# Stochastic Structural Dynamics

## Lecture-35

## Probabilistic methods in earthquake engineering-4

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**Spatial variability of earthquake ground motions and response of multi - supported structures**

- Long span bridges, large dams, pipelines, tunnels, ···

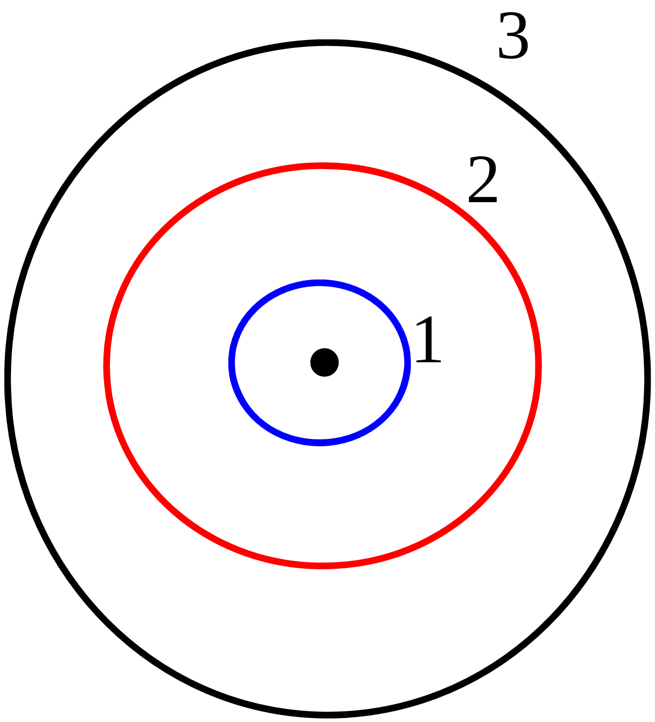
Reference : A Zerva, 2009, Spatial variations of seismic ground motions, CRC Press, Boca Raton

**SMART array at Taiwan (~ 1980 - s)**

| Circle | Radius km | SMA-s |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0.2 | 4 |
| 2 | 1.0 | 12 |
| 3 | 2.0 | 12 |

Two more stations at 2.8 and 4.8 km south of the centre.

Tri-axial accelerometers at every station

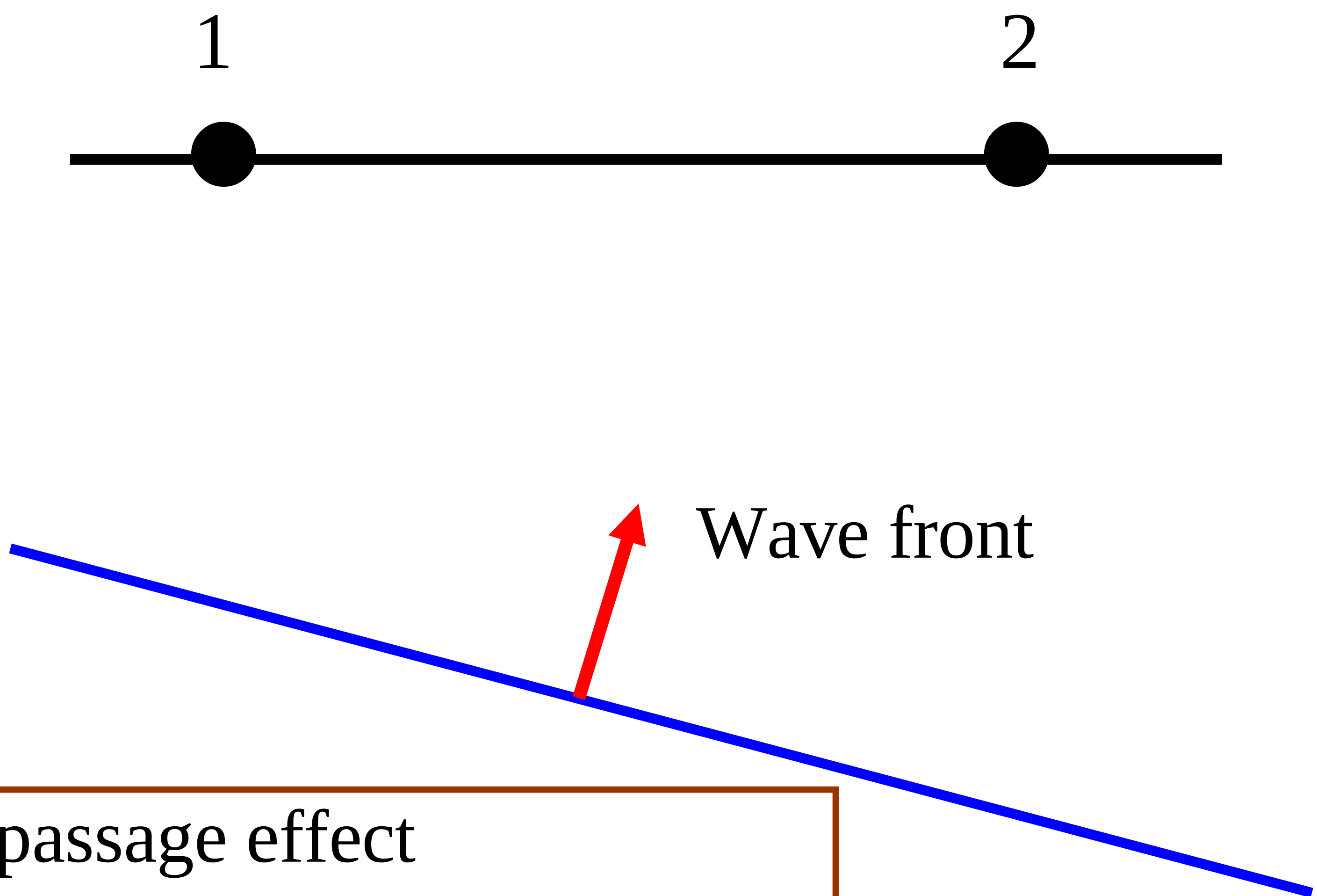


Focus: land based structures
(and not secondary systems like piping and rotors in industrial complexes)

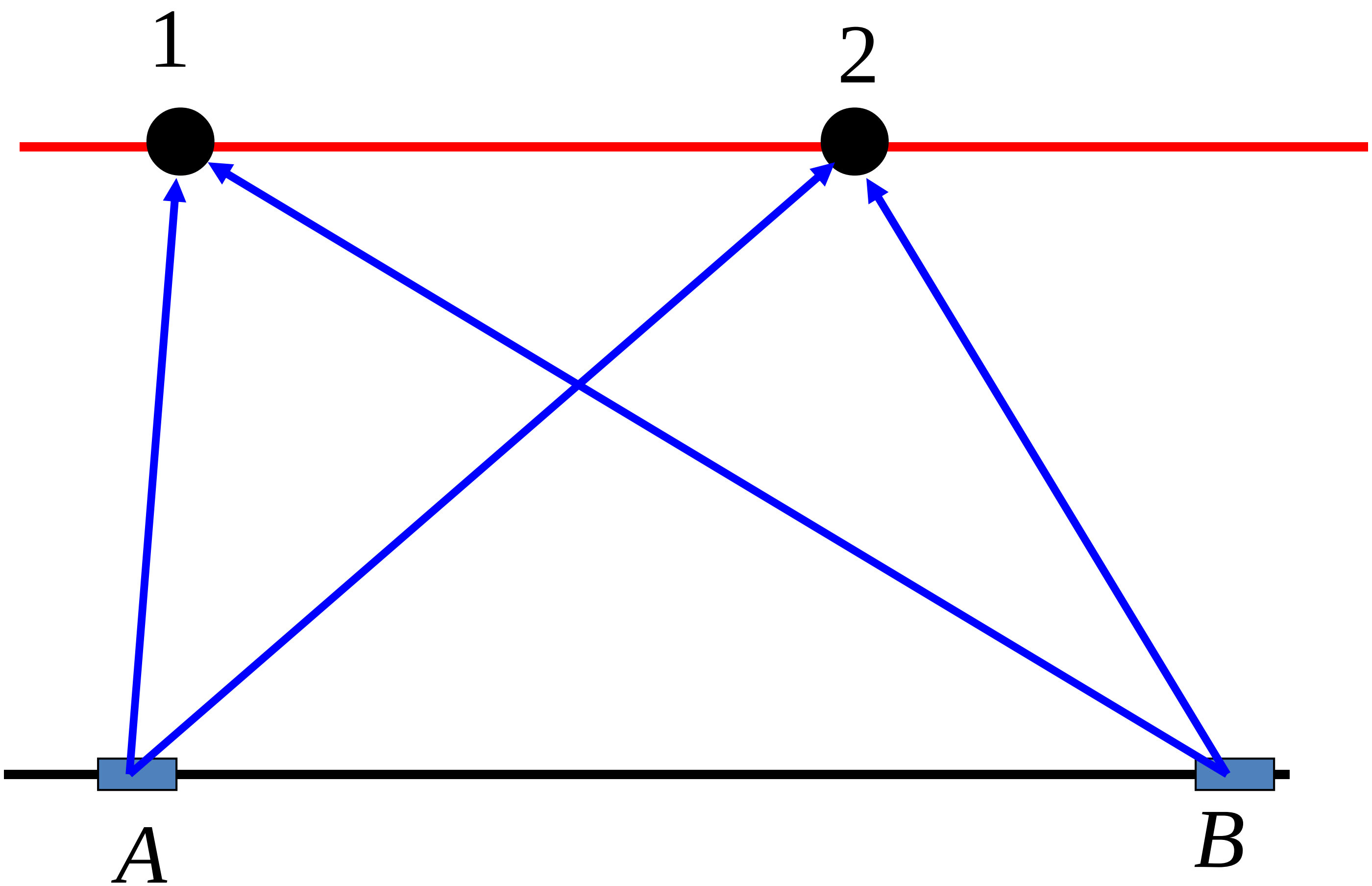
The assumption of uniform support motions is not guranteed to provide conservative estimates of response.

