INDIAN INSTITUTE OF SCIENCE

Water Resources Systems:
Modeling Techniques and Analysis

Lecture - 7

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Summary of the previous lecture

- Function with equality constraints
  - Lagrange multipliers

$$L = f(X) - \sum_{j=1}^{m} \lambda_j g_j(X)$$

Necessary condition:

$$\frac{\partial L}{\partial x_i} = 0; \quad \frac{\partial L}{\partial \lambda_j} = 0$$

Sufficiency condition: $|D| = 0$

$$L_{ij} = \left.\frac{\partial^2 L}{\partial x_i \partial x_j}\right|_{(X^*, \lambda^*)}; \quad g_{ji} = \left.\frac{\partial g_j(X)}{\partial x_i}\right|_{X^*}$$

- Function with inequality constraints

Minimize $f(X)$

s.t.

$g_j(X) \leq 0 \qquad j = 1, 2, \ldots.. m$

Kuhn – Tucker conditions:

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots.. n$$

$$\lambda_j g_j = 0$$

$$g_j \leq 0 \qquad j = 1, 2, \ldots.. m$$

$$\lambda_j \geq 0$$

Example – 1

Maximize

$$f\left(X\right) = -x_1^2 - x_2^2 + 4x_1 + 4x_2 - 8$$

s.t.

$$x_1 + 2x_2 \le 4$$

$$2x_1 + x_2 \le 5$$

K-T conditions

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{2} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1, 2$$

$$\left.\begin{aligned} \lambda_j g_j &= 0 \\ g_j &\le 0 \\ \lambda_j &\le 0 \end{aligned}\right\} j = 1, 2$$

Example – 1 (Contd.)

Let $$f\left(X\right) = -x_1^2 - x_2^2 + 4x_1 + 4x_2 - 8$$

$$g_1\left(X\right) = x_1 + 2x_2 - 4$$

$$g_2\left(X\right) = 2x_1 + x_2 - 5$$

K-T conditions

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{2} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad i = 1,\ 2$$

$$\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0$$

Example – 1 (Contd.)

$$g_1(X) = x_1 + 2x_2 - 4$$

$$\frac{\partial g_1}{\partial x_1} = 1; \qquad \frac{\partial g_1}{\partial x_2} = 2$$

$$g_2(X) = 2x_1 + x_2 - 5$$

$$\frac{\partial g_2}{\partial x_1} = 2; \qquad \frac{\partial g_2}{\partial x_2} = 1$$

Example – 1 (Contd.)

$$f\left(X\right) = -x_1^2 - x_2^2 + 4x_1 + 4x_2 - 8$$

$$\frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_1} = 0$$

$$\frac{\partial g_1}{\partial x_1} = 1; \qquad \frac{\partial g_1}{\partial x_2} = 2$$

$$\frac{\partial g_2}{\partial x_1} = 2; \qquad \frac{\partial g_2}{\partial x_2} = 1$$

$$-2x_1 + 4 + \lambda_1 + 2\lambda_2 = 0 \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} + \lambda_2 \frac{\partial g_2}{\partial x_2} = 0$$

$$-2x_2 + 4 + 2\lambda_1 + \lambda_2 = 0 \quad \text{—— (2)}$$

Example – 1 (Contd.)

$$g_1(X) = x_1 + 2x_2 - 4$$

$$g_2(X) = 2x_1 + x_2 - 5$$

$$\lambda_j g_j = 0 \qquad j = 1, 2$$

$$\lambda_1 (x_1 + 2x_2 - 4) = 0 \quad \text{———} \quad \textcircled{3}$$

$$\lambda_2 (2x_1 + x_2 - 5) = 0 \quad \text{———} \quad \textcircled{4}$$

4 equations and 4 unknowns

Example – 1 (Contd.)

