# Stochastic Structural Dynamics

## Lecture-36

## Fatigue failure & Vibration energy flow models

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# Two Applications

- Models for accumulated fatigue damage
- Vibration energy flow

## Empirical background

**Fatigue :**

Loss of mechanical integrity of the structure due to reversal of stresses.

- Progressive fracture
- Fracture of a structural member due to repeated cycles of load.

Fatigue is the primary mode of failure for metals subjected to oscillatory loads. A major source of failures in aircrafts, railway vehicles, ships, bridges, and rotors.

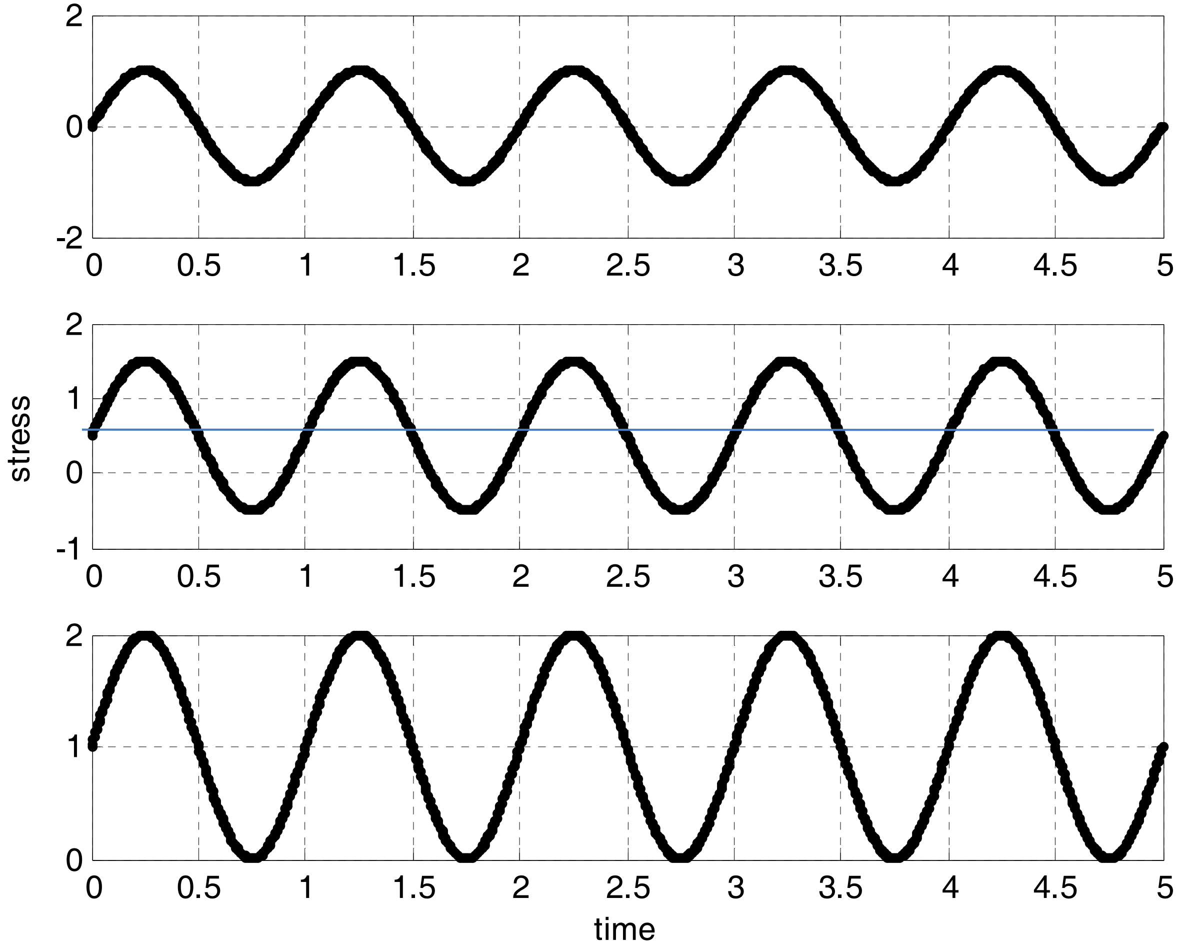
Components that can carry high constant amplitude loads fail under a substantially lower magnitude fluctuating load.

During fatigue failure, maximum stresses could be well below the tensile strength of the material but the structure fails after oscillating for a finite number of cycles.
That is, at failure, the response levels could be well below the limits of first passage failure.

**Aim**: To obtain a probabilistic description of fatigue damage in structures which are driven by random excitations.

- What is the expected rate at which fatigue damage accumulates?
- What is the PDF of the life of the strucutre?
- What is the influence of randomness in structural properties?
- How to characterize the reliability of structure against fatigue failure?

# Fatigue stress time histories




Completely reversed cyclic stress

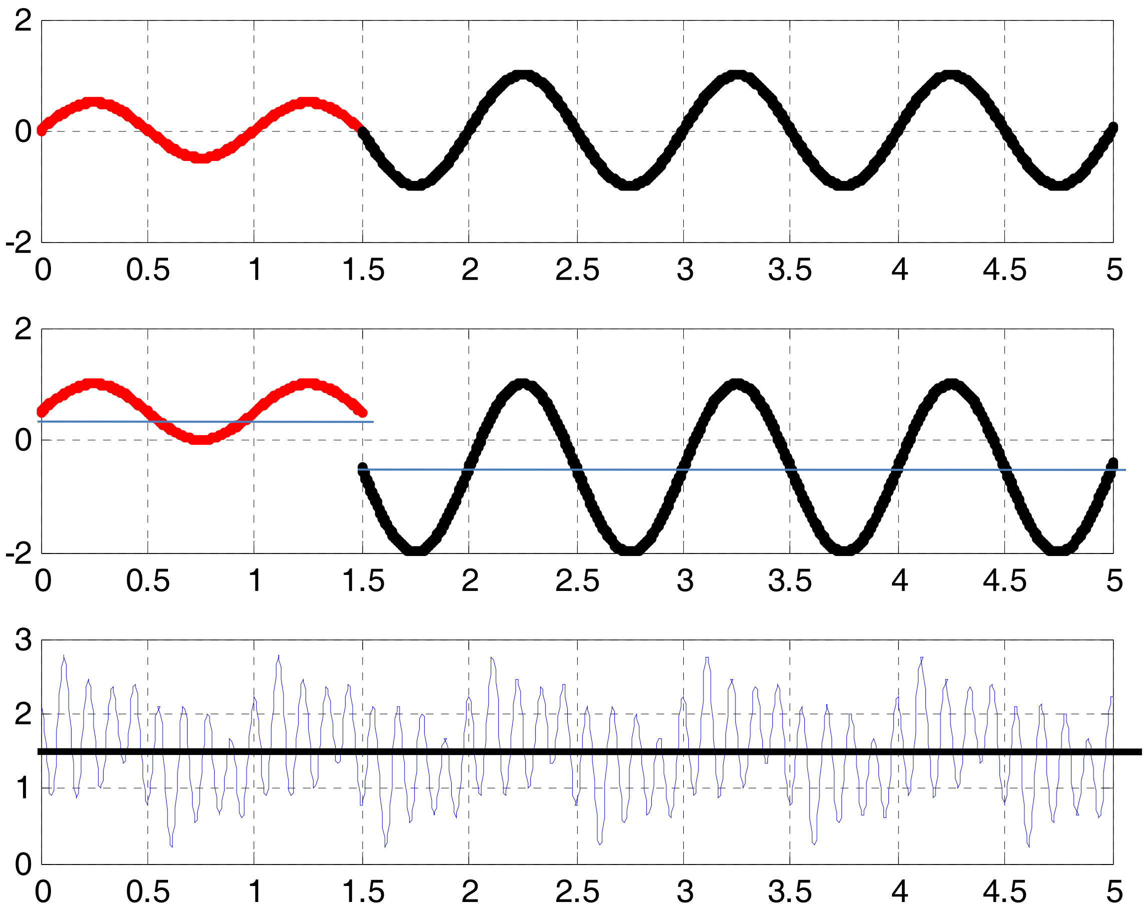
Nonzero mean stress

Released tension (zero minimum)

$$S_{\max}, S_{\min}, S_{\text{mean}} = \text{ maximum, minimum and arithmetic mean of stress}$$