**Why spatial variability occurs?**

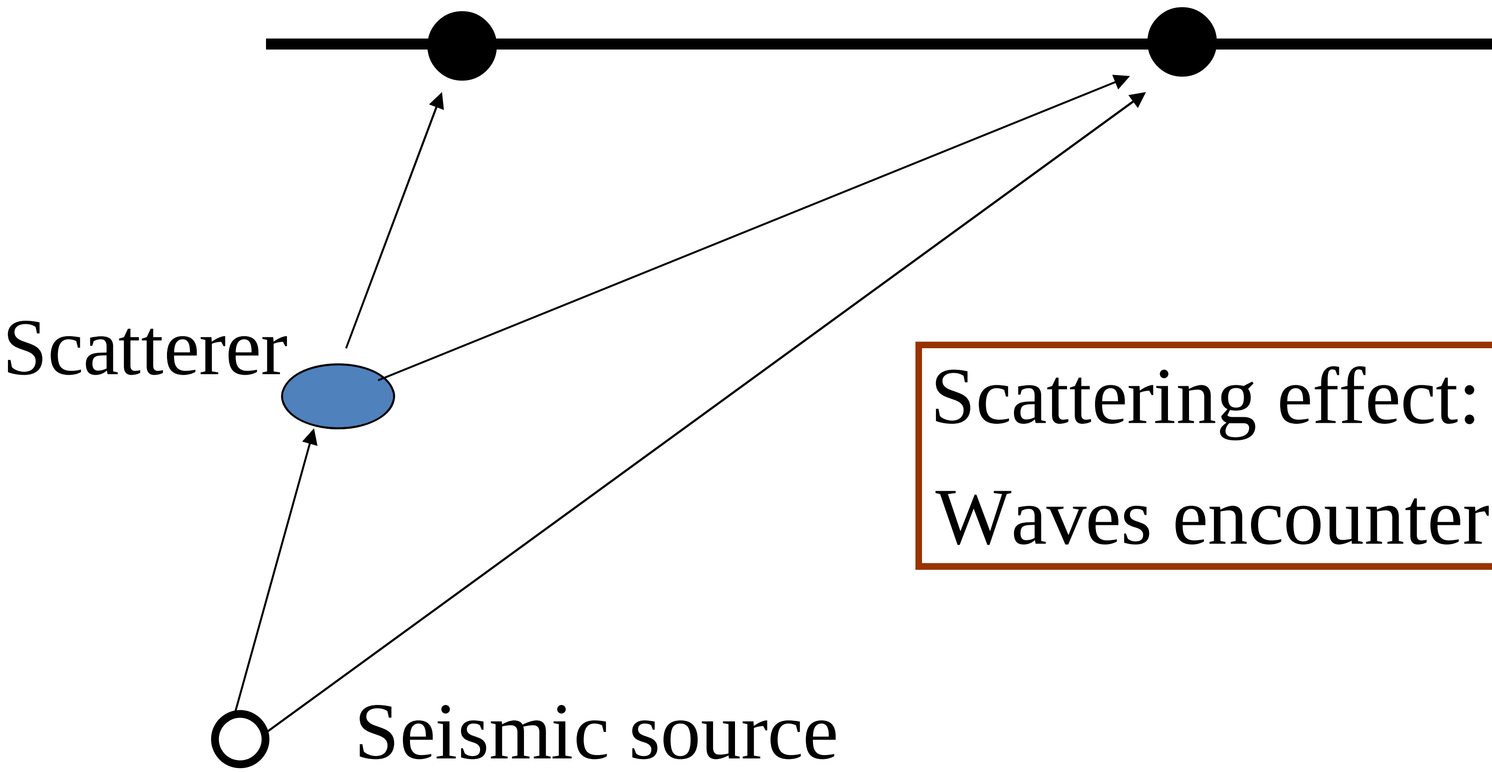
- Wave passage effect
- Extended source effect
- Scattering effect
- Attenuation effect




Wave passage effect
Inclined incidence of plane waves
leading to time delays in arrival

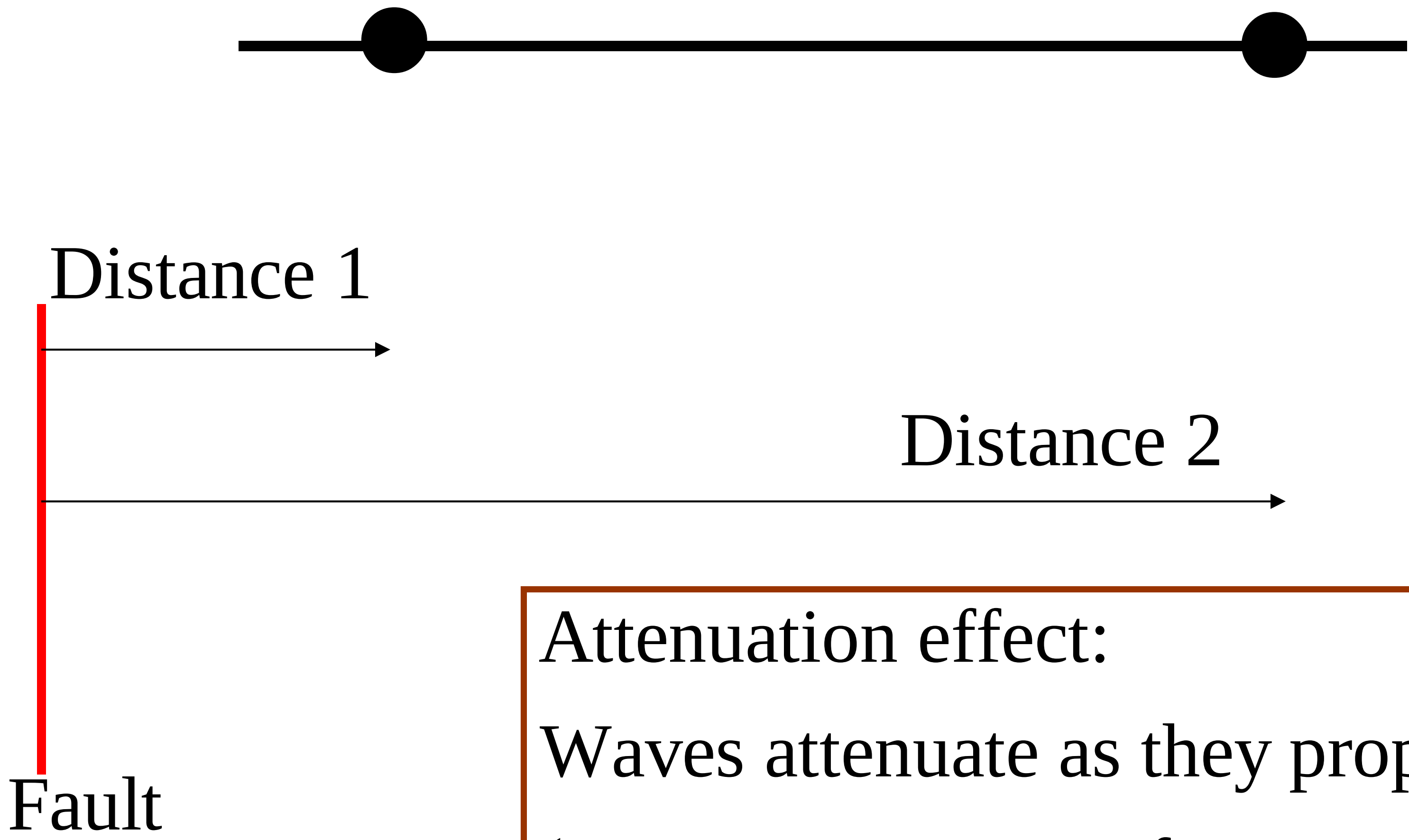



Extended source effect: as rupture propagates along an extended fault, it transmits energy that arrives delayed on the ground surface.