Using Eq.1 and Eq.2

$$-2x_1 + 4 + \lambda_1 + 2\lambda_2 = 0$$

$$-2x_2 + 4 + 2\lambda_1 + \lambda_2 = 0$$

multiplying eq.1 with 2 and subtracting eq.2, rearranging the terms

$$2\lambda_1 + 4\lambda_2 = 4x_1 - 8$$

$$2\lambda_1 + \lambda_2 = 2x_2 - 4$$

$$\overline{3\lambda_2 = 4x_1 - 2x_2 - 4;} \qquad \lambda_2 = \frac{4x_1 - 2x_2 - 4}{3}$$

Example – 1 (Contd.)

Substituting in Eq.2,

$$\lambda_1 = \frac{-2x_1 + 4x_2 - 4}{3}$$

$$\lambda_1\left(x_1 + 2x_2 - 4\right) = 0$$

$$\lambda_2\left(2x_1 + x_2 - 5\right) = 0$$

Substituting λ_1 and λ_2 in Eq.3 and Eq.4,

$$\left(-2x_1 + 4x_2 - 4\right)\left(x_1 + 2x_2 - 4\right) = 0 \quad \text{———} \quad (5)$$

$$\left(4x_1 - 2x_2 - 4\right)\left(2x_1 + x_2 - 5\right) = 0 \quad \text{———} \quad (6)$$

Four solutions possible

Example – 1 (Contd.)

First terms in both Eq.5 and Eq.6,

$$\left(-2x_1 + 4x_2 - 4\right) = 0$$

$$\left(4x_1 - 2x_2 - 4\right) = 0$$

Solving the equations gives,

$$x_1 = 2; \qquad x_2 = 2$$

Check for conditions

$$g_j \le 0 \qquad\qquad j = 1, 2$$

$$\lambda_j \le 0$$

Example – 1 (Contd.)

$$\lambda_1 = \frac{-2x_1 + 4x_2 - 4}{3} = 0$$

$$\lambda_2 = \frac{4x_1 - 2x_2 - 4}{3} = 0$$

$\lambda_j \le 0$

Condition is satisfied

$$g_1(X) = x_1 + 2x_2 - 4 = 2$$

$$g_2(X) = 2x_1 + x_2 - 5 = 1$$

$g_j \le 0$

Condition is not satisfied

Check for other solutions

Example – 1 (Contd.)

Second terms in both Eq.5 and Eq.6,

$$\left(x_1 + 2x_2 - 4\right) = 0$$

$$\left(4x_1 - 2x_2 - 4\right) = 0$$

Solving the equations gives,

$$x_1 = \frac{8}{5}; \qquad x_2 = \frac{6}{5}$$

Check for conditions

$$g_j \leq 0 \qquad\qquad j = 1, 2$$

$$\lambda_j \leq 0$$

Example – 1 (Contd.)

$$\left.\begin{aligned}\lambda_1 &= \frac{-2x_1 + 4x_2 - 4}{3} = -\frac{4}{5} \\ \lambda_2 &= \frac{4x_1 - 2x_2 - 4}{3} = 0\end{aligned}\right\}$$

$\lambda_j \leq 0$

Condition is satisfied

$$\left.\begin{aligned}g_1(X) &= x_1 + 2x_2 - 4 = 0 \\ g_2(X) &= 2x_1 + x_2 - 5 = -\frac{3}{5}\end{aligned}\right\}$$

$g_j \leq 0$

Condition is satisfied

Therefore, $\left(\frac{8}{5}, \frac{6}{5}\right)$ satisfies all K-T conditions

Example – 1 (Contd.)