$$\Delta S = S_{\max} - S_{\min} = \text{ stress range}$$

$$S_a = \frac{\Delta S}{2} = \text{ alternating stress amplitude}$$

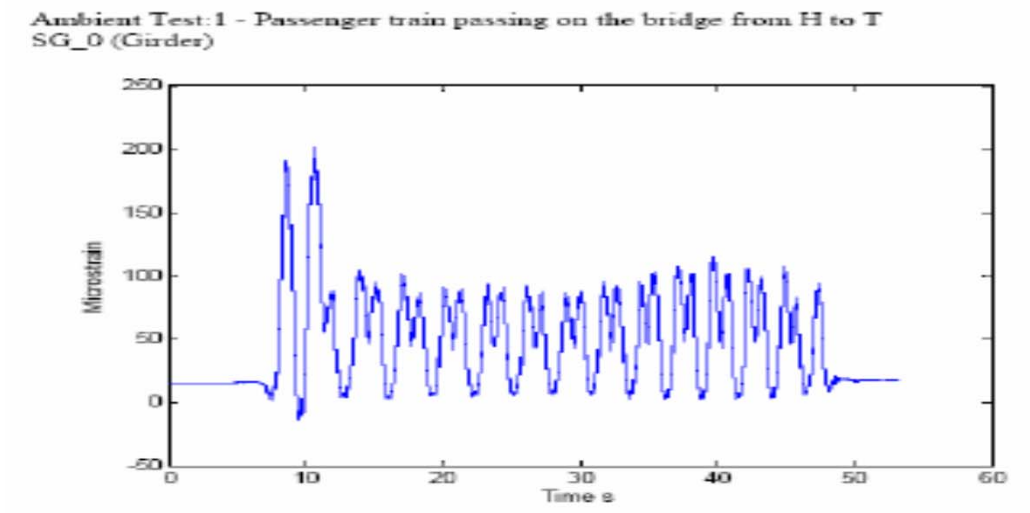



Varying amplitude
zero mean stress

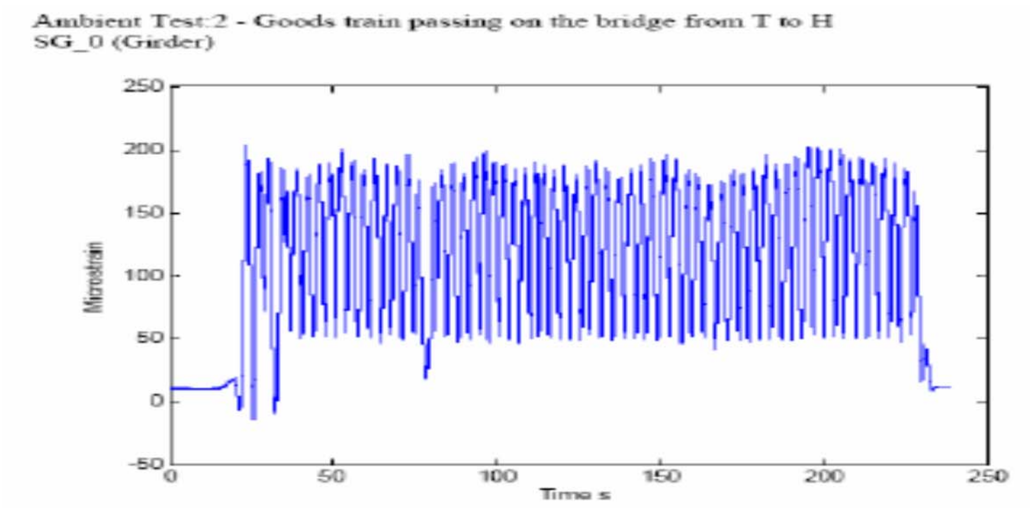
Varying amplitude
varying mean stress

Random
nonzero mean

# Measured strain time histories on steel railway bridges




(a)




(b)

SN curve

## S - N curves (Wohler curves )

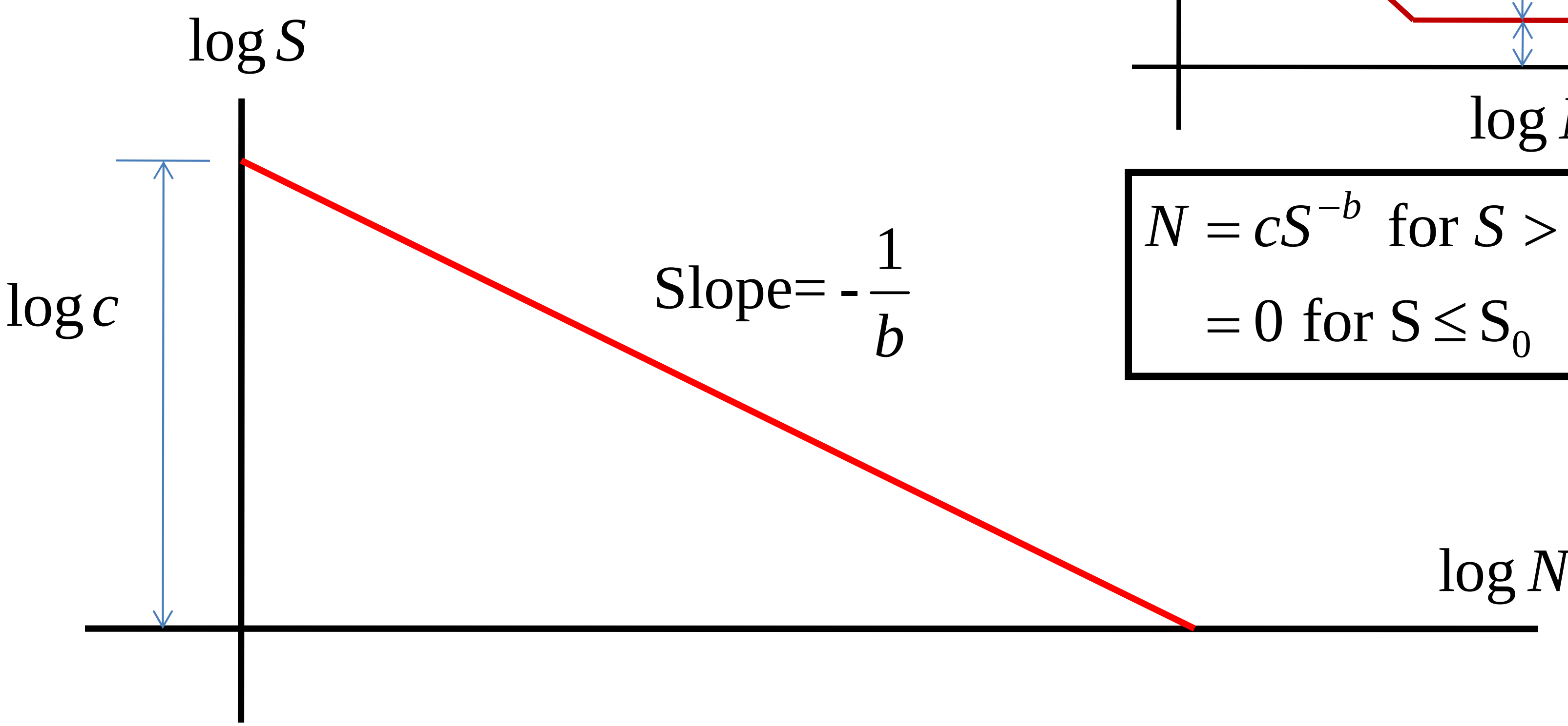
- Plot of cyclic stress level versus the number of cycles to failure.
- Test specimen: cylindrical; subjected to uni-axial cyclic stress or small cantilever beam under bending
- Stress amplitudes are kept constant

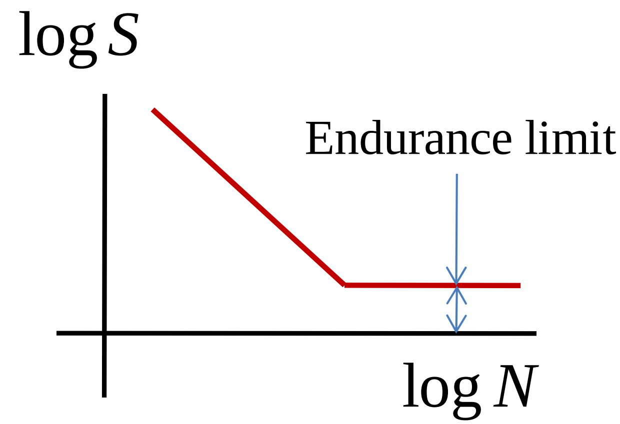
Note: In fatigue testing real life stress cycles can also be simulated.

# Model for S - N curve

$$NS^{b} = c; b, c > 0$$

$$\Rightarrow \log N + b \log S = \log c$$




$$N = cS^{-b} \text{ for } S > S_0$$
$$= 0 \text{ for } S \le S_0$$

## Remarks

- Does not deal with physical phenomena within material.
- It does not separate the crack initiation and crack propagation stages.
- Considers only the total life to fracture.