Scattering effect:

Waves encounter scatterers



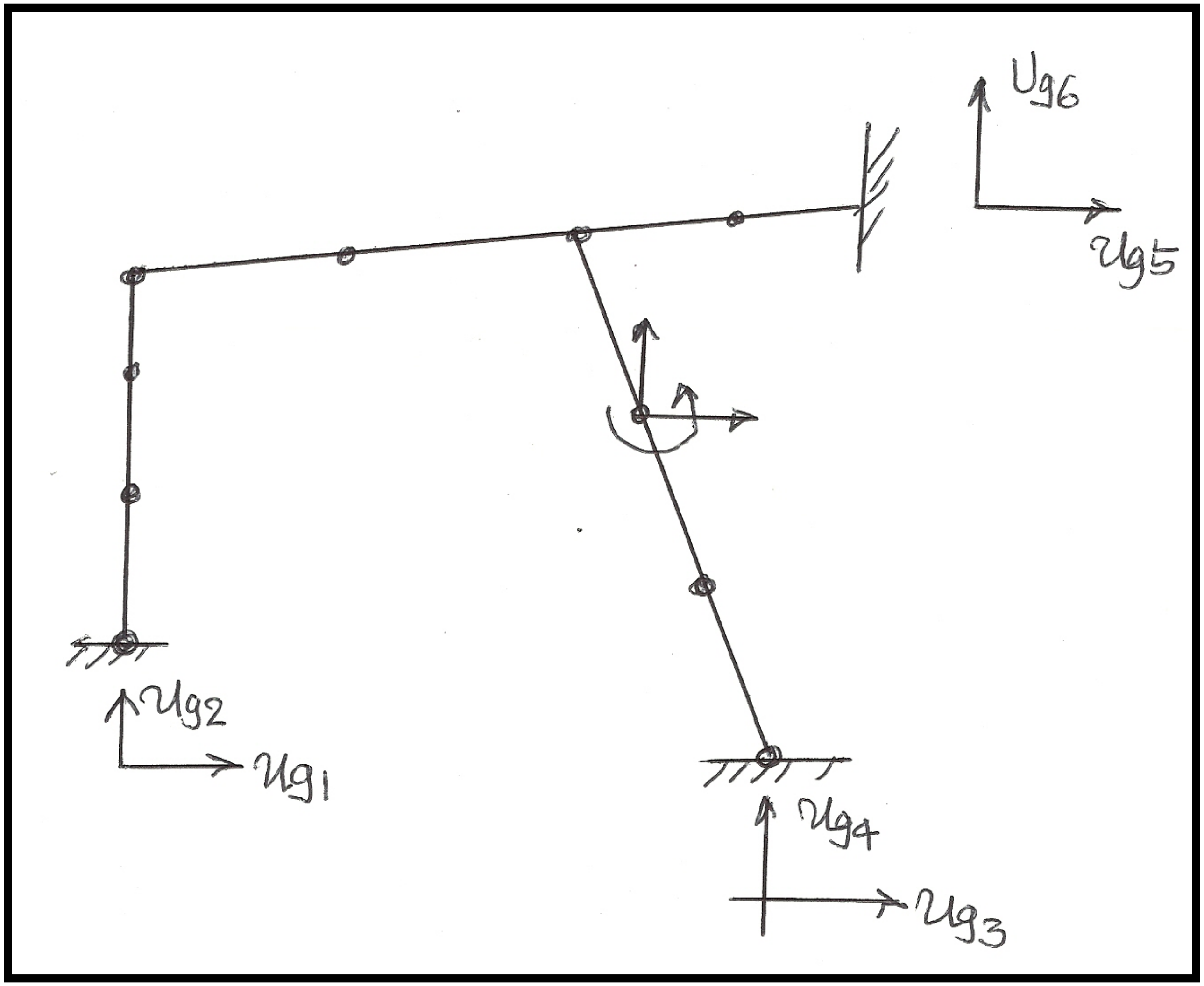

Attenuation effect:

Waves attenuate as they propagate (not very important for engineering structures)

**Questions**

- What are the phenomenological features associated with response of structures subjected to spatially varying ground motions?
- When it is important to consider them?
- How to model spatially varying ground motions as random processes?
  - Based on data
  - Based on phenomenological considerations
- How to develop modal combination rules when the inputs are specifed in terms of response spectra?

# Structures under differential support motions




Model $\boldsymbol{U_g}(\boldsymbol{t})$ as a vector random process

**Recall**

**Description of two random processes**

Covariance matrix

$$C(t_1,t_2)=\begin{bmatrix} C_{UU}(t_1,t_2) & C_{UV}(t_1,t_2) \\ C_{VU}(t_1,t_2) & C_{VV}(t_1,t_2) \end{bmatrix}$$

$$C(\tau)=\begin{bmatrix} C_{UU}(\tau) & C_{UV}(\tau) \\ C_{VU}(\tau) & C_{VV}(\tau) \end{bmatrix}$$

$$C_{UV}(\tau)=\langle U(t)V(t+\tau)\rangle=\langle V(t+\tau)U(t)\rangle$$

$$\Rightarrow C_{UV}(\tau)=C_{VU}(-\tau)$$

PSD matrix

$$S(\omega)=\begin{bmatrix} S_{UU}(\omega) & S_{UV}(\omega) \\ S_{VU}(\omega) & S_{VV}(\omega) \end{bmatrix}$$

$$S_{UU}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle X_T(\omega)\,X_T^*(\omega)\right\rangle \Rightarrow S_{UU}(\omega)=S_{UU}(-\omega)$$

$$S_{UV}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle U_T(\omega)V_T^*(\omega)\right\rangle \Rightarrow$$

$$S_{UV}(-\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle U_T(-\omega)V_T^*(-\omega)\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle U_T^*(\omega)V_T(\omega)\right\rangle=S_{VU}(\omega)$$

$$S_{UV}^*(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle U_T^*(\omega)V_T(\omega)\right\rangle=\lim_{T\to\infty}\frac{1}{T}\left\langle U_T(-\omega)V_T^*(-\omega)\right\rangle=S_{UV}(-\omega)$$

$$S_{UV}(\omega) = \int_{-\infty}^{\infty} R_{UV}(\tau)\exp(i\omega\tau)\,d\tau$$

$$R_{UV}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{UV}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$S_{UV}(\omega) = \left|S_{UV}(\omega)\right|\exp\left[-i\phi(\omega)\right]$$

$$\left|S_{UV}(\omega)\right| = \text{amplitude of cross PSD function}$$

$$\phi(\omega) = \text{phase spectrum}$$

$$p_{UV}(\omega) = \mathrm{Re}\left[S_{UV}(\omega)\right] = \text{co-spectrum}$$

$$q_{UV}(\omega) = \mathrm{Im}\left[S_{UV}(\omega)\right] = \text{quadrature spectrum}$$

Complex coherency function

$$\mathrm{coh}_{UV}(\omega) = \frac{S_{UV}(\omega)}{\sqrt{S_{UU}(\omega) S_{VV}(\omega)}}$$

$$\mathrm{coh}_{UV}(\omega) = \left|\mathrm{coh}_{UV}(\omega)\right| \exp\left(-i\theta(\omega)\right) \text{Coherency}$$

$$\left|\mathrm{coh}_{UV}(\omega)\right| = \frac{\left|S_{UV}(\omega)\right|}{\sqrt{S_{UU}(\omega) S_{VV}(\omega)}}$$

$$0 \leq \left|\mathrm{coh}_{UV}(\omega)\right| \leq 1$$

$$\left|\mathrm{coh}_{UV}(\omega)\right| = 0$$

$\Rightarrow$ lack of linear dependency between two processes

Two processes are linearly related

$$\left|\mathrm{coh}_{UV}(\omega)\right| = 1$$

**Semi - empirical model** A Der Kiureghian, 1996,
A coherency model for spatially varying ground motion,
EESD, 25,99-111.

**Phenomena leading to spatial variability**

- Incoherency effect:
  Scattering in heterogeneous medium and differential superpositioning of waves arriving from extended source.
- Wave passage effect (time delays)
- Attenuation effect
- Site-response effect

Consider two stations $k$ and $l$ and the ground accelerations $a_k(t)$ and $a_l(t)$. Let these be modeled as zero mean, stationary Gaussian random processes. The coherency function is given by

$$\gamma_{kl}(\omega)=\begin{cases}\dfrac{G_{a_k a_l}(\omega)}{\sqrt{G_{a_k a_k}(\omega)G_{a_l a_l}(\omega)}} & \text{for } G_{a_k a_k}(\omega)G_{a_l a_l}(\omega)\neq 0 \\ 0 & \text{for } G_{a_k a_k}(\omega)G_{a_l a_l}(\omega)\overset{0}{\cancel{\neq}} 0\end{cases}$$

$$\gamma_{kl}(\omega)=\left|\gamma_{kl}(\omega)\right|\exp\left[i\theta_{kl}(\omega)\right]$$

$$\theta_{kl}(\omega)=\tan^{-1}\frac{\operatorname{Im}\left[\gamma_{kl}(\omega)\right]}{\operatorname{Re}\left[\gamma_{kl}(\omega)\right]}$$

**Digress**

$$a(t)=\sum_{i=1}^{n} A_i \cos(\omega_i t+\phi_i)$$

$$A_i \sim N(0,\sigma_i^2); A_i \perp A_j \forall i \neq j \in [1,n]$$

$$\phi_i \sim \text{iid U}(0,2\pi); \phi_i \perp A_j \forall i, j \in [1,n]$$

$$\Rightarrow \langle a(t) \rangle = 0 \ \& \ \langle a(t)a(t+\tau) \rangle = \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 \cos\omega_i\tau$$

$$G_{aa}(\omega)=\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\delta(\omega-\omega_i)$$

This PSD can be taken to be a discrete approximation to

$$\mathrm{G}_{\tilde{a}\tilde{a}}(\omega)=\frac{\sigma_i^2}{2\Delta\omega} \ \text{ for } \omega_i-\frac{1}{2}\Delta\omega \leq \omega < \omega_i+\frac{1}{2}\Delta\omega$$

Consider two stations $k$ and $l$ and the ground accelerations $a_k(t)$ and $a_l(t)$.

$$a_k(t) = \sum_{i=1}^{N} A_i f_k(\omega_i, r_k)\cos(\omega_i t + \phi_i)$$

$$a_l(t) = \sum_{i=1}^{N} \left(p_{kl,i} A_i + q_{kl,i} B_i\right) f_l(\omega_i, r_l)\cos\left[\omega_i \left(t - \tau_{kl,i}\right) + \phi_i + \varepsilon_{kl,i}\right]$$

$(A_i, B_i) =$ random variables; $A_i \perp B_j \forall i, j = 1, 2, \cdots, N$.