First term in Eq.5 and second term in Eq.6,

$$\left(-2x_1 + 4x_2 - 4\right) = 0$$

$$\left(2x_1 + x_2 - 5\right) = 0$$

Solving the equations gives,

$$x_1 = \frac{8}{5}; \qquad x_2 = \frac{9}{5}$$

Check for conditions

$$g_j \leq 0 \qquad\qquad j = 1, 2$$

$$\lambda_j \leq 0$$

Example – 1 (Contd.)

$$\lambda_1 = \frac{-2x_1 + 4x_2 - 4}{3} = 0$$

$$\lambda_2 = \frac{4x_1 - 2x_2 - 4}{3} = -\frac{2}{5}$$

$\lambda_j \leq 0$

Condition is satisfied

$$g_1(X) = x_1 + 2x_2 - 4 = \frac{6}{5}$$

$$g_2(X) = 2x_1 + x_2 - 5 = 0$$

$g_j \leq 0$

Condition is not satisfied

Check for other solutions

Example – 1 (Contd.)

Second term in Eq.5 and second term in Eq.6,

$$\left(x_1 + 2x_2 - 4\right) = 0$$

$$\left(2x_1 + x_2 - 5\right) = 0$$

Solving the equations gives,

$$x_1 = 2; \qquad x_2 = 1$$

Check for conditions

$$g_j \leq 0 \qquad\qquad j = 1, 2$$

$$\lambda_j \leq 0$$

Example – 1 (Contd.)

$$\lambda_1 = \frac{-2x_1 + 4x_2 - 4}{3} = -\frac{4}{3}$$

$$\lambda_2 = \frac{4x_1 - 2x_2 - 4}{3} = \frac{2}{3}$$

$\lambda_j \leq 0$

Condition is not satisfied

$$g_1(X) = x_1 + 2x_2 - 4 = 0$$

$$g_2(X) = 2x_1 + x_2 - 5 = 0$$

$g_j \leq 0$

Condition is satisfied

Example – 1 (Contd.)

Hence the optimal solution to the problem is

$$x_1^* = \frac{8}{5}; \qquad x_2^* = \frac{6}{5}$$

LINEAR PROGRAMMING

Linear Programming

- Linear programming (LP) is an optimization method applicable for solution of problems with objective function and constraints as a linear functions of decision variables.
- Linear equations may be in the form of equalities or inequalities.
- During World War II, George B. Dantzig formulated general LP problem for allocating resources, and devised the simplex method of solution.
- LP is considered as a revolutionary development that permits us to make optimal decisions in complex situations.

Source: “Engineering and optimization-theory and practice” by Singiresu S. Rao, 1996, John Wiley & Sons

Linear Programming

- Simplex method is most efficient and popular method.
- Karmakar’s method is 50 times faster than simplex method – developed in 1984.
- Some applications of LP in water resources
    - Reservoir operation
    - Irrigation scheduling
    - Reservoir capacity determination
    - Screening of alternatives in river basin development.
    - Conjunctive use of surface and ground water
    - Resource allocation
- 25% of all problems solved in computers.

Linear Programming

General form of a LP

Linear objective function …. $f(X)$ is linear function of X

linear constraints …. $g_j(X)$ are all linear functions of X

non-negativity of variables $X \geq 0$

LINEAR PROGRAMMING
Graphical Solution

LP – Graphical Solution

- Graphical method gives a physical picture of certain geometrical characteristics of LP problem.
- Complex as the no. of variables increases.

Example – 2

Maximize

$$Z = 3x_1 + 5x_2$$

s.t.