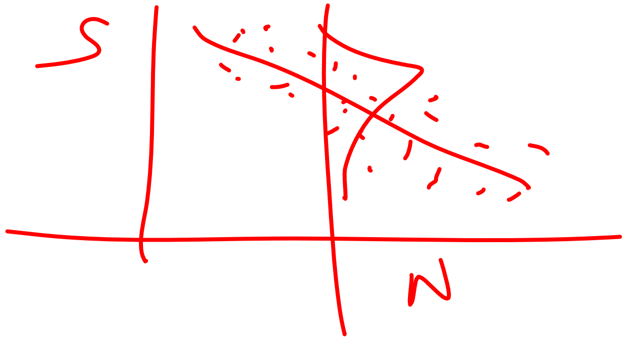
Laboratory conditions → Field conditions

**Factors**

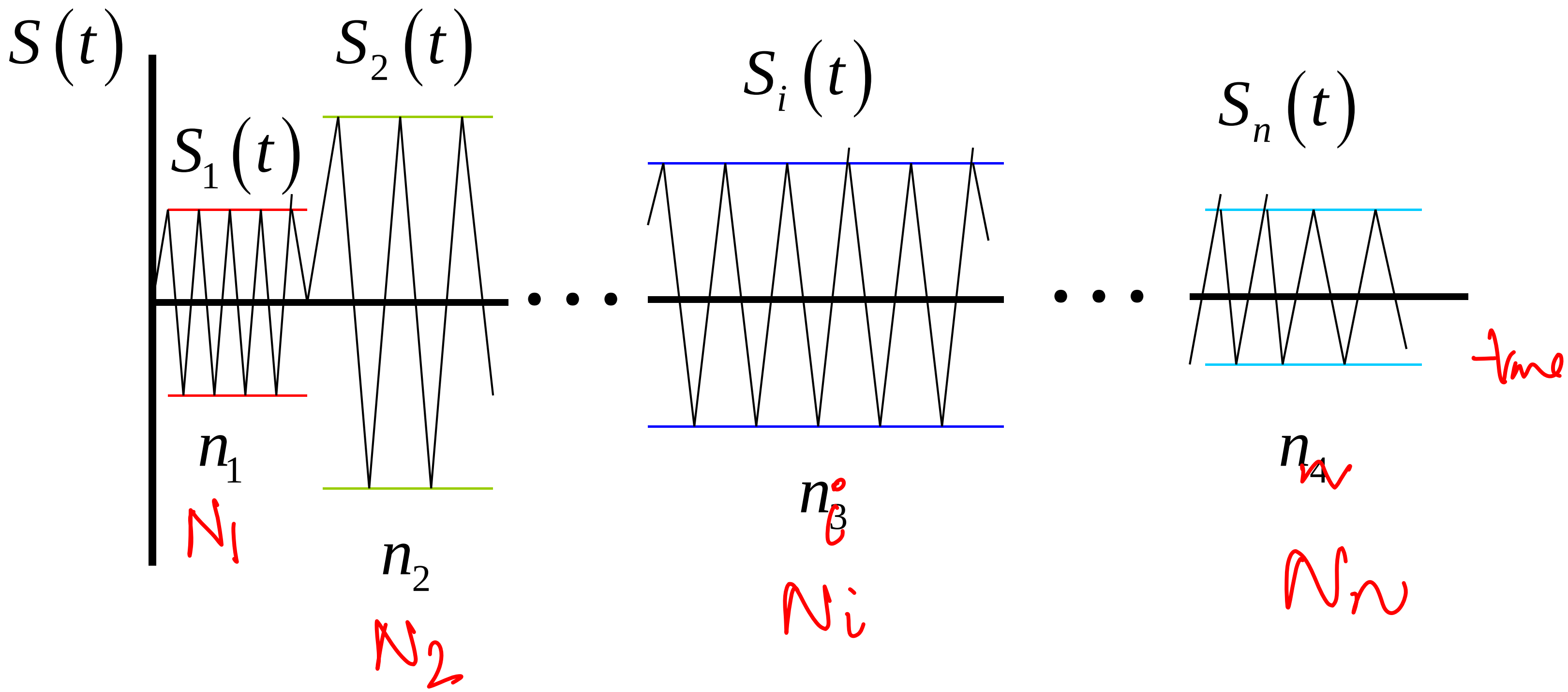
- Nonzero mean stress
- Varying stress amplitudes
- Environmental conditions: temperature, humidity, corrosive media...
- Size, shape and surface finish
- Frequency of cycling

**Remarks**

- Large scatter is observed; reflects the influence of uncertainties; S-N-P curves
- N follows lognormal or Weibull distribution
- Endurance limit: stress level below which the specimen seems to last indefinitely



# Palmgren-Miner rule



| | | | | |
|---|---|---|---|---|
| Stress level | $S_1$ | $S_2$ | $S_3$ | |
| No. of cycles | $n_1$ | $n_2$ | $n_3$ | |
| No. of cycles to failure | $N_1$ | $N_2$ | $N_3$ | |
| Incremental damage | $\frac{n_1}{N_1}$ | $\frac{n_2}{N_2}$ | $\frac{n_3}{N_3}$ | $\cdots$ |
| Cumulative damage | $\frac{n_1}{N_1}$ | $\frac{n_1}{N_1}+\frac{n_2}{N_2}$ | $\frac{n_1}{N_1}+\frac{n_2}{N_2}+\frac{n_3}{N_3}$ | |

Accumulated damage at the end of m - th cycle

$$\Delta = \sum_{i=1}^{m} \frac{n_i}{N_i} = \sum_{i=1}^{m} \frac{n_i S_i^b}{c}$$

Condition for failure

$$\Delta = 1$$

**Safe Limit**

$$\sum_{i=1}^{m} \frac{n_i S_i^b}{c} \leq 1$$

**Remarks on PM theory**

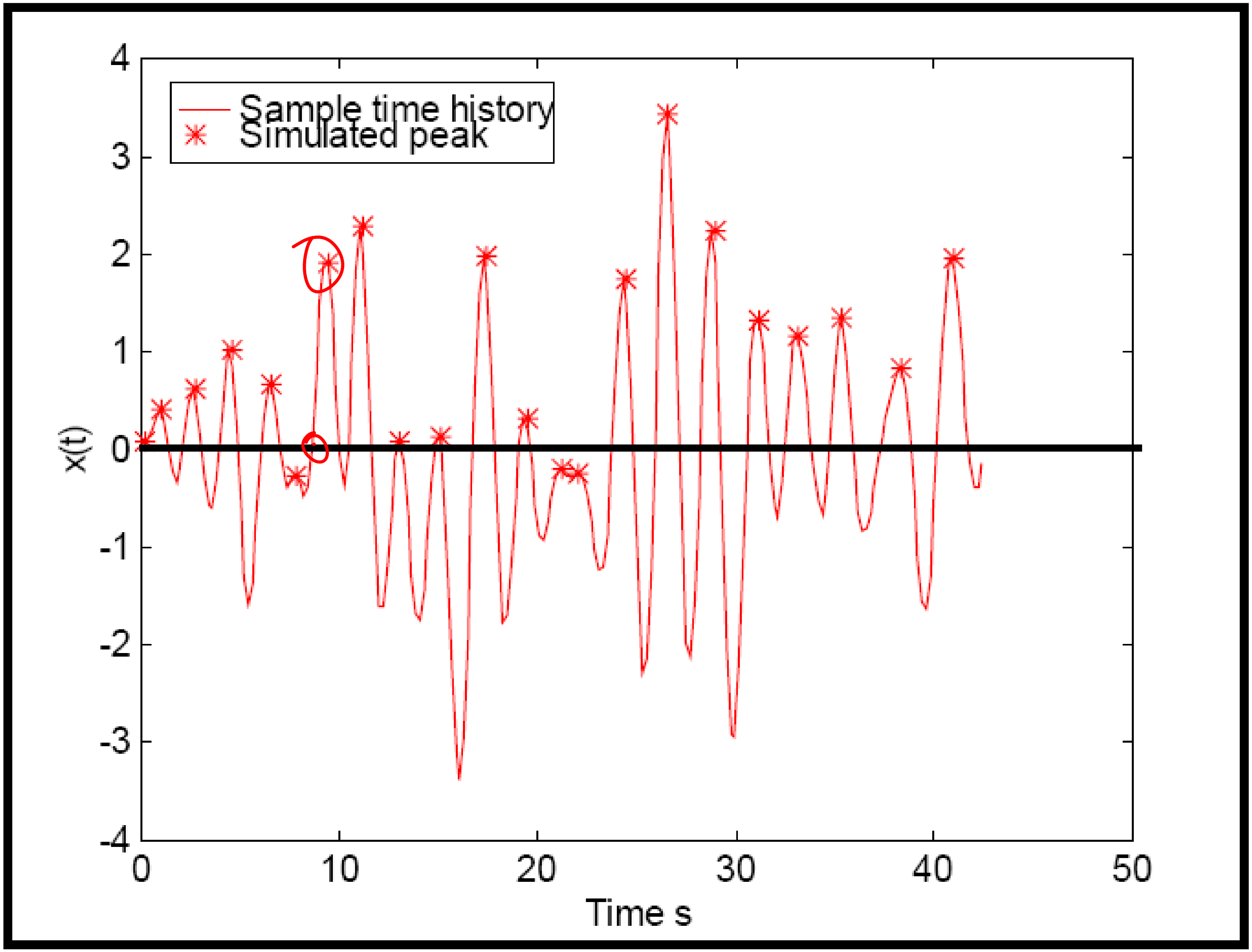
- Order in which stresses are applied does not matter (linear damage accumulation).

In reality failure is sensitive to order of loading.

- Lack of assessment of variability.
- Damage is assumed to accumulate at the same rate at a given stress level without regard to past history.
- Experiments: Miner's sum: 0.25 to 4. For random time histories (no ordered sequence of high/low amplitudes) Miner's sum~ 0.6 to 1.6.

**Extension of PM theory to random stress time histories**

- Assumption: stress time history is a zero mean, narrow band, Gaussian, random process
- There are no "discrete" stress levels



Sample time history
Simulated peak
x(t)
Time s
4
3
2
1
0
-1
-2
-3
-4
0
10
20
30
40
50

$$D(T) = \int_0^T \chi(t)\,dt$$

$\chi(t) =$ rate of accumulation of damage

$$\chi(t) = \frac{n(t)S^b(t)}{c} \qquad \left[ \text{Fashioned after } D = \frac{n_i S_i^b}{c} \right]$$

Interpretation

$S(t) =$ peak magnitude

$n(t) =$ rate of peaks

= rate of zero crossings for narrow band processes

The joint density function $p_{Sn}(s,n;t)$ is not available.