$A_i \perp A_j \forall i \neq j; B_i \perp B_j \forall i \neq j$

$\langle A_i \rangle = \langle B_i \rangle = 0; \langle A_i^2 \rangle = \langle B_i^2 \rangle$

$\phi_i$ : iid $\sim U[0, 2\pi]; \varepsilon_{kl,i}$ : iid $\sim N(0, \alpha_{kl,i}); \phi_i \perp \varepsilon_{kl,j} \perp A_r \perp B_s \forall i, j, r, s \in [1, N]$

$\tau_{kl,i}$ : arrival time delay of the $i$-th component from station $k$ to $l$

$p_{kl,i}, q_{kl,i}$ : deterministic constants; $p_{kl,i}^2 + q_{kl,i}^2 = 1$; $p_{kl,i} = \cos\beta_{kl,i}; q_{kl,i} = \sin\beta_{kl,i}$

$r_m =$ source to site distance, $m = k, l$;

$f_k(\omega_i, r_k)$ : attenuation law; $0 \leq f_k(\omega_i, r_k) \leq 1$

$$\gamma_{kl}(\omega) = \cos\left[\beta(d_{kl},\omega)\right]\exp\left[-\frac{1}{2}\alpha^2(d_{kl},\omega)\right]$$

$$\exp\left\{i\left[\theta_{kl}^{\text{wave passage}}(\omega) + \theta_{kl}^{\text{site response}}(\omega)\right]\right\}$$

$$\theta_{kl}^{\text{wave passage}}(\omega) = -\frac{\omega d_{kl}^{L}}{v_{\text{app}}(\omega)}$$

$$\theta_{kl}^{\text{site response}}(\omega) = \tan^{-1}\frac{\operatorname{Im}\left[H_k(\omega)H_l(-\omega)\right]}{\operatorname{Re}\left[H_k(\omega)H_l(-\omega)\right]}$$

$d_{kl}$ = distance between sites $k$ and $l$;

$d_{kl}^{L}$ = projection of $d_{kl}$ along the direction of wave propagation

$v_{\text{app}}(\omega)$ = apparent shear wave velocity

$H_m(\omega)$ = bed rock to surface transfer funciton; $m = k, l$

Data based models (Zerva 2009)

- Are the random fields isotropic? That is, does the covariance between two stations depend upon the separation distance and not on the direction? Does the direction of wave propagation matter in this context?
- The notion of principal axes is taken to be valid for the array data. That is, for ground motion propagating in the general epicentral direction, the components of ground motions can be taken to be uncorrelated in the physical direction.

Loh and others

- One dimensional isotropic models

$$\left|\gamma\left(\xi,\omega\right)\right| = \exp\left[-a\left(\omega\right)\xi\right]$$

$$\left|\gamma\left(\xi,\omega\right)\right| = \exp\left[-a\frac{\omega\xi}{2\pi c}\right]$$

$$\left|\gamma\left(\xi,\omega\right)\right| = \exp\left[\left(-a-b\omega^{2}\right)\xi\right]$$

$$\left|\gamma\left(\xi,\omega\right)\right| = \exp\left[\left(-a-b\omega\right)\xi^{\nu}\right]$$

$\xi =$ distance between sites; $\omega$=frequency rad/s

- Directionally dependent coherency model

$$\left|\gamma(\xi,\omega)\right| = \exp\left[\left(-a_1 - b_1\omega^2\right)\left|\xi\cos\theta\right|\right]\exp\left[\left(-a_2 - b_2\omega^2\right)\left|\xi\sin\theta\right|\right]$$

$\theta =$ angle between direction of wave propagation and line joining the sites

Typical values of model parameters

$a_1 = 0.02; b_1 = 0.0025; a_2 = 0.02; b_2 = 0.0012$

$\xi$ : measured in km

Harichandran and Vanmarcke model

$$\left|\gamma(\xi, f)\right| = A\exp\left[-\frac{2B\xi}{av(f)}\right] + (1-A)\exp\left[-\frac{2B\xi}{v(f)}\right]$$

$$v(f) = k\left[1+\left(\frac{f}{f_0}\right)^b\right]^{-\frac{1}{2}}; B = 1 - aA + A$$

$$A = 0.736; a = 0.147; k = 5210 \text{ m}; f_0 = 1.09 \text{ Hz}; b = 2.78$$

$f$ : frequency in Hz

Hao and others (Anisotropic model)

$$\left|\gamma\left(\xi_1,\xi_2,f\right)\right| = \exp\left(-\beta_1\xi_1 - \beta_2\xi_2\right)$$
$$\exp\left\{-\left[\alpha_1\left(f\right)\sqrt{\xi_1} + \alpha_2\left(f\right)\sqrt{\xi_2}\right] f^2\right\}$$

$f$ = frequency Hz

$\xi_1, \xi_2$ = the projected distance of the station separation vector along the normal to the direction of propagation

$$\alpha_1\left(f\right) = \frac{a}{f} + bf + c; \alpha_2\left(f\right) = \frac{d}{f} + ef + g$$

$\beta_1 = 2.25\times10^{-4}; \beta_2 = 5.1\times10^{-4}; a = 1.07\times10^{-2}; b = 2.65\times10^{-5}$

$c = -1.0\times10^{-4}; d = 6.66\times10^{-3}; e = 5.88\times10^{-3}; g = 1.1\times10^{-3}$

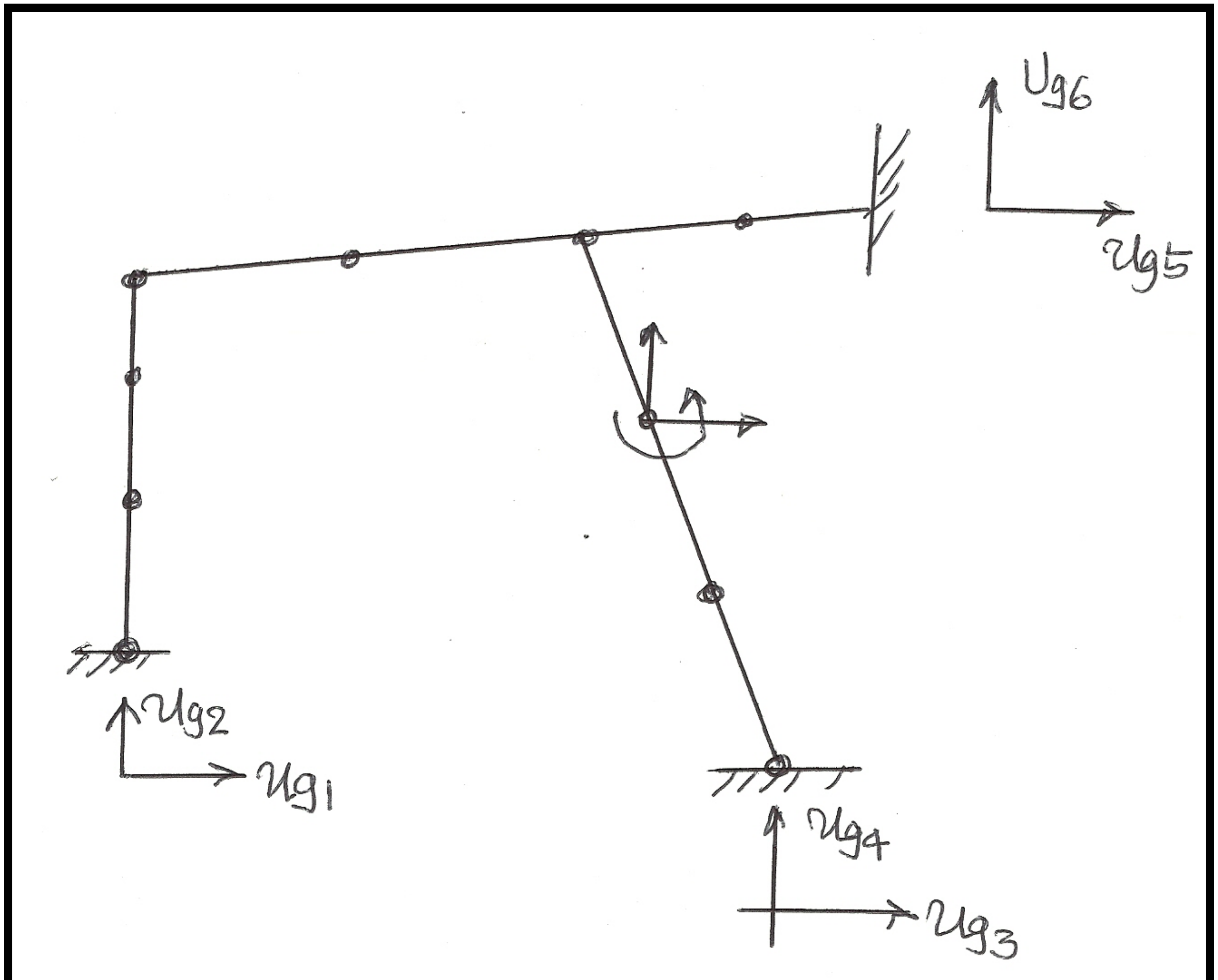
Abrahmson and others

$$\left|\gamma(\omega, f)\right| = \tanh\{(2.54 - 0.012\xi)$$

$$\left[\exp\left[(-0.115 - 0.00084\xi) f\right] + \frac{f^{-0.878}}{3}\right] + 0.35\}$$

