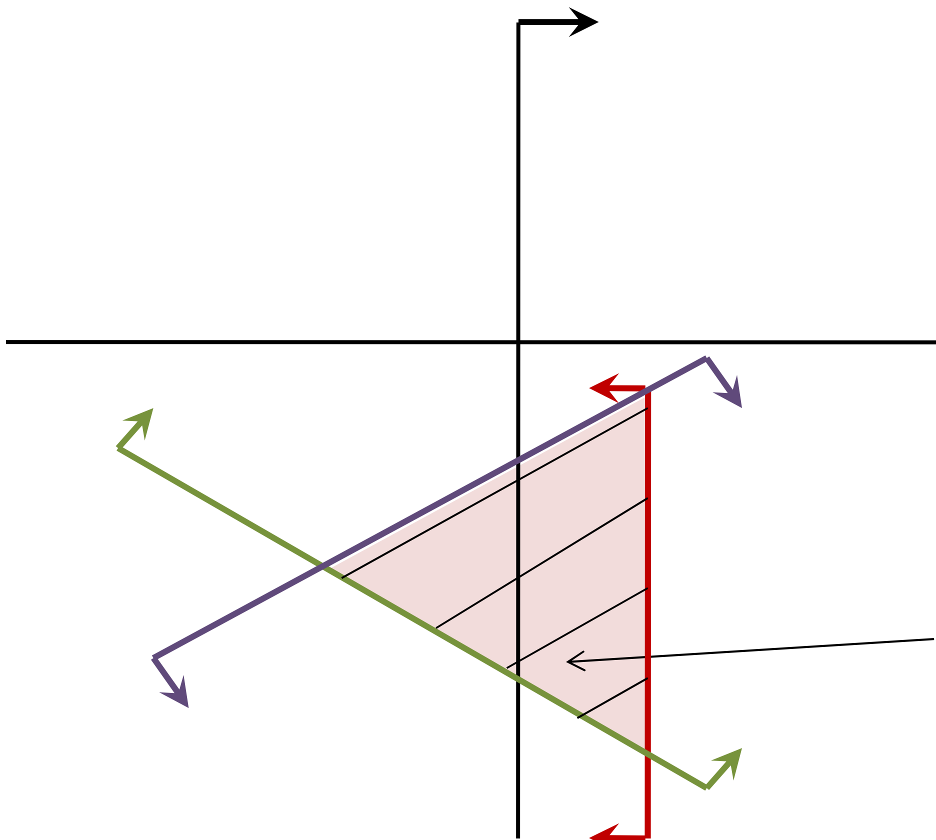
Infeasible solution

Constraints are such that they do not have intersecting feasible region



LP – Graphical Solution

Infeasible solution



Although the constraints form an intersecting region, the region violates non-negativity conditions

Linear Programming

General form of LP:

Maximize Z

s.t.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

.

.

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

$$x_1 \geq 0;\ x_2 \geq 0;\ ...\ ;x_n \geq 0$$

Linear Programming

i.e.,

Maximize Z

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad i = 1, 2, \ldots\ldots m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots\ldots n$$

Linear Programming

- In a LP problem, there are m equations and n decision variables.
- If $m > n$, there would be $m - n$ redundant equations which can be eliminated.
- If $m = n$, either there is a unique solution in which case there can be no optimization or there is no solution to the problem, in which case the constraints are inconsistent.
- If $m < n$, it corresponds to an under-determined set of linear equations; if there is one solution, then there are infinite number of solutions. The problem is to find the optimal solution from among these infinite no. of solutions.

Linear Programming

- The characteristics of a LP problem stated in general form are

  1. The objective function is of the “maximization” type.
  2. All the constraints are of equality type.
  3. All the decision variables are non-negative.

Linear Programming

1. The objective function is of the "maximization" type:

- The minimization of a function is equivalent to the maximization of the negative of the same function.

  for example,

  Minimize $\sum_{j=1}^{n} c_j x_j$

  can be expressed as

  Maximize $\sum_{j=1}^{n} -(c_j x_j)$

Linear Programming

2. All the constraints are of equality type:

Inequality constraint of the form,

$$a_{k1}x_1 + a_{k2}x_2 + \dots\dots\dots + a_{kn}x_n \leq b_k$$

may be converted to an equality constraint by adding a non-negative variable

$$a_{k1}x_1 + a_{k2}x_2 + \dots\dots\dots + a_{kn}x_n + x_{n+1} = b_k$$

x_{n+1} is called a slack variable

$$x_{n+1} \geq 0$$

Linear Programming

For example,

$$3x_1 + 2x_2 \leq 18$$

may be converted to

$$3x_1 + 2x_2 + x_3 = 18$$

$$x_3 \geq 0$$

Slack variable

Linear Programming

if the constraint is of greater than type,

$$a_{k1}x_1 + a_{k2}x_2 + ... + a_{kn}x_n \geq b_k$$

a non-negative variable x_{n+1} is subtracted from the LHS to make it an equality constraint

$$a_{k1}x_1 + a_{k2}x_2 + ... + a_{kn}x_n - x_{n+1} = b_k$$

x_{n+1} is called a surplus variable

$$x_{n+1} \geq 0$$

Linear Programming

For example,

$$x_1 + x_2 \geq 5$$

may be converted to

$$x_1 + x_2 - x_4 = 5$$

$$x_4 \geq 0$$

Surplus variable

Linear Programming

3. All the decision variables are non-negative

- In most of the engineering problems, decision variables represent some physical variables; hence most variables are non-negative.
- Occasionally some problems may involve variables unrestricted in sign. In such cases, the non-negativity condition must be forced in the problem formulation.
- An unrestricted variable can be written as the difference of two non-negative variables.