**What we know?**

- Peak magnitudes are Rayleigh distributed (approximation for narrow band processes)

- $\langle n \rangle = \dfrac{\sigma_2}{2\pi\sigma_1}$

Adhoc assumption: $p_{Sn}(s,n;t) = p_S(s;t)\, p_n(n;t)$

$$\Rightarrow \langle \chi(t) \rangle = \left\langle \frac{n(t) S^b(t)}{c} \right\rangle = \frac{1}{c} \langle n(t) \rangle \langle S^b(t) \rangle$$

$$= \frac{1}{c} \frac{\sigma_2}{2\pi\sigma_1} \int_0^\infty s^b \frac{s}{\sigma_1^2} \exp\left( -\frac{s^2}{2\sigma_1^2} \right) ds$$

Recall:

$$\Gamma(\nu) = 2\int_0^\infty y^{2\nu-1}\exp\left(-y^2\right)dy$$

gamma fcn

$$\Rightarrow$$

$$\left\langle \chi(t) \right\rangle = \frac{1}{c}\frac{\sigma_2}{2\pi\sigma_1}\Gamma\left(\frac{b+2}{2}\right)\left(\sigma_1\sqrt{2}\right)^b$$

$$\left\langle D(T) \right\rangle = \int_0^T \left\langle \chi(t) \right\rangle dt = \frac{T}{c}\frac{\sigma_2}{2\pi\sigma_1}\Gamma\left(\frac{b+2}{2}\right)\left(\sigma_1\sqrt{2}\right)^b$$

$$\left\langle D(T)\right\rangle = \frac{T}{c}\frac{\sigma_2}{2\pi\sigma_1}\Gamma\left(\frac{b+2}{2}\right)\left(\sigma_1\sqrt{2}\right)^b$$

We could visualize a $T^*$ such that

$$\left\langle D(T)\right\rangle = 1 \Rightarrow T^* = \frac{1}{\left\langle \chi(t)\right\rangle}.$$

$T^*$ can be taken as the approximate mean fatigue life.

**Remarks**

- b and c have been assumed to be deterministic constants. They can be treated as random variables in which case the above expectation has to be interpreted as a conditional expectation.
- Is $T^*$ indeed the expected fatigue life?

  Let $L$ be the life time. It is a random variable.

  Condtion for failure is $\int_0^L \chi(t)\,dt = 1 \Rightarrow \left\langle \int_0^L \chi(t)\,dt \right\rangle = 1.$

$$\int_0^L \chi(t)\,dt = 1 \Rightarrow \left\langle \int_0^L \chi(t)\,dt \right\rangle = 1$$

This does not mean that $\int_0^{\langle L \rangle} \langle \chi(t) \rangle dt = 1.$

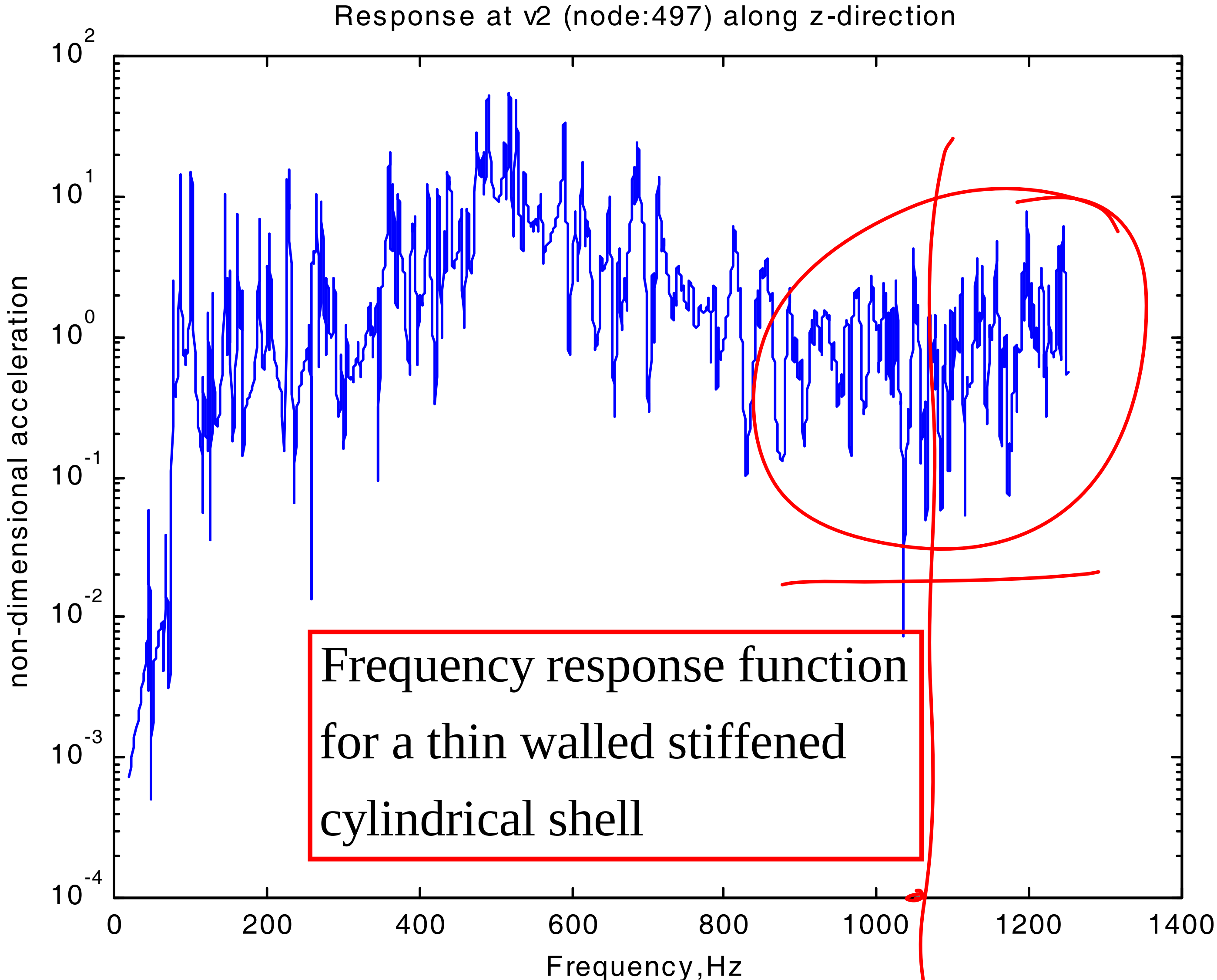
But this is what has been done.

Conclusion: $T^*$ is not the exact expected fatigue life.

- An expression for the variance of $\chi(t)$ is also available.
- Counting algorithms for broad band time histories are available:
    - range pair counting
    - rainflow counting

**Analysis of Vibration in high frequency regime using random vibration principles**

We consider that a structure is vibrating in the *high frequency regime* if its response at any frequency consists of significant contributions from a large number of modes




Frequency response function for a thin walled stiffened cylindrical shell

# Energy flow models: statistical energy analysis (SEA)

- The structural behavior at high frequencies is very sensitive to minor changes in structural parameters and details of modeling. A deterministic approach to modeling structural system parameters is inappropriate.

- Description of dynamic behavior of structural joints, with increasing frequencies becomes difficult. This calls for experimental approaches to characterize structural behavior of joints.

# Energy flow models: statistical energy analysis (SEA)

- Detailed characterization of structural response in terms of spatio-temporal variations of displacement / stress/strain fields becomes unwieldy. Macro-level descriptions that involve space-time-frequency averaged response quantities, such as, vibration energy content in spatial domains, within a structure, may be adequate.

- Thus, method of analysis aimed at establishing spatial distribution of vibration energy stored in the structure become appropriate in analysis of high frequency vibrations.