$$\xi < 100m$$

# Structures under differential support motions




Model $\boldsymbol{U}_{\boldsymbol{g}}(\boldsymbol{t})$ as a vector random process

$$\begin{bmatrix} M & M_g \\ M_g^t & M_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{u}^T \\ \ddot{u}_g \end{Bmatrix} + \begin{bmatrix} C & C_g \\ C_g^t & C_{gg} \end{bmatrix} \begin{Bmatrix} \dot{u}^T \\ \dot{u}_g \end{Bmatrix} + \begin{bmatrix} K & K_g \\ K_g^t & K_{gg} \end{bmatrix} \begin{Bmatrix} u^T \\ u_g \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_g(t) \end{Bmatrix}$$

$$\ddot{u}^T \sim N \times 1$$

$$\ddot{u}_g, p_g(t) \sim N_g \times 1; N_T = N + N_g$$

$$M, C, K \sim N \times N$$

$$M_g, C_g, K_g \sim N \times N_g$$

$$M_{gg}, C_{gg}, K_{gg} \sim N_g \times N_g$$

Pseudo-dynamic response

$$\begin{bmatrix} K & K_g \\ K_g^t & K_{gg} \end{bmatrix} \begin{Bmatrix} u^p \\ u_g \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_g^p(t) \end{Bmatrix}$$

$$Ku^p + K_g u_g = 0 \Rightarrow u^p = -K^{-1} K_g u_g(t) = \Gamma u_g(t)$$

$$\Gamma = -K^{-1} K_g$$

$$p_g^p(t) = K_g^t u^p + K_{gg} u_g = \left[ -K_g^t K^{-1} K_g + K_{gg} \right] u_g(t)$$

Total response= pseudo-dynamic response + dynamic response

$$\begin{Bmatrix} u^T \\ u_g \end{Bmatrix} = \begin{Bmatrix} u^p(t) \\ u_g(t) \end{Bmatrix} + \begin{Bmatrix} u(t) \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} M & M_g \\ M_g^t & M_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{u} + \ddot{u}^p(t) \\ \ddot{u}_g \end{Bmatrix} + \begin{bmatrix} C & C_g \\ C_g^t & C_{gg} \end{bmatrix} \begin{Bmatrix} \dot{u} + \dot{u}^p \\ \dot{u}_g \end{Bmatrix} +$$

$$\begin{bmatrix} K & K_g \\ K_g^t & K_{gg} \end{bmatrix} \begin{Bmatrix} u + u^p \\ u_g \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_g(t) \end{Bmatrix}$$

$$\Rightarrow$$

$$M\ddot{u} + C\dot{u} + Ku = p_{eff}(t)$$

$$p_{eff}(t) = -\underline{M\ddot{u}^p(t)} - M_g\ddot{u}_g - C\dot{u}^p(t) - C_g\dot{u}_g$$

$$= -M\Gamma\ddot{u}_g(t) - M_g\ddot{u}_g - \Gamma C\dot{u}_g(t) - C_g\dot{u}_g$$

$$= -\left[M\Gamma + M_g\right]\ddot{u}_g(t) - \left[C\Gamma + C_g\right]\dot{u}_g(t)$$

$$M\ddot{u}+C\dot{u}+Ku=-\left[M\Gamma+M_g\right]\ddot{u}_g(t)-\left[C\Gamma+C_g\right]\dot{u}_g(t)$$

Special Case

Mass matrix is diagonal $\Rightarrow \mathrm{M}_g=0$ //

$C$ is proportional to $K$ $(C=\alpha K)$

$$\Rightarrow\left[C\Gamma+C_g\right]=\alpha\left[K\Gamma+K_g\right]=\alpha\left[-KK^{-1}K_g+K_g\right]=0$$

$\Rightarrow$

$$M\ddot{u}+C\dot{u}+Ku=-\underline{\underline{M\Gamma\ddot{u}_g(t)}} \quad //$$

$\Gamma \sim N\times N_g$

$\ddot{u}_g(t)\sim N_g\times 1$

Note: If all supports suffer the same motion, $N_g=1$

$$\Gamma=\{1 \quad 1 \quad \cdots \quad 1\}^t$$

## Random vibration analysis in frequency domain

$$M\ddot{u}+C\dot{u}+Ku=-\left[M\Gamma+M_g\right]\ddot{u}_g(t)-\left[C\Gamma+C_g\right]\dot{u}_g(t)=p(t)$$

$u_g(t)\sim N_g\times 1$: vector of stationary random process with zero mean and PSD matrix

$$S_{gg}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle U_{gT}(\omega)U_{gT}^{*t}(\omega)\right\rangle$$

$$p(t)=-\left[M\Gamma+M_g\right]\ddot{u}_g(t)-\left[C\Gamma+C_g\right]\dot{u}_g(t)$$

$$P_T(\omega)=\omega^2\left[M\Gamma+M_g\right]U_{gT}(\omega)-i\omega\left[C\Gamma+C_g\right]U_{gT}(\omega)$$

$$=\left[\omega^2\left[M\Gamma+M_g\right]-i\omega\left[C\Gamma+C_g\right]\right]U_{gT}(\omega)$$

$$P_T^{*t}(\omega)=U_{gT}^{*t}(\omega)\left[\omega^2\left[\Gamma^tM+M_g\right]+i\omega\left[\Gamma^tC+C_g\right]\right]$$

$$S_{pp}(\omega) = \lim_{T\to\infty} \frac{1}{T} < \left[\omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] U_{gT}(\omega)$$

$$U_{gT}^{*t}(\omega)\left[\omega^2\left[\Gamma^t M + M_g\right] + i\omega\left[\Gamma^t C + C_g\right]\right] >$$

$$S_{pp}(\omega) = \left[\omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right]$$

$$S_{gg}(\omega)\left[\omega^2\left[\Gamma^t M + M_g\right] + i\omega\left[\Gamma^t C + C_g\right]\right]$$

$$\Rightarrow$$

$$S_{UU}(\omega) = H(\omega) S_{pp}(\omega) H^{*t}(\omega)$$

$$\left[H(\omega)\right] = \left[-\omega^2 M + i\omega C + k\right]^{-1} = \left[\sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}\right]$$

Pseudo-dynamic response

$$u^p = -K^{-1}K_g u_g(t) = \Gamma u_g(t)$$

$$\Gamma = -K^{-1}K_g$$

$$S_{u^p u^p}(\omega) = \Gamma S_{gg}(\omega)\Gamma^t$$

Total response

$$u^T(t) = u^p(t) + u(t)$$

$$= \Gamma u_g(t) + u(t)$$

$$U_T^T(\omega) = \Gamma U_{gT}(\omega) + U_T(\omega)$$

$$= \Gamma U_{gT}(\omega) + H(\omega) P_T(\omega)$$

$$P_T(\omega) = \left[\omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] U_{gT}(\omega)$$

$$\Rightarrow$$

$$U_T^T(\omega) = \left[\Gamma + \omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] U_{gT}(\omega)$$

Total response

$$U_T^T(\omega) = \left[\Gamma + \omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right]U_{gT}(\omega)$$

$$\Rightarrow$$

$$S_{TT}(\omega) = \left[\Gamma + \omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right]S_{gg}(\omega)$$

$$\left[\Gamma + \omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right]^t$$

Variance of total response=

variance of pseudo-dynamic response+

variance of dynamic response+

contributions due to correlation between

pseudo-dynamic and dynamic responses

**Modal combination rules**

A Der Kiureghian and A Neuenhofer, 1993, EESD, 21,713-740

$$\begin{bmatrix} M & M_c \\ M_c^t & M_g \end{bmatrix}\begin{Bmatrix} \ddot{\mathrm{x}} \\ \ddot{\mathrm{u}} \end{Bmatrix} + \begin{bmatrix} C & C_c \\ C_c^t & C_g \end{bmatrix}\begin{Bmatrix} \dot{\mathrm{x}} \\ \dot{\mathrm{u}} \end{Bmatrix} + \begin{bmatrix} K & K_c \\ K_c^t & K_g \end{bmatrix}\begin{Bmatrix} \mathrm{x} \\ \mathrm{u} \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$\mathrm{x} : n \times 1;$