Linear Programming

- For example, x_i is unrestricted in sign; it is replaced by two variables x_{i1} and x_{i2} such that

$$x_i = x_{i1} - x_{i2}$$

where

$$x_{i1} \geq 0$$

$$x_{i2} \geq 0$$

x_i will be negative if $x_{i2} > x_{i1}$

LINEAR PROGRAMMING
Simplex Method

LP – Simplex Method

Motivation for the Simplex method

Max $$Z = 6x_1 + 8x_2$$

s.t.

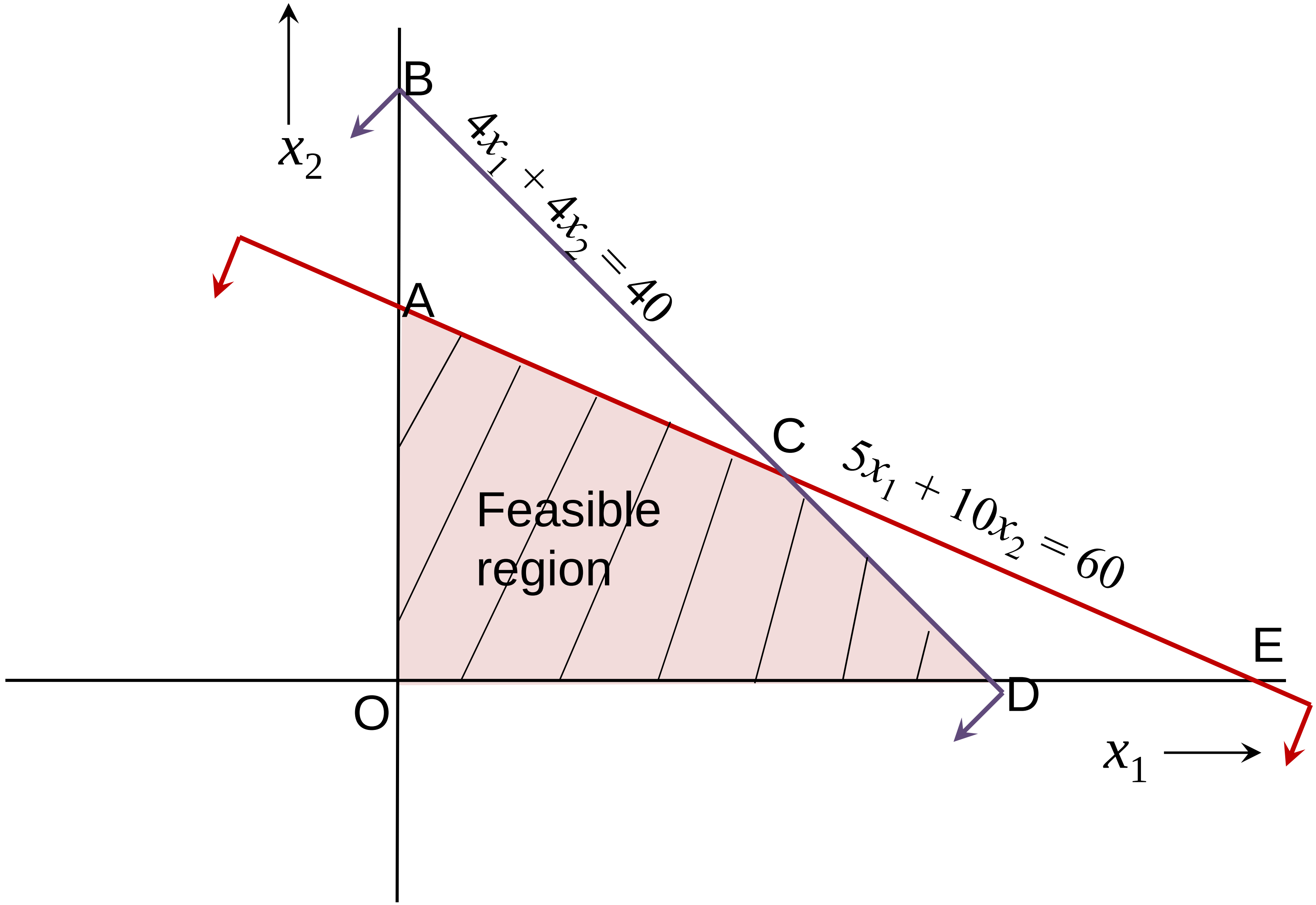
$$5x_1 + 10x_2 \leq 60$$

$$4x_1 + 4x_2 \leq 40$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

LP – Simplex Method



LP – Simplex Method

The constraints are converted as

$$5x_1 + 10x_2 \leq 60$$

$$4x_1 + 4x_2 \leq 40$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$\left.\begin{aligned} 5x_1 + 10x_2 + x_3 &= 60 \\ 4x_1 + 4x_2 + x_4 &= 40 \end{aligned}\right\}$$ 2 equations and 4 unknowns

$$x_1 \geq 0; \quad x_2 \geq 0$$

$$x_3 \geq 0; \quad x_4 \geq 0$$

x_3, x_4 are slack variables