# SEA can be viewed as a branch of linear vibration theory with following distinguishing features

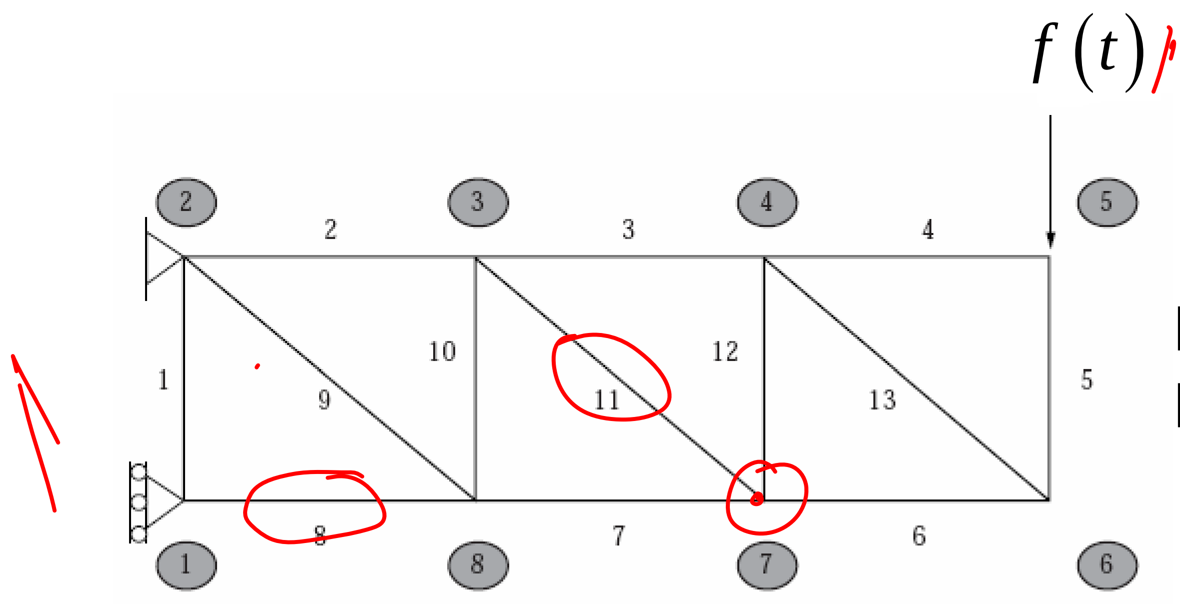
- The built-up structure is taken to be random in nature. It is divided into a set of subsystems and the subsystem natural frequencies are taken to be identical and independently random variables distributed uniformly in the frequency range of interest.

- The external excitations, that are often random in nature, are specified in terms of power input and the governing equation for system behavior are described in terms of power balance between subsystems.

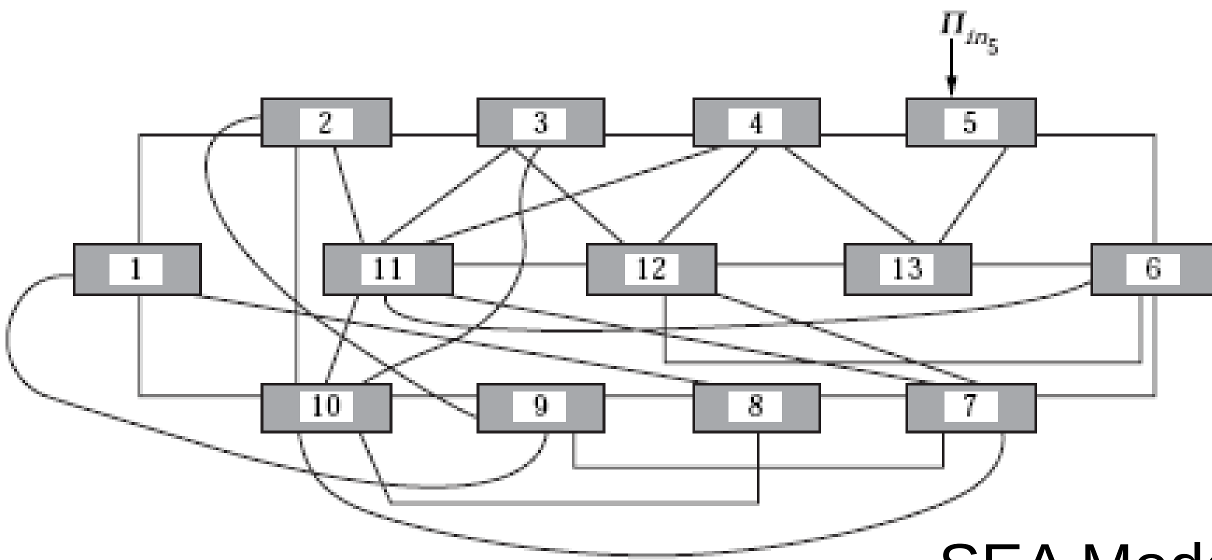
- The primary objective of the response analysis is to determine spatial distribution of total vibration energy residing in the system.

# References

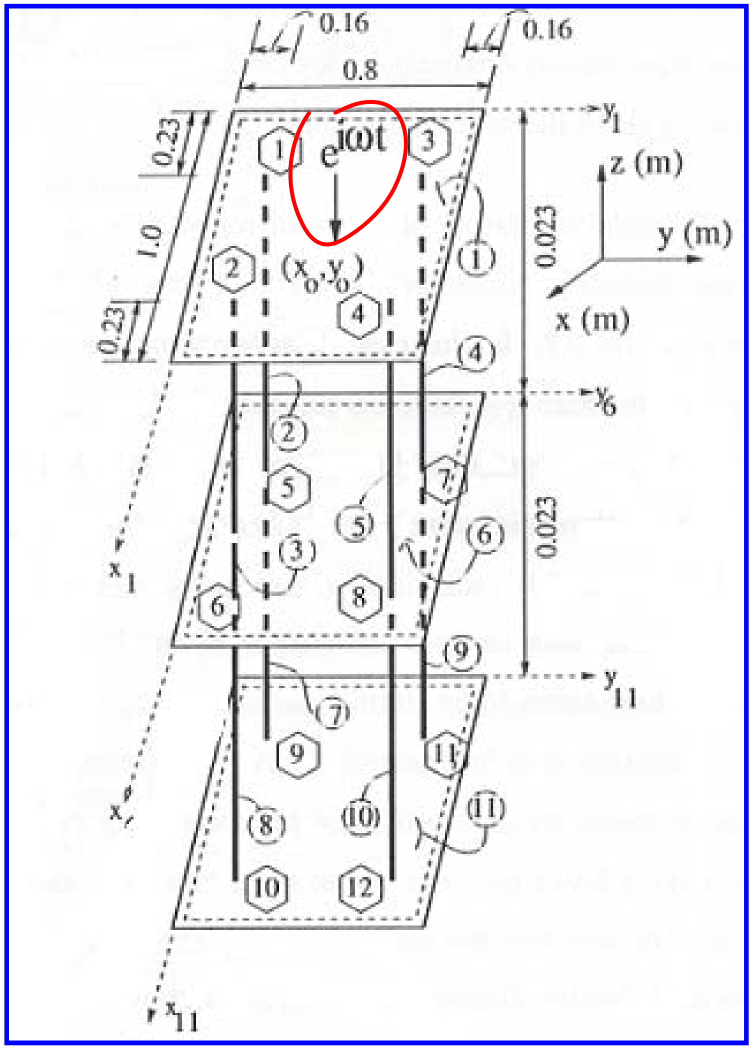

- R H Lyon and R G DeJong, 1995, Theory and application of statistical energy analysis, 2nd Edition, Butterworth-Heinmann, MA.
- J Woodhouse, 1981, An introduction to statistical energy analysis of structural vibration, Applied Acoustics, 14, 455-469.
- C S Manohar and R A Ibrahim, 1999, Progress in structural dynamics with stochastic parameter variations, 1987-98, Applied Mechanics Reviews, 52(5), 177-197.



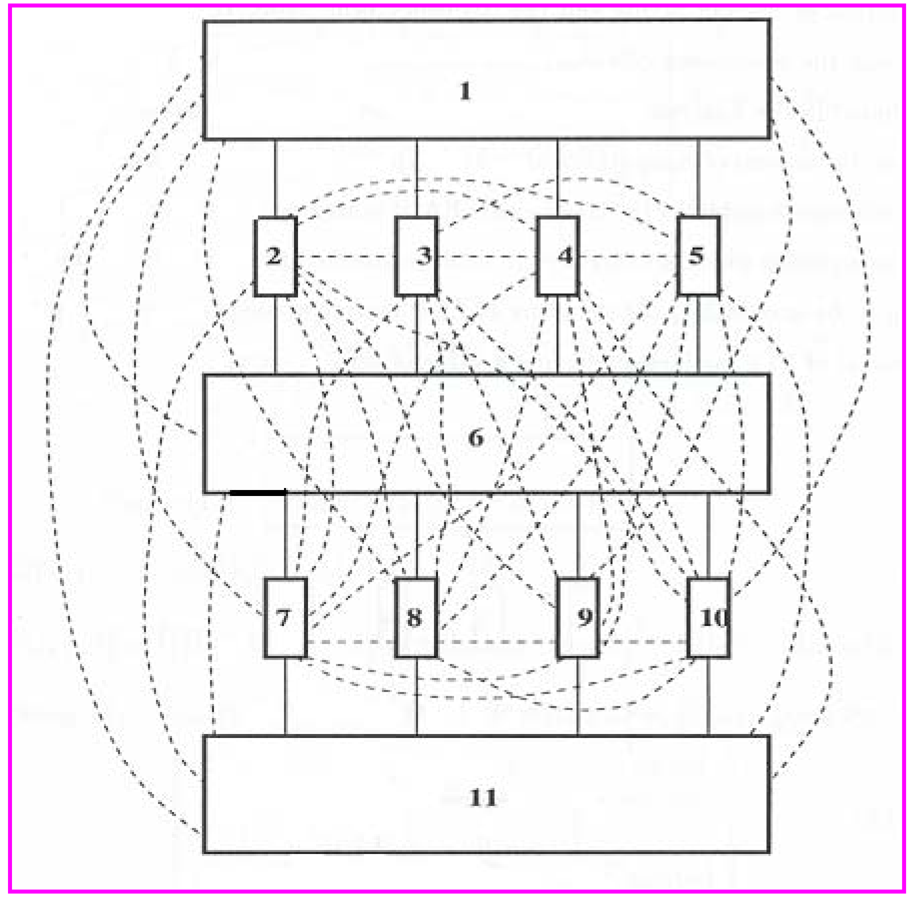

Randomly Excited truss




SEA Model




Schematic diagram of the plate assembly; dimensions not to scale




SEA Model

# SEA: Inputs

- Input power (derivable in terms of the input power spectrum and input power receptance)
- “High” frequency range (20-20kHz)
- Typically random but could be deterministic

# SEA: system modeling

- Collection of energy storing elements called subsystems
- Essentially linear and randomly parametered
- Mass, stiffness and damping are modeled as being random or alternatively modal characteristics are modeled as random.