$\mathrm{u}, F : m \times 1$

$M, C, K : n \times n$

$M_g, C_g, K_g : m \times m$

$M_c, C_c, K_c : n \times m$

$\mathrm{x} = \mathrm{x}^s + \mathrm{x}^d$

[Total response=pseudodynamic response+dynamic response]

$\mathrm{x}^s = -K^{-1}K_c\mathrm{u} = R\mathrm{u}$

$M\ddot{\mathrm{x}}^d + C\dot{\mathrm{x}}^d + K\mathrm{x} = -\left(MR + M_c\right)\ddot{\mathrm{u}} - \left(-CR + C_c\right)\dot{\mathrm{u}} \approx -\left(MR + M_c\right)\ddot{\mathrm{u}}$

$$M\ddot{\mathrm{x}}^d + C\dot{\mathrm{x}}^d + K\mathrm{x} \approx -\left(MR + M_c\right)\ddot{\mathrm{u}}$$

$$\mathrm{x}^d = \Phi y$$

$\phi^t M \phi = I \quad \phi^t K \phi = \Lambda \quad \phi^t C \phi$ diagonal

$$\ddot{y}_i + 2\eta_i\omega_i\dot{y}_i + \omega_i^2 y_i = \sum_{k=1}^{m}\beta_{ki}\ddot{u}_k(t)$$

$\beta_{kl} = \phi_i^t\left(Mr_k + M_c I_k\right); r_k = k^{\mathrm{th}}$ column of $R$;

$I_k = k^{th}$ column of $k \times k$ identity matrix.

Define: $\ddot{s}_{ki} + 2\eta_i\omega_i\dot{s}_{ki} + \omega_i^2 s_{ki} = \ddot{u}_k(t) \Rightarrow y_i(t) = \sum_{k=1}^{m}\beta_{ki}s_i(t)$

Response quantity of interest: $z(t) = q^t\mathrm{x}(t) = q^t\left[\mathrm{x}^s(t) + \mathrm{x}^d(t)\right]$

$$z(t) = \underbrace{\sum_{k=1}^{m}a_k u_k(t)}_{} + \underbrace{\sum_{k=1}^{m}\sum_{j=1}^{n}b_{ki}s_{ki}(t)}_{}$$

$a_k = q^t r_k k = 1,2,\cdots,m; b_{ki} = q^t\phi_i\beta_{kl}k = 1,2,\cdots,m; i = 1,2,\cdots,n$

$$G_{ZZ}(\omega)=\sum_{k=1}^{m}\sum_{l=1}^{m}a_k a_l G_{u_k u_l}(\omega)+2\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{j=1}^{n}a_k b_{ij}H_j(-\mathrm{i}\omega)G_{u_k \ddot{u}_l}(\omega)$$

$$+\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ki}b_{lj}H_i(\mathrm{i}\omega)H_j(-\mathrm{i}\omega)G_{\ddot{u}_k \ddot{u}_l}(\omega)$$

$$H_i(\mathrm{i}\omega)=\left[\omega_i^2-\omega^2+\mathrm{i}2\eta\omega\omega_i\right]^{-1}$$

$$\sigma_z^2=\sum_{k=1}^{m}\sum_{l=1}^{m}a_k a_l \rho_{u_k u_l}\sigma_{u_k}\sigma_{u_l}+2\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{j=1}^{n}a_k b_{ij}\rho_{u_k s_{ij}}\sigma_{u_k}\sigma_{s_{ij}}$$

$$+\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ki}b_{lj}\rho_{s_{ki}s_{lj}}\sigma_{s_{ki}}\sigma_{s_{lj}}$$

$$\sigma_{u_k}^2=\int_{-\infty}^{\infty}G_{u_k u_l}(\omega)\,d\omega;\quad \sigma_{s_{ki}}^2=\int_{-\infty}^{\infty}\left|H_i(-\mathrm{i}\omega)\right|^2 G_{\ddot{u}_k \ddot{u}_k}(\omega)\,d\omega$$

$$\rho_{u_k u_l} = \frac{1}{\sigma_{u_k}\sigma_{u_l}} \int_{-\infty}^{\infty} G_{u_k u_l}(\mathrm{i}\omega)\, d\omega$$

$$\rho_{u_k s_{lj}} = \frac{1}{\sigma_{u_k}\sigma_{s_{lj}}} \int_{-\infty}^{\infty} H_j(-\mathrm{i}\omega) G_{u_k \ddot{u}_l}(\mathrm{i}\omega)\, d\omega$$

$$\rho_{s_{ki} s_{lj}} = \frac{1}{\sigma_{s_{ki}}\sigma_{s_{lj}}} \int_{-\infty}^{\infty} H_i(\mathrm{i}\omega) H_j(-\mathrm{i}\omega) G_{\ddot{u}_k \ddot{u}_l}(\mathrm{i}\omega)\, d\omega$$

$$\left.\begin{aligned} \ddot{s}_{ki} + 2\eta_i\omega_i\dot{s}_{ki} + \omega_i^2 s_{ki} &= \ddot{u}_k(t) \\ \ddot{s}_{lj} + 2\eta_i\omega_i\dot{s}_{lj} + \omega_i^2 s_{lj} &= \ddot{u}_l(t) \end{aligned}\right\}$$

$$\gamma_{kl}(\mathrm{i}\omega) = \frac{G_{\ddot{u}_k \ddot{u}_l}(\mathrm{i}\omega)}{\sqrt{G_{\ddot{u}_k \ddot{u}_k}(\omega) G_{\ddot{u}_l \ddot{u}_l}(\omega)}} = \left|\gamma_{kl}(\mathrm{i}\omega)\right| \exp\left[\mathrm{i}\theta_{kl}(\mathrm{i}\omega)\right]$$

Take

$$\gamma_{kl}(\mathrm{i}\omega) = \exp\left[-\left(\frac{\alpha\omega d_{kl}}{v_s}\right)^2\right] \exp\left(\mathrm{i}\frac{\omega d_{kl}^L}{v_{app}}\right)$$

$v_s$ = shear wave velocity of the medium

$v_{app}$ = surface apparent wave velocity

$$G_{\ddot{u}_k \ddot{u}_l}(\mathrm{i}\omega) = \gamma_{kl}(\mathrm{i}\omega) \sqrt{G_{\ddot{u}_k \ddot{u}_k}(\omega) G_{\ddot{u}_l \ddot{u}_l}(\omega)}$$

$$G_{u_k \ddot{u}_l}(\mathrm{i}\omega) = -\frac{1}{\omega^2} \gamma_{kl}(\mathrm{i}\omega) \sqrt{G_{\ddot{u}_k \ddot{u}_k}(\omega) G_{\ddot{u}_l \ddot{u}_l}(\omega)}$$

$$G_{u_k u_l}(\mathrm{i}\omega) = \frac{1}{\omega^4} \gamma_{kl}(\mathrm{i}\omega) \sqrt{G_{\ddot{u}_k \ddot{u}_k}(\omega) G_{\ddot{u}_l \ddot{u}_l}(\omega)}$$

$$G_{\ddot{u}_k\ddot{u}_k}(\omega) = G_{kk}\underbrace{\frac{\omega_{fk}^4 + 4\eta_{fk}^4\omega_{fk}^2\omega^2}{\left(\omega_{fk}^2-\omega^2\right)^2 + 4\eta_{fk}^4\omega_{fk}^2\omega^2}}_{\text{Soil}}\underbrace{\frac{\omega^4}{\left(\omega_{gk}^2-\omega^2\right)^2 + 4\eta_{gk}^4\omega_{gk}^2\omega^2}}_{\text{High pass filter}}$$

$\rho_{u_k u_l}, \rho_{u_k s_{lj}}, \rho_{s_{ki} s_{lj}}$ can now be computed.