$$x_1 \le 4$$

$$2x_2 \le 12$$

$$3x_1 + 2x_2 \le 18$$

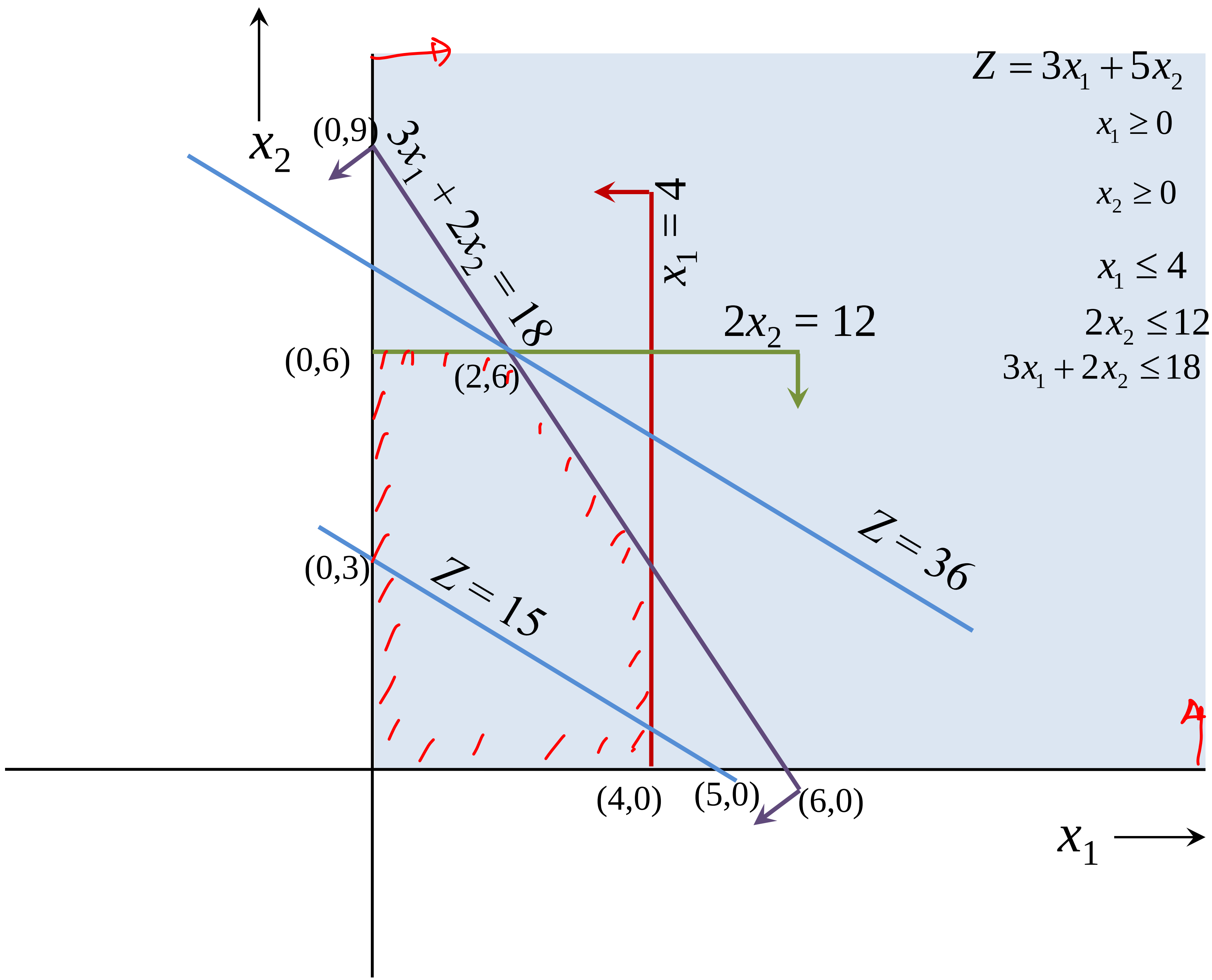
Constraints

$$x_1 \ge 0$$

$$x_2 \ge 0$$

Decision variables

Non-negativity



$Z = 3x_1 + 5x_2$
$x_1 \geq 0$
$x_2 \geq 0$
$x_1 \leq 4$
$2x_2 \leq 12$
$3x_1 + 2x_2 \leq 18$
x_2
x_1
(0,9)
(0,6)
(0,3)
(2,6)
(4,0)
(5,0)
(6,0)
$3x_1 + 2x_2 = 18$
$x_1 = 4$
$2x_2 = 12$
$Z = 36$
$Z = 15$