# SEA: Response quantities

- Steady state, time averaged, total energy stored in each subsystem - often averaged over frequency and ensemble of random realizations

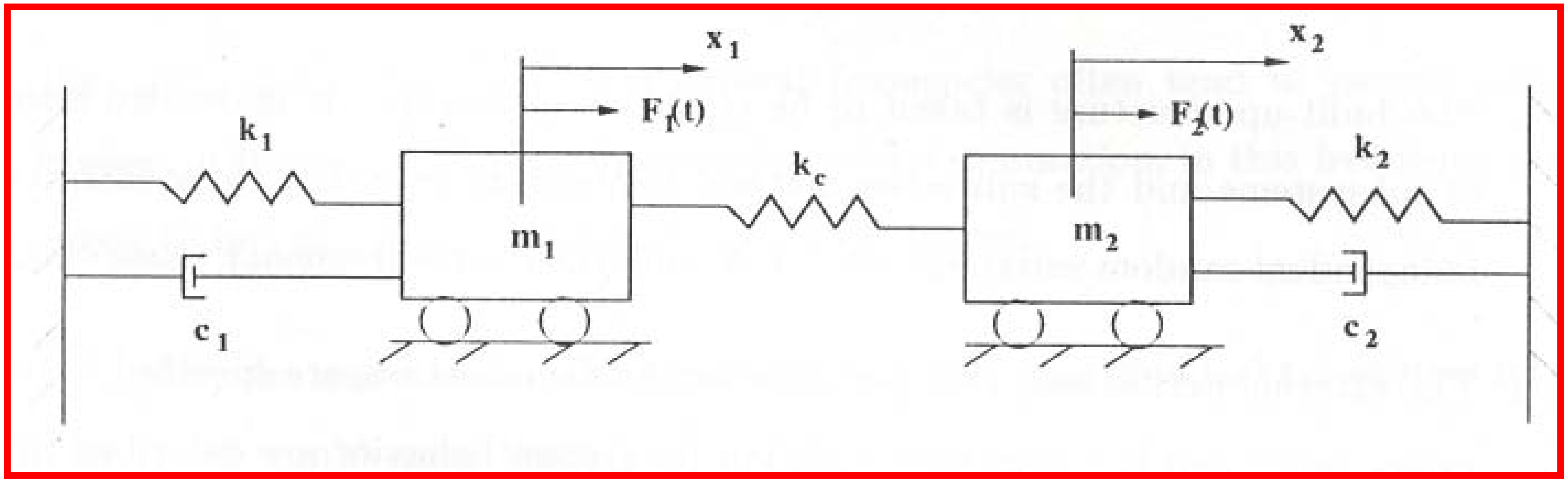
# SEA: Governing equations

- Represents condition of power balance and has the form: $\{\pi_{ini}\} = [\eta_{ij}]\{E_i\}$
- This is essentially a representation in the steady state.
- The equation is in the frequency domain

# SEA: Theoretical foundations:

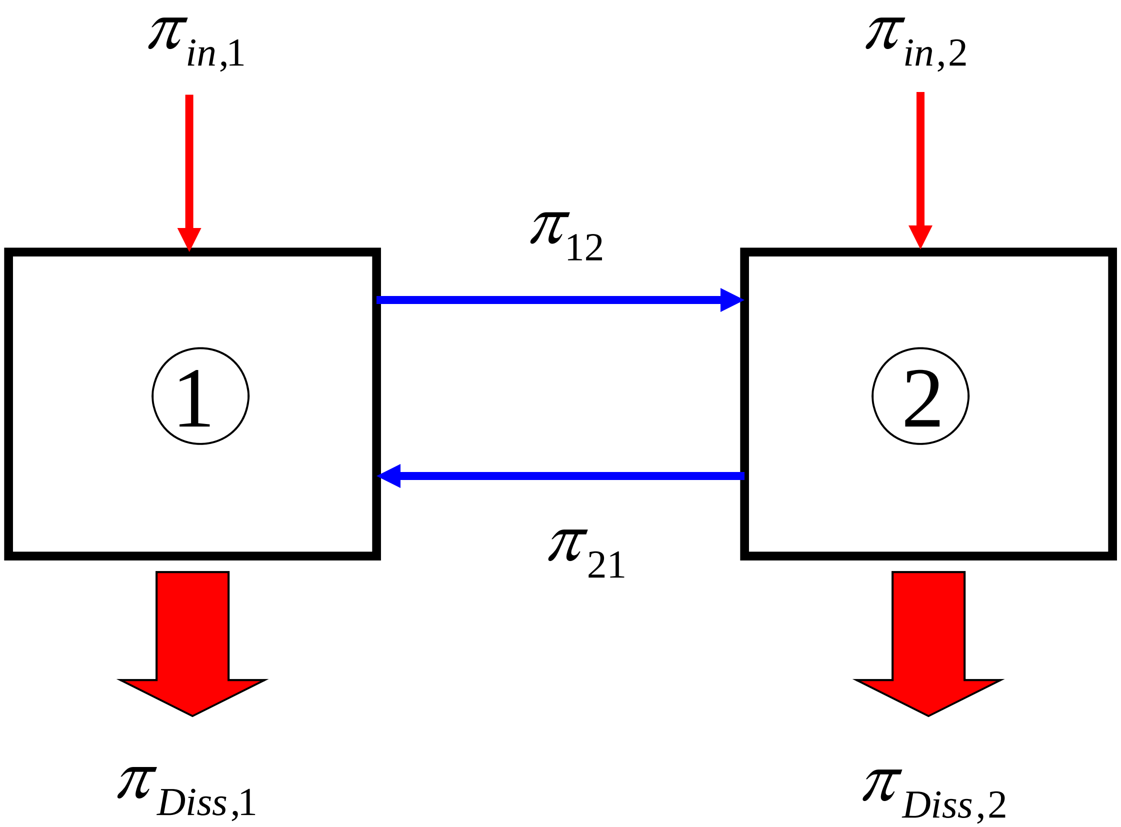
- No sound foundation exists
- No systematic convergence criterion exists.

## Two coupled oscillators: the case of exact results in SEA




$F_1(t) \& F_2(t)$: zero mean stationary Gaussian mutually independent random processes (broad banded; ideally white noise processes)

# Power balance




$$\pi_{in,1} = \pi_{12} - \pi_{21} + \pi_{diss,1}$$

$$\pi_{in,2} = \pi_{21} - \pi_{12} + \pi_{diss,2}$$

Equations of motion

$$m_1\ddot{y}_1 + c_1\dot{y}_1 + k_1 y_1 + k_c(y_1 - y_2) = F_1(t)$$

$$m_2\ddot{y}_2 + c_2\dot{y}_2 + k_2 y_2 + k_c(y_2 - y_1) = F_2(t)$$

$$\langle F_1(t)\rangle = 0; \langle F_1(t)F_1(t+\tau)\rangle = I_1\delta(\tau)$$

$$\langle F_2(t)\rangle = 0; \langle F_2(t)F_2(t+\tau)\rangle = I_2\delta(\tau)$$

$$\langle F_1(t)F_2(t+\tau)\rangle = 0$$

The equation can be recast as

$$m_1\ddot{y}_1 + c_1\dot{y}_1 + (k_1 + k_c)y_1 = F_1(t) + k_c y_2(t)$$

$$m_2\ddot{y}_2 + c_2\dot{y}_2 + (k_2 + k_c)y_2 = F_2(t) + k_c y_1(t)$$

As $t \to \infty$, the system reaches steady state.

**Digress**

$$m\ddot{x}+c\dot{x}+kx=f\left(t\right)$$

$$\left\langle f\left(t\right)\right\rangle=0;\left\langle f\left(t\right)f\left(t+\tau\right)\right\rangle=\delta\left(\tau\right)$$

Consider $t\rightarrow\infty$

$$S_{xf}\left(\omega\right)=\lim_{t\rightarrow\infty}\frac{1}{T}\left\langle X_T\left(\omega\right)F_T^*\left(\omega\right)\right\rangle$$

$$=\lim_{t\rightarrow\infty}\frac{1}{T}\left\langle H\left(\omega\right)F_T\left(\omega\right)F_T^*\left(\omega\right)\right\rangle$$

$$=H\left(\omega\right)S_{FF}\left(\omega\right)=H\left(\omega\right)S_0$$

$$R_{xf}\left(\tau\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H\left(\omega\right)S_0\exp\left(-i\omega\tau\right)d\omega$$

$$\Rightarrow R_{xf}\left(\tau\right)=\frac{S_0h\left(\tau\right)}{2\pi}\Rightarrow R_{xf}\left(0\right)=\frac{S_0h\left(0\right)}{2\pi}=0$$

**Recall**

If $X(t)$ and $Y(t)$ are jointly stationary, we have shown that

$$\left\langle \frac{d^n X(t+\tau)}{dt^n} \frac{d^m Y(t)}{dt^m} \right\rangle = (-1)^m \frac{d^{n+m} R_{XY}(\tau)}{d\tau^{n+m}}$$

$$m_1\ddot{y}_1 + c_1\dot{y}_1 + (k_1 + k_c)y_1 = F_1(t) + k_c y_2(t)$$

$$m_2\ddot{y}_2 + c_2\dot{y}_2 + (k_2 + k_c)y_2 = F_2(t) + k_c y_1(t)$$

Consider $t \to \infty$.