**Response spectrum method**

Recall: response spectrum definitions and limiting behavior

$$\ddot{s}_{ki} + 2\eta_i \omega_i \dot{s}_{ki} + \omega_i^2 s_{ki} = \ddot{u}_k(t)$$

$$D_k(\omega_i, \eta_i) = \mathrm{E}\left[\max_t |s_{ki}(t)|\right] = \cancel{u_{k,\max}}$$

$$\lim_{\omega_k \to 0} D_k(\omega_i, \eta_i) \to \mathrm{E}\left[\max_t |u_k(t)|\right] = \cancel{\ddot{u}_{k,\max}} \quad u_{k,\max}$$

$$\lim_{\omega_k \to \infty} \omega_k^2 D_k(\omega_i, \eta_i) \to \mathrm{E}\left[\max_t |\ddot{u}_k(t)|\right] \rightarrow \ddot{u}_{k,\max} \quad PGA$$

Peak factors

$$u_{k,\max} = p_{u_k} \sigma_{u_k}; D_k(\omega_i, \eta_i) = p_{s_{ki}} \sigma_{s_{ki}}$$

$$E\left[\max |z(t)|\right] = p_z \sigma_z$$

$$\sigma_z^2 = \sum_{k=1}^{m}\sum_{l=1}^{m} a_k a_l \rho_{u_k u_l} \sigma_{u_k} \sigma_{u_l} + 2\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{j=1}^{n} a_k b_{ij} \rho_{u_k s_{ij}} \sigma_{u_k} \sigma_{s_{ij}}$$

$$+\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ki} b_{lj} \rho_{s_{ki} s_{lj}} \sigma_{s_{ki}} \sigma_{s_{lj}}$$

$$E\left[\max\left|z(t)\right|\right] = [\sum_{k=1}^{m}\sum_{l=1}^{m} a_k a_l \rho_{u_k u_l} \frac{p_z^2}{p_{u_k} p_{u_l}} u_{k,\max} u_{l,\max}$$

$$+2\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{j=1}^{n} a_k b_{ij} \rho_{u_k s_{ij}} \frac{p_z^2}{p_{u_k} p_{s_{lj}}} u_{k,\max} D_l\left(\omega_j, \eta_j\right)$$

$$+\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ki} b_{lj} \rho_{s_{ki} s_{lj}} \frac{p_z^2}{p_{s_{ki}} p_{s_{lj}}} D_k\left(\omega_k, \eta_k\right) D_l\left(\omega_j, \eta_j\right)]^{\frac{1}{2}}$$

Since peak factors are weakly dependent on frequency and

$$\frac{p_z^2}{p_{u_k} p_{u_l}} \approx 1, \frac{p_z^2}{p_{u_k} p_{s_{lj}}} \approx 1, \frac{p_z^2}{p_{s_{ki}} p_{s_{lj}}} \approx 1, \text{ we get}$$

$$E\left[\max\left|z(t)\right|\right] = [\sum_{k=1}^{m}\sum_{l=1}^{m} a_k a_l \rho_{u_k u_l} u_{k,\max} u_{l,\max}$$

$$+2\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{j=1}^{n} a_k b_{ij} \rho_{u_k s_{ij}} u_{k,\max} D_l\left(\omega_j, \eta_j\right)$$

$$+\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ki} b_{lj} \rho_{s_{ki} s_{lj}} D_k\left(\omega_k, \eta_k\right) D_l\left(\omega_j, \eta_j\right)]^{\frac{1}{2}}$$

**Remarks**

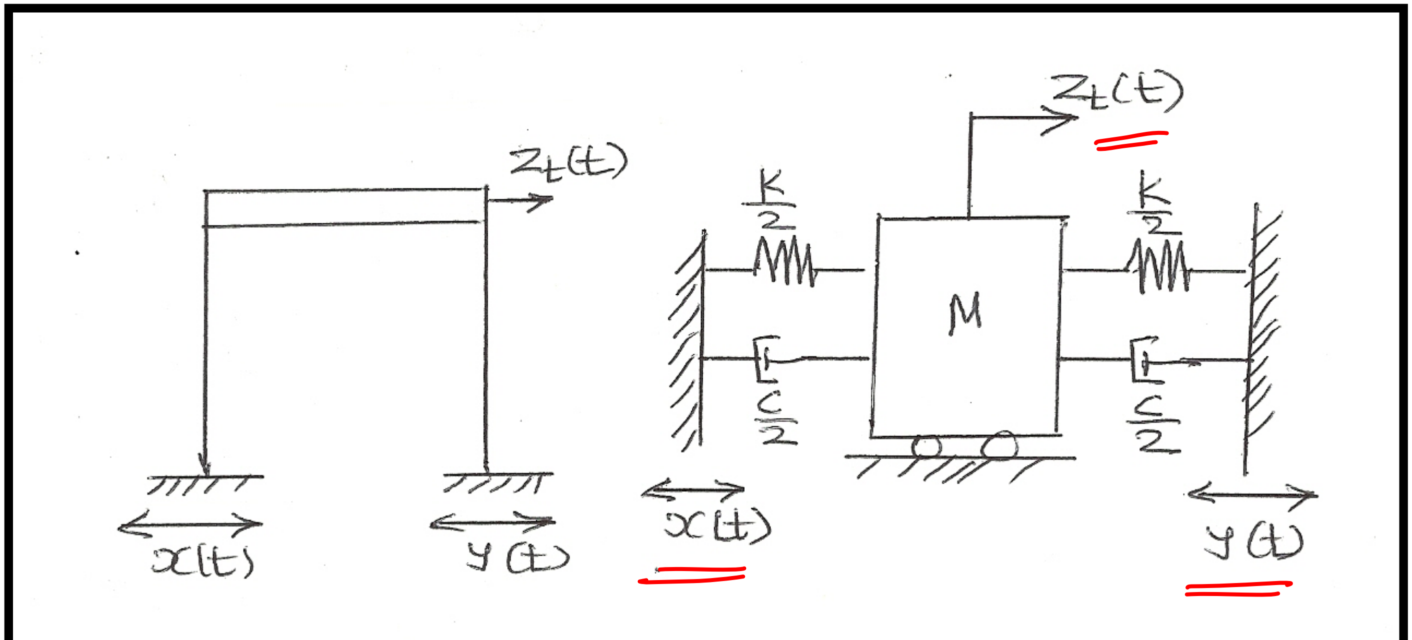
- The implementation of this rule requires the knowledge of the PSD compatible response spectrum and knowledge of coherency function.
- Generalization to include multi-component nature of excitation and separation of response into pseudo-dynamic and dynamic components could be achieved.
- The idea of existence of principal axes for excitation could be assumed and these axes could be assumed to be the same for all recording stations

Optimal cross PSD function models for earthquake excitations

What is the nature of cross PSD functions that lead to the highest response?

A Sarkar and C S Manohar, 1998, JSV, 212(3), 525-546

# Doubly supported SDOF system under differential ground motions




What is relative displacment?

Total response=pseudo-dynamic response+dynamic response

$$m\ddot{z}_t + \frac{c}{2}[\dot{z}_t - \dot{x}] + \frac{c}{2}[\dot{z}_t - \dot{y}] + \frac{k}{2}[z_t - x] + \frac{k}{2}[z_t - y] = 0$$

$$m\ddot{z}_t + c\left[\dot{z}_t - \left(\frac{\dot{x}+\dot{y}}{2}\right)\right] + k\left[z_t - \left(\frac{x+y}{2}\right)\right] = 0$$

Pseudo-dynamic response

$$k\left[z_{ps} - \left(\frac{x+y}{2}\right)\right] = 0 \Rightarrow z_{ps} = \left(\frac{x+y}{2}\right)$$

Dynamic response

$$z(t) = z_t(t) - z_{ps}(t) = z_t(t) - \left(\frac{x+y}{2}\right)$$

$$\Rightarrow$$

$$m\ddot{z} + c\dot{z} + kz = -m\left(\frac{\ddot{x}+\ddot{y}}{2}\right)$$

**Description of input**

$\ddot{x}(t)$ & $\ddot{y}(t)$ are zero mean, stationary, Gaussian random processes with PSD matrix $S(\omega)$.

$$S(\omega)=\begin{bmatrix} S_{xx}(\omega) & S_{xy}(\omega) \\ S_{yx}(\omega) & S_{yy}(\omega) \end{bmatrix}$$

$$S_{xy}(\omega)=\left|S_{xy}(\omega)\right|\exp\left[-i\phi_{xy}(\omega)\right]$$

$$=\left|S_{xy}(\omega)\right|\left\{\cos\phi_{xy}(\omega)-i\sin\phi_{xy}(\omega)\right\}$$

# Force in the left spring

$$F = \frac{k}{2}\left[z_t(t) - x(t)\right]$$
$$= \frac{k}{2}\left[z + \frac{x+y}{2} - x\right]$$
$$= \frac{k}{4}\left[2z - (x-y)\right]$$

Define $g(t) = \frac{4F}{k} = \left[2z - (x-y)\right]$

**Question**

What is the psd of $g(t)$

and what is its variance?

$$
\boxed{
\begin{aligned}
&S_{gg}(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle \left| g_T(\omega) \right|^2 \right\rangle \\
&g_T(\omega) = 2z_T(\omega) - \left[ x_T(\omega) + y_T(\omega) \right] \\
&z_T(\omega) = H_0(\omega) \frac{\omega^2}{2} \left[ x_T(\omega) + y_T(\omega) \right] \\
&\Rightarrow \\
&S_{gg}(\omega) = S_{xx}(\omega) H_1(\omega) + S_{yy}(\omega) H_2(\omega) + \left| S_{xy}(\omega) \right| H_3(\omega)
\end{aligned}
}
$$

$$H_1(\omega) = \{\frac{1}{\omega^4} + \frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2} + \frac{2\left(\omega^2 - \omega_n^2\right)}{\omega^2\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2\right]}\}\left|H_f(\omega)\right|^2$$

$$H_2(\omega) = \{\frac{1}{\omega^4} + \frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2} - \frac{2\left(\omega^2 - \omega_n^2\right)}{\omega^2\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2\right]}\}\left|H_f(\omega)\right|^2$$

$$H_3(\omega) = \{-\frac{2\cos\phi_{xy}(\omega)}{\omega^4} + \frac{2\cos\phi_{xy}(\omega)}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2}$$

$$+\frac{8\eta\omega\omega_n \sin\phi_{xy}(\omega)}{\omega^2\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2\right]}\}\left|H_f(\omega)\right|^2$$

$$\sigma_g^2 = \int_0^\infty \left[\underline{S_{xx}}(\omega)H_1(\omega) + \underline{S_{yy}(\omega)H_2}(\omega) + \left|S_{xy}(\omega)\right|\underline{H_3}(\omega)\right]d\omega$$

Rearranging the terms we get

$$H_1(\omega) = \frac{\left(2\omega^2 - \omega_n^2\right)^2 + \left(2\eta\omega\omega_n\right)^2}{\omega^4\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2\right]} \left|H_f(\omega)\right|^2$$

$$H_2(\omega) = \frac{\omega_n^2\left(\omega_n^2 + 4\eta^2\omega^2\right)}{\omega^4\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2\right]} \left|H_f(\omega)\right|^2$$

$$\Rightarrow$$

$$H_1(\omega) \geq 0 \;\&\; H_2(\omega) \geq 0$$

$$\sigma_g^2 = \int_0^\infty \left[ S_{xx}(\omega) H_1(\omega) + S_{yy}(\omega) H_2(\omega) + \left| S_{xy}(\omega) \right| H_3(\omega) \right] d\omega$$

Question

What is the optimal $S_{xy}(\omega)$ which produces the highest variance $\sigma_g^2$?

