

## Water Resources Systems: Modeling Techniques and Analysis

Lecture - 5 Course Instructor: Prof. P. P. MUJUMDAR Department of Civil Engg., IISc.

# Summary of the previous lecture

• Optimization of a function of a single variable necessary condition f'(x) = 0

Sufficiency condition  $f''(x)|_{x_0} < 0$   $f''(x)|_{x_0} > 0$ 

Function of multiple variables

necessary condition  $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \dots = \frac{\partial f}{\partial x_n} = 0$ Sufficiency condition: Hessian matrix  $H[f(X)] = \begin{bmatrix} \frac{\partial^2 f(X)}{\partial x_1^2} & \frac{\partial^2 f(X)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(X)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(X)}{\partial x_2^2} \end{bmatrix}$ 

- H positive definite at  $X = X_0 \dots$  Minimum
- H negative definite at  $X = X_0 \dots$  Maximum

#### Example – 1

Examine the function for convexity/concavity and determine the values at extreme points

$$f(X) = -x_1^2 - x_2^2 - 4x_1 - 8$$

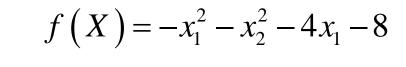
The stationary point is obtained by solving

$$\frac{\partial f}{\partial x_1} = -2x_1 - 4 = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = -2x_2 = 0$$
$$x_1 = -2, \ x_2 = 0$$
$$X = (-2, 0)$$

• Hessian matrix is

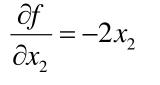
$$H\left[f\left(X\right)\right] = \begin{bmatrix} \frac{\partial^{2} f\left(X\right)}{\partial x_{1}^{2}} & \frac{\partial^{2} f\left(X\right)}{\partial x_{1} \partial x_{2}} \\ \frac{\partial^{2} f\left(X\right)}{\partial x_{2} \partial x_{1}} & \frac{\partial^{2} f\left(X\right)}{\partial x_{2}^{2}} \end{bmatrix}_{\left(-2,0\right)}$$

Hessian matrix evaluated at stationary point (-2,0)



$\frac{\partial f}{\partial t} = -2x_1 - 4$	$\partial^2 f = 0$
$\partial x_1$	$\frac{1}{\partial x_1 \partial x_2} = 0$

 $\frac{\partial^2 f}{\partial x_1^2} = -2$ 



 $\frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$ 

 $\frac{\partial^2 f}{\partial x_2^2} = -2$ 

### Hessian matrix is $H \left[ f \left( X \right) \right] = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

Eigen values of Hessian matrix:

$$\begin{vmatrix} \lambda I - H \begin{bmatrix} f(X) \end{bmatrix} \end{vmatrix} = 0$$
$$\begin{vmatrix} \lambda I - H \end{bmatrix} = \begin{bmatrix} \lambda + 2 & 0 \\ 0 & \lambda + 2 \end{bmatrix}$$
$$(\lambda + 2)^{2} = 0$$

Eigen values are  $\lambda_1 = -2, \lambda_2 = -2$ 

• As both the eigen values are negative, the matrix is negative definite

Hence the function has local maximum at X = (-2, 0)

As the Hessain matrix does not depend on  $x_1$  and  $x_2$  and it is negative definite matrix, the function is strictly concave and therefore the local maximum is also the global maximum

#### Example – 2

Determine the extreme values of the function

$$f(X) = x_1^3 + x_2^3 - 3x_1 - 12x_2 + 20$$

The stationary point is obtained by solving

$$\frac{\partial f}{\partial x_1} = 3x_1^2 - 3 = 0 \qquad x_1 = \pm 1$$
  
and  
$$\frac{\partial f}{\partial x_2} = 3x_2^2 - 12 = 0 \qquad x_2 = \pm 2$$

Four solutions

$$X = (-1, -2), (1, 2), (1, -2) \text{ and } (-1, 2)$$

• Hessian matrix is

$$H\left[f\left(X\right)\right] = \begin{bmatrix} \frac{\partial^{2} f\left(X\right)}{\partial x_{1}^{2}} & \frac{\partial^{2} f\left(X\right)}{\partial x_{1} \partial x_{2}} \\ \frac{\partial^{2} f\left(X\right)}{\partial x_{2} \partial x_{1}} & \frac{\partial^{2} f\left(X\right)}{\partial x_{2}^{2}} \end{bmatrix}$$

$$f(X) = x_1^3 + x_2^3 - 3x_1 - 12x_2 + 20$$
$$\frac{\partial f}{\partial x_1} = 3x_1^2 - 3 \qquad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1$$