It can be verified that (Exercise)

$$\left\langle F_1(t)y_1(t)\right\rangle = 0; \left\langle F_1(t)y_2(t)\right\rangle = 0; \left\langle F_1(t)y_2(t)\right\rangle = 0; \left\langle F_1(t)y_2(t)\right\rangle = 0$$

$$\left\langle y_1(t)\dot{y}_1(t)\right\rangle = 0; \left\langle y_2(t)\dot{y}_2(t)\right\rangle = 0; \left\langle \dot{y}_1(t)\ddot{y}_1(t)\right\rangle = 0; \left\langle \dot{y}_2(t)\ddot{y}_2(t)\right\rangle = 0$$

$$\left\langle y_1(t)\ddot{y}_1(t)\right\rangle = -\left\langle \dot{y}_1^2\right\rangle; \left\langle y_2(t)\ddot{y}_2(t)\right\rangle = -\left\langle \dot{y}_2^2\right\rangle;$$

$$\left\langle \ddot{y}_1(t)\dot{y}_2(t)\right\rangle = -\left\langle \dot{y}_1(t)\ddot{y}_2(t)\right\rangle; \left\langle y_1(t)\dot{y}_2(t)\right\rangle = -\left\langle \dot{y}_1(t)y_2(t)\right\rangle$$

$$\left\langle \ddot{y}_1(t)y_2(t)\right\rangle = -\left\langle \dot{y}_1(t)\dot{y}_2(t)\right\rangle; \left\langle y_1(t)\ddot{y}_2(t)\right\rangle = -\left\langle \dot{y}_1(t)\dot{y}_2(t)\right\rangle$$

$$\left\langle F_1(t)\dot{y}_2(t)\right\rangle = 0; \left\langle F_2(t)\dot{y}_1(t)\right\rangle = 0;$$

We are interested in

$$\langle \pi_{1,in} \rangle = \langle F_1(t)\dot{y}_1(t) \rangle; \langle \pi_{2,in} \rangle = \langle F_2(t)\dot{y}_2(t) \rangle$$

$$\langle \pi_{1,diss} \rangle = c_1 \langle \dot{y}_1^2 \rangle; \langle \pi_{2,diss} \rangle = c_2 \langle \dot{y}_2^2 \rangle$$

$$\langle \pi_{12} \rangle = k_c \langle y_1 \dot{y}_2 \rangle; \langle \pi_{21} \rangle = k_c \langle y_2 \dot{y}_1 \rangle$$

Since $\langle y_1(t)\dot{y}_2(t) \rangle = -\langle \dot{y}_1(t) y_2(t) \rangle$ it follows

$$\langle \pi_{12} \rangle = -\langle \pi_{21} \rangle$$

$$
\begin{Bmatrix} \langle F_1\dot{y}_1\rangle \\ \langle F_2\dot{y}_2\rangle \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & k_c & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 & -k_c & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1 & k_1+k_c & m_1 \\ 0 & 0 & 0 & 0 & 0 & c_2 & -k_2-k_c & m_1 \\ -m_1 & 0 & k_1+k_c & 0 & -k_c & 0 & 0 & 0 \\ 0 & -m_2 & 0 & k_2+k_c & -k_c & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_c & k_1+k_c & -m_1 & -c_1 & 0 \\ 0 & 0 & -k_c & 0 & k_2+k_c & -m_2 & c_2 & 0 \end{bmatrix} \begin{Bmatrix} \langle \dot{y}_1^2\rangle \\ \langle \dot{y}_2^2\rangle \\ \langle y_1^2\rangle \\ \langle y_2^2\rangle \\ \langle y_1 y_2\rangle \\ \langle \dot{y}_1\dot{y}_2\rangle \\ \langle y_1\dot{y}_2\rangle \\ \langle \ddot{y}_1\dot{y}_2\rangle \end{Bmatrix}
$$

First two rows:

$$c_1 \left\langle \dot{y}_1^2 \right\rangle + k_c \left\langle y_1 \dot{y}_2 \right\rangle = \left\langle F_1(t)\, \dot{y}_1(t) \right\rangle$$

$$c_2 \left\langle \dot{y}_2^2 \right\rangle - k_c \left\langle y_2 \dot{y}_1 \right\rangle = \left\langle F_2(t)\, \dot{y}_2(t) \right\rangle$$

$$\Rightarrow$$

$$\left\langle \pi_{1,diss} \right\rangle + \left\langle \pi_{12} \right\rangle = \left\langle \pi_{1,in} \right\rangle$$

$$\left\langle \pi_{2,diss} \right\rangle - \left\langle \pi_{12} \right\rangle = \left\langle \pi_{2,in} \right\rangle$$

Law of conservation of energy

Rows 5 and 6 $\Rightarrow$

$$-m_1\left\langle \dot{y}_1^2 \right\rangle + \left(k_1 + k_c\right)\left\langle y_1^2 \right\rangle - k_c\left\langle y_1 y_2 \right\rangle = 0$$

$$-m_2\left\langle \dot{y}_2^2 \right\rangle + \left(k_1 + k_c\right)\left\langle y_2^2 \right\rangle - k_c\left\langle y_1 y_2 \right\rangle = 0$$

$\Rightarrow$

$$-\left\langle KE_1 \right\rangle + \left\langle PE_1 \right\rangle = 2k_c\left\langle y_1 y_2 \right\rangle$$

$$-\left\langle KE_2 \right\rangle + \left\langle PE_2 \right\rangle = 2k_c\left\langle y_1 y_2 \right\rangle$$

Remark

In the absence of coupling $\left(k_c = 0\right)$ we get

$$\left\langle KE_1 \right\rangle = \left\langle PE_1 \right\rangle; \ \left\langle KE_2 \right\rangle = \left\langle PE_2 \right\rangle$$

For $k_c << k_1, k_2$ (light coupling/weak coupling)

$$\langle y_1 \dot{y}_2 \rangle = \frac{k_c^2 \left[ \frac{c_1}{m_1} + \frac{c_2}{m_2} \right] \left[ \frac{m_1}{c_1} \langle F_1 \dot{y}_1 \rangle - \frac{m_2}{c_2} \langle F_2 \dot{y}_2 \rangle \right]}{\left[ \frac{k_1 + k_c}{m_1} - \frac{k_2 + k_c}{m_2} \right]^2 + \left[ \frac{c_1}{m_1} + \frac{c_2}{m} \right] \left[ \frac{c_1 (k_2 + k_c) + c_2 (k_1 + k_c)}{m_1 m_2} \right]}$$

$\Rightarrow$

$$\langle \pi_{12} \rangle = h_{12} \left[ \frac{m_1}{c_1} \langle \pi_{1,in} \rangle - \frac{m_2}{c_2} \langle \pi_{2,in} \rangle \right]$$

$$h_{12} = \frac{k_c^2 \left[ \frac{c_1}{m_1} + \frac{c_2}{m_2} \right]}{\left[ \frac{k_1 + k_c}{m_1} - \frac{k_2 + k_c}{m_2} \right]^2 + \left[ \frac{c_1}{m_1} + \frac{c_2}{m} \right] \left[ \frac{c_1 (k_2 + k_c) + c_2 (k_1 + k_c)}{m_1 m_2} \right]}$$