**Case - 1** Assume that the phase spectrum $\phi_{xy}(\omega)$ is given

We have

$$\sigma_g^2 = \int_0^\infty \left[ S_{xx}(\omega) H_1(\omega) + S_{yy}(\omega) H_2(\omega) + \left| S_{xy}(\omega) \right| H_3\left\{\omega, \phi_{xy}(\omega)\right\} \right] d\omega$$

$$0 \le \left| S_{xy}(\omega) \right| \le \sqrt{S_{xx}(\omega) S_{yy}(\omega)}$$

Clearly, $\sigma_g^2$ would reach its highest value if

$$\left| S_{xy}(\omega) \right| = 0 \forall \omega \ni H_3\left\{\omega, \phi_{xy}(\omega)\right\} \le 0$$

$$\left| S_{xy}(\omega) \right| = \sqrt{S_{xx}(\omega) S_{yy}(\omega)} \forall \omega \ni H_3\left\{\omega, \phi_{xy}(\omega)\right\} > 0$$

Conversely $\sigma_g^2$ would reach its least value if

$$\left| S_{xy}(\omega) \right| = 0 \forall \omega \ni H_3\left\{\omega, \phi_{xy}(\omega)\right\} > 0$$

$$\left| S_{xy}(\omega) \right| = \sqrt{S_{xx}(\omega) S_{yy}(\omega)} \forall \omega \ni H_3\left\{\omega, \phi_{xy}(\omega)\right\} \le 0$$

**Remark**

The least favorable and the most favorable responses are produced neither by fully coherent motions nor by fully incoherent motions. Instead special form of CPSD functions exist which produce these optimal responses

Case 2: $\left|S_{xy}(\omega)\right|$ and $\phi_{xy}(\omega)$ are not known

$$\sigma_g^2 = \int_0^\infty \left[ S_{xx}(\omega) H_1(\omega) + S_{yy}(\omega) H_2(\omega) + \left|S_{xy}(\omega)\right| H_3\left(\omega, \phi_{xy}(\omega)\right) \right] d\omega$$

$$H_3(\omega) = \{ -\frac{2\cos\phi_{xy}(\omega)}{\omega^4} + \frac{2\cos\phi_{xy}(\omega)}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2}$$

$$+ \frac{8\eta\omega\omega_n \sin\phi_{xy}(\omega)}{\omega^2 \left[ \left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2 \right]} \} \left|H_f(\omega)\right|^2$$

Let $H_3(\omega) = R(\omega)\cos\left[\phi_{xy}(\omega) - \alpha(\omega)\right]$

$$H_3(\omega) = R(\omega)\cos\left[\phi_{xy}(\omega) - \alpha(\omega)\right]$$

$$R(\omega) = \sqrt{g_1^2(\omega) + g_2^2(\omega)}$$

$$\alpha(\omega) = \tan^{-1}\left\{\frac{g_1(\omega)}{g_2(\omega)}\right\}$$

$$g_1(\omega) = \left\{-\frac{2}{\omega^4} + \frac{2}{\left(\omega^2 - \omega_0^2\right)^2 + \left(2\eta\omega\omega_0\right)^2}\right\}\left|H_f(\omega)\right|^2$$

$$g_2(\omega) = \left\{\frac{8\eta\omega\omega_0}{\omega^2\left[\left(\omega^2 - \omega_0^2\right)^2 + \left(2\eta\omega\omega_0\right)^2\right]}\right\}\left|H_f(\omega)\right|^2$$

$$\sigma_g^2 = \int_0^\infty \left[ S_{xx}(\omega) H_1(\omega) + S_{yy}(\omega) H_2(\omega) + \left| S_{xy}(\omega) \right| H_3(\omega) \right] d\omega$$

$$H_3(\omega) = R(\omega) \cos\left[ \phi_{xy}(\omega) - \alpha(\omega) \right]$$

$$0 \le \left| S_{xy}(\omega) \right| \le \sqrt{S_{xx}(\omega) S_{yy}(\omega)}$$

Clearly, for $\sigma_g^2$ to be maximum

$$\left| S_{xy}(\omega) \right| = \sqrt{S_{xx}(\omega) S_{yy}(\omega)} \;\&\; \cos\left[ \phi_{xy}(\omega) - \alpha(\omega) \right] = 1$$

$$\Rightarrow \phi_{xy}(\omega) = \alpha(\omega) = \tan^{-1}\left\{ \frac{g_1(\omega)}{g_2(\omega)} \right\}$$ produces the least favorable response.

Conversely, for $\sigma_g^2$ to be minimum

$$\left|S_{xy}(\omega)\right| = \sqrt{S_{xx}(\omega)S_{yy}(\omega)} \ \& \cos\left[\phi_{xy}(\omega)-\alpha(\omega)\right] = -1$$

$$\Rightarrow \phi_{xy}(\omega)-\alpha(\omega) = \pi \quad //$$

$$\Rightarrow \phi_{xy}(\omega) = \pi + \alpha(\omega) = \pi + \tan^{-1}\left\{\frac{g_1(\omega)}{g_2(\omega)}\right\}$$

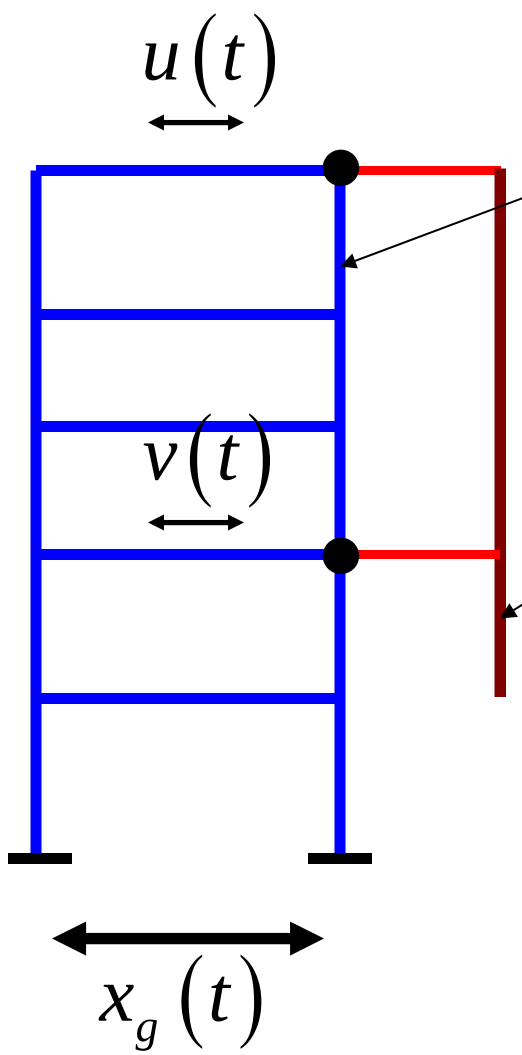
produces the most favorable response.

**Remark**

The optimal responses are produced by fully coherent motions but the phase spectrum depends upon frequency in a specific manner.

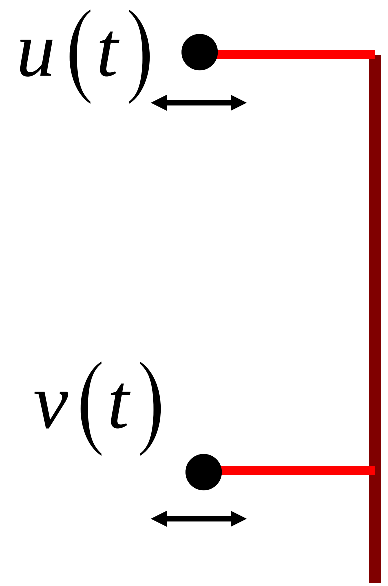
## Problems of spatial variation of support motions in secondary systems of industrial structures

Example: piping networks




Primary system

Secondary system




Multiply supported structure subjected to differential support motions