#### Example - 2 (Contd.) $f(X) = x_1^3 + x_2^3 - 3x_1 - 12x_2 + 20$ $= 3x_2^2 - 12$ $\frac{\partial^2 f}{\partial x_1^2} = 0$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 - 12 \qquad \qquad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$$
$$\frac{\partial^2 f}{\partial x_2^2} = 6x_2$$

Hessian matrix is

$$H\left[f\left(X\right)\right] = \begin{bmatrix} 6x_1 & 0\\ 0 & 6x_2 \end{bmatrix}$$

# Hessian matrix is $H\left[f\left(X\right)\right] = \begin{bmatrix} 6x_1 & 0\\ 0 & 6x_2 \end{bmatrix}$

Eigen values of Hessian matrix:

$$\begin{vmatrix} \lambda I - H \begin{bmatrix} f(X) \end{bmatrix} \end{vmatrix} = 0$$
$$\begin{vmatrix} \lambda I - H \end{vmatrix} = \begin{bmatrix} \lambda - 6x_1 & 0 \\ 0 & \lambda - 6x_2 \end{bmatrix} = 0$$
$$(\lambda - 6x_1)(\lambda - 6x_2) = 0$$

Eigen values are  $\lambda_1 = 6x_1, \lambda_2 = 6x_2$ 

Hessian matrix at  

$$H\left[f\left(X\right)\right] = \begin{bmatrix} 6x_1 & 0 \\ 0 & 6x_2 \end{bmatrix} (1,2)$$
Eigen values are  $\lambda_1 = 6x_1, \lambda_2 = 6x_2$ 

$$\lambda_1 = 6, \ \lambda_2 = 12$$

All the eigen values of Hessian matrix are positive, hence the matrix is positive definite at X = (1, 2)

Therefore the function has a local minimum at this point

$$f_{\min}(X) = 1^3 + 2^3 - 3 \times 1 - 12 \times 2 + 20 = 2$$

Hessian matrix at  

$$H\left[f\left(X\right)\right] = \begin{bmatrix} 6x_1 & 0\\ 0 & 6x_2 \end{bmatrix} \begin{pmatrix} -1, -2 \end{pmatrix}$$

Eigen values are  $\lambda_1 = 6x_1, \lambda_2 = 6x_2$ 

$$\lambda_1 = -6, \ \lambda_2 = -12$$

All the eigen values of Hessian matrix are negative, hence the matrix is negative definite at X = (-1, -2)

Therefore the function has a local maximum at this point  $f_{\max}(X) = (-1)^3 + (-2)^3 - 3 \times (-1) - 12 \times (-2) + 20$ 

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Hessian matrix at

$$H\left[f\left(X\right)\right] = \begin{bmatrix} 6x_1 & 0\\ 0 & 6x_2 \end{bmatrix} (-1,2) \text{ or } (1,-2)$$

Eigen values are -6 and 12 (or 6 and -12)

The H matrix is neither positive definite nor negative definite at these two points

### **Constrained Optimization**

Constrained Optimization:

f(X) is a function of n variables represented by vector X
 = (x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>, ..... x<sub>n</sub>)

Maximize or Minimize f(X)Subject to (s.t.)  $g_j(X) \le 0$  j = 1, 2, ..., m $m \le n$ 

f(X) and g(X) may or may not be linear functions

• If *m* > *n* the problem is over defined and there will be no solution unless redundant constrains are present

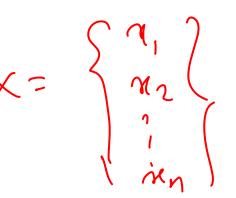
**Constrained Optimization:** 

- Function with equality constraints
- Function with inequality constraints

Maximize or Minimize f(X)(s.t.)  $g_j(X) = 0$   $\int f(X) = 0$   $\int f(X) = 0$ Function with equality constraints

Two methods discussed

- **Direct substitution**
- Lagrange multipliers



Direct substitution:

Reduce the problem to an unconstrained problem by expressing *m* variables in terms of the remaining (*n* – *m*) variables.

For example,

3 variables :  $x_1, x_2, x_3$ 

2 constraints

 $x_2$ ,  $x_3$  may be expressed in terms of  $x_1$  and render the problem as unconstrained problem with only  $x_1$  involved

Limitation:

- With higher no. of variables and constraints this method becomes quite cumbersome.
- Constraint equations are often non-linear difficult to solve them simultaneously.