For $k_c << k_1, k_2$ (light coupling/weak coupling)

$$\langle \pi_{1,in} \rangle = \langle \pi_{1,diss} \rangle = c_1 \langle \dot{y}_1^2 \rangle = \frac{2c_1}{m_1} \langle KE_1 \rangle$$

$$\langle \pi_{2,in} \rangle = \langle \pi_{2,diss} \rangle = c_2 \langle \dot{y}_1^2 \rangle = \frac{2c_1}{m_2} \langle KE_2 \rangle$$

$$\langle KE_1 \rangle = \langle PE_1 \rangle$$

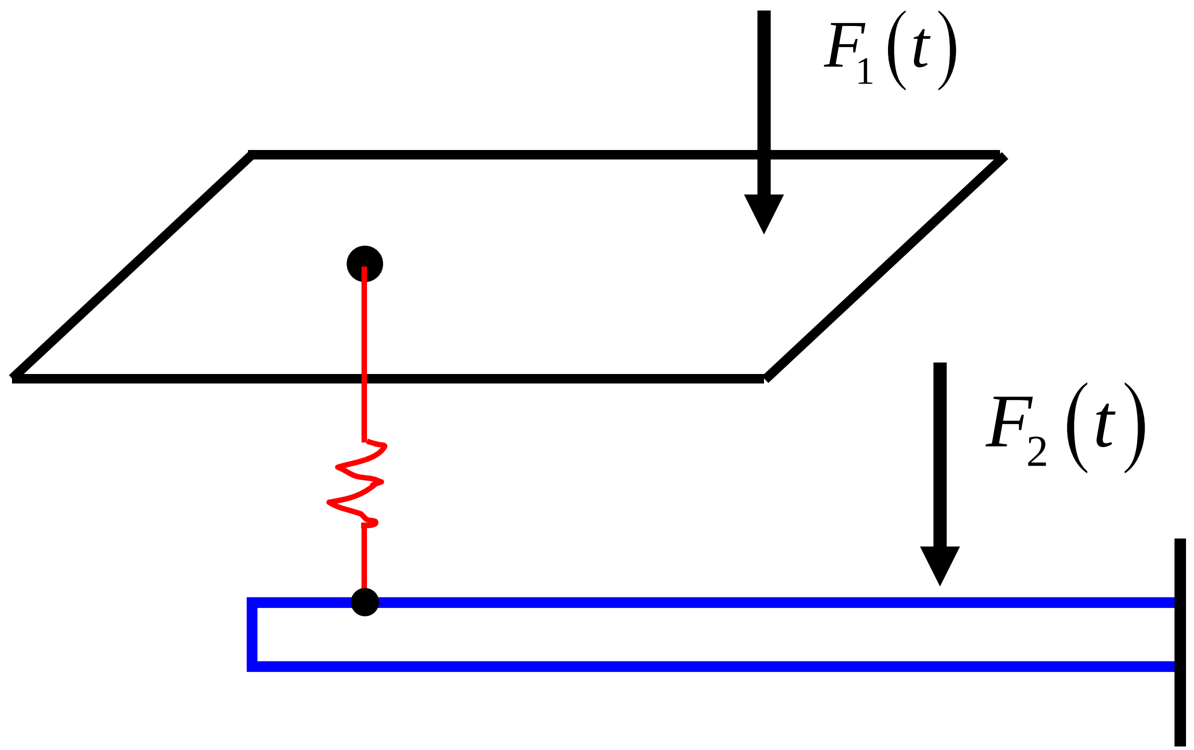
$$\langle KE_2 \rangle = \langle PE_2 \rangle$$

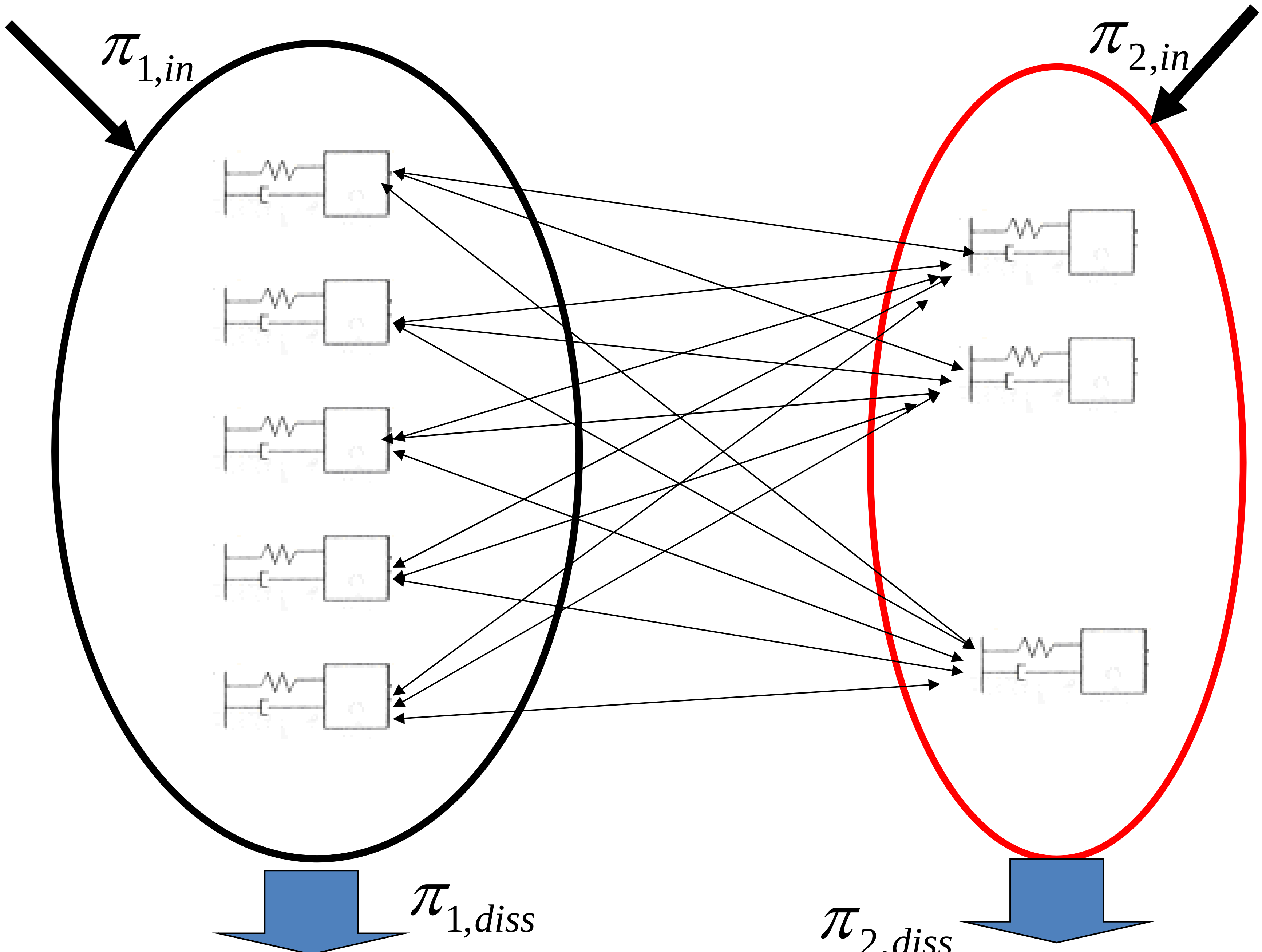
$\Rightarrow$ Fundamental SEA result

$$\langle \pi_{12} \rangle = h_{12} \left[ \langle KE_1 + PE_1 \rangle - \langle KE_2 + PE_2 \rangle \right]$$

$h_{12}$ = coupling loss factor

# Extension to coupled multi-modal systems




$\pi_{1,in}$
$\pi_{2,in}$
$\pi_{1,diss}$
$\pi_{2,diss}$

$$\left\langle \pi_{\alpha\beta} \right\rangle = \omega \left[ \frac{1}{\Delta\omega} \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} \eta_{ij} \right] \left\{ \frac{\left\langle E_\alpha \right\rangle}{n_\alpha} - \frac{\left\langle E_\beta \right\rangle}{n_\beta} \right\}$$

$\omega =$ central frequency

$\Delta\omega$=frequency bandwidth

$\left\langle E_\alpha \right\rangle =$ total average energy in subsystem $\alpha$

$n_\alpha =$ number of modes per unit frequency interval for the subsystem $\alpha$.

Energy flow between conservatively coupled linear subsystems excited by broad band random excitation is proportional to the difference between subsystem average modal energies

**SEA equation**

$$\omega \underbrace{\begin{bmatrix} \left(\eta_i + \eta_{ij}\right) & -\eta_{ji} \\ -\eta_{ij} & \left(\eta_j + \eta_{ji}\right) \end{bmatrix}}_{\text{Matrix of coupling loss factors}} \underbrace{\begin{Bmatrix} E_i \\ E_j \end{Bmatrix}}_{\text{Unknowns}} = \underbrace{\begin{Bmatrix} \Pi_{ini} \\ \Pi_{inj} \end{Bmatrix}}_{\text{Input power}}$$

**SEA equation for a system of $n$ - subsystems**

$$\omega\begin{bmatrix} \left(\eta_1+\sum_{\substack{i=1\\ i\neq 1}}^{N_s}\eta_{1i}\right) & -\eta_{12} & \cdots & -\eta_{1N_s} \\ -\eta_{21} & \left(\eta_2+\sum_{\substack{i=1\\ i\neq 2}}^{N_s}\eta_{2i}\right) & \cdots & -\eta_{2N_s} \\ \vdots & \vdots & \ddots & \vdots \\ -\eta_{N_s1} & -\eta_{N_s2} & \cdots & \left(\eta_{N_s}+\sum_{\substack{i=1\\ i\neq N_s}}^{N_s}\eta_{N_si}\right) \end{bmatrix}\begin{Bmatrix} E_1 \\ E_2 \\ \vdots \\ E_{N_s} \end{Bmatrix}=\begin{Bmatrix} \Pi_{in1} \\ \Pi_{in2} \\ \vdots \\ \Pi_{inN_s} \end{Bmatrix}$$

**Further reading**

- F J Fahy, 1994, Statistical energy analysis : a critical review, Philosophical Transactions of Royal Society of London, A346, 431 – 447.
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- A J Keane, 1992, Energy flow between arbitrary configurations of conservatively coupled multi-modal elastic subsystems, Proceedings of Royal Society of London, A, 436, 537-568.