#### Example – 3

Minimize the function

$$f(X) = x_1^2 + x_2^2 + 4x_1x_2$$

s.t.

$$x_1 + x_2 - 4 = 0$$

Solution:

$$x_1 = 4 - x_2$$

The modified function is

$$f(X) = (4 - x_2)^2 + x_2^2 + 4(4 - x_2)x_2$$
$$= 16 + 8x_2 - 2x_2^2$$

$$\frac{\partial f}{\partial x_2} = 8 - 4x_2$$
$$\frac{\partial f}{\partial x_2} = 0$$
$$8 - 4x_2 = 0$$
$$x_2 = 2$$

$$\frac{\partial^2 f}{\partial x_2^2} = -4 < 0,$$
  
Global maximum occurs at  $x_2 = 2$ 

Lagrange multipliers:

Maximize or Minimize f(X)s.t.  $g_j(X) = 0$  j = 1, 2, ..., m

- Introduce one additional variable corresponding to each constraint.
- Lagrange function *f(X)* is written as

$$L = f(X) - \frac{\lambda_j g_j(X)}{f_j(X)} \qquad L = f(X) - \sum_{j \in J} \lambda_j g_j(X)$$

- When  $g_j(X) = 0$ , optimizing *L* is same as optimizing f(X)
- The problem is transformed to unconstrained optimization problem

M

$$L = f(X) - \lambda_1 g_1(X) - \lambda_2 g_2(X) - \dots - \lambda_m g_m(X)$$

 The problem of *n* variables with *m* constraints is changed to a single problem of (*n* + *m*) variables with no constraints.

Necessary condition: For a function f(X) subject to the constraints  $g_j(X) = 0$ , j = 1, 2, ..., m to have a relative optimum at a point  $X^*$  is that the first partial derivatives of the Langrange function with respect to each of its arguments must be zero.

$$L = f(X) - \sum_{j=1}^{m} \lambda_j g_j (X)$$
$$\frac{\partial L}{\partial x_i} = 0 \qquad i = 1, 2, \dots, n$$
$$\frac{\partial L}{\partial \lambda_i} = 0 \qquad j = 1, 2, \dots, m$$

# **Optimization using Calculus**

The (n + m) simultaneous equations are solved to get a solution,  $(X^*, \lambda^*)$ .

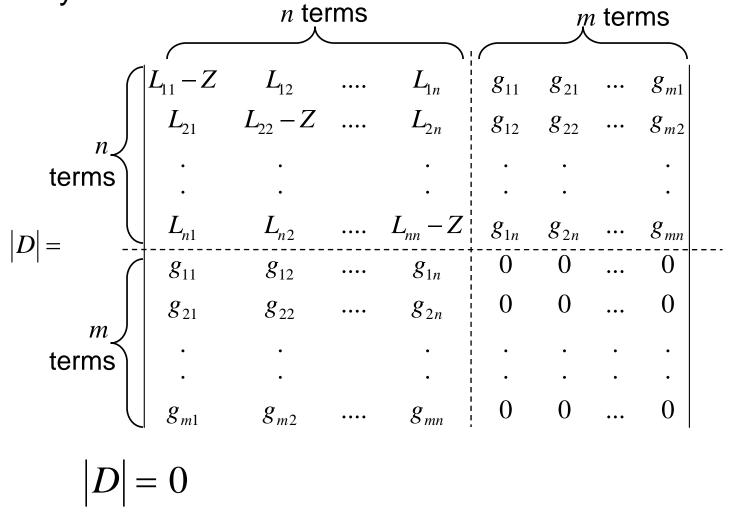
Sufficiency condition:

The second partial derivatives are denoted by

$$L_{ij} = \frac{\partial^2 L}{\partial x_i \partial x_j} \bigg|_{(X^*, \lambda^*)} \qquad i = 1, 2, \dots, n$$
$$g_{ij} = \frac{\partial g_j (X)}{\partial x_i} \bigg|_{X^*} \qquad j = 1, 2, \dots, m$$

# **Optimization using Calculus**

Sufficiency condition:



# **Optimization using Calculus**

Leads to a polynomial in Z of the order (n - m)

Solve for Z

If all *Z* values are positive ..... X\* corresponds to minimum

If all Z values are negative ..... X\* corresponds to maximum

If some values are positive and some are negative ... X\* is neither a minimum nor a maximum.