Stochastic Structural Dynamics

Lecture-33

Probabilistic methods in earthquake engineering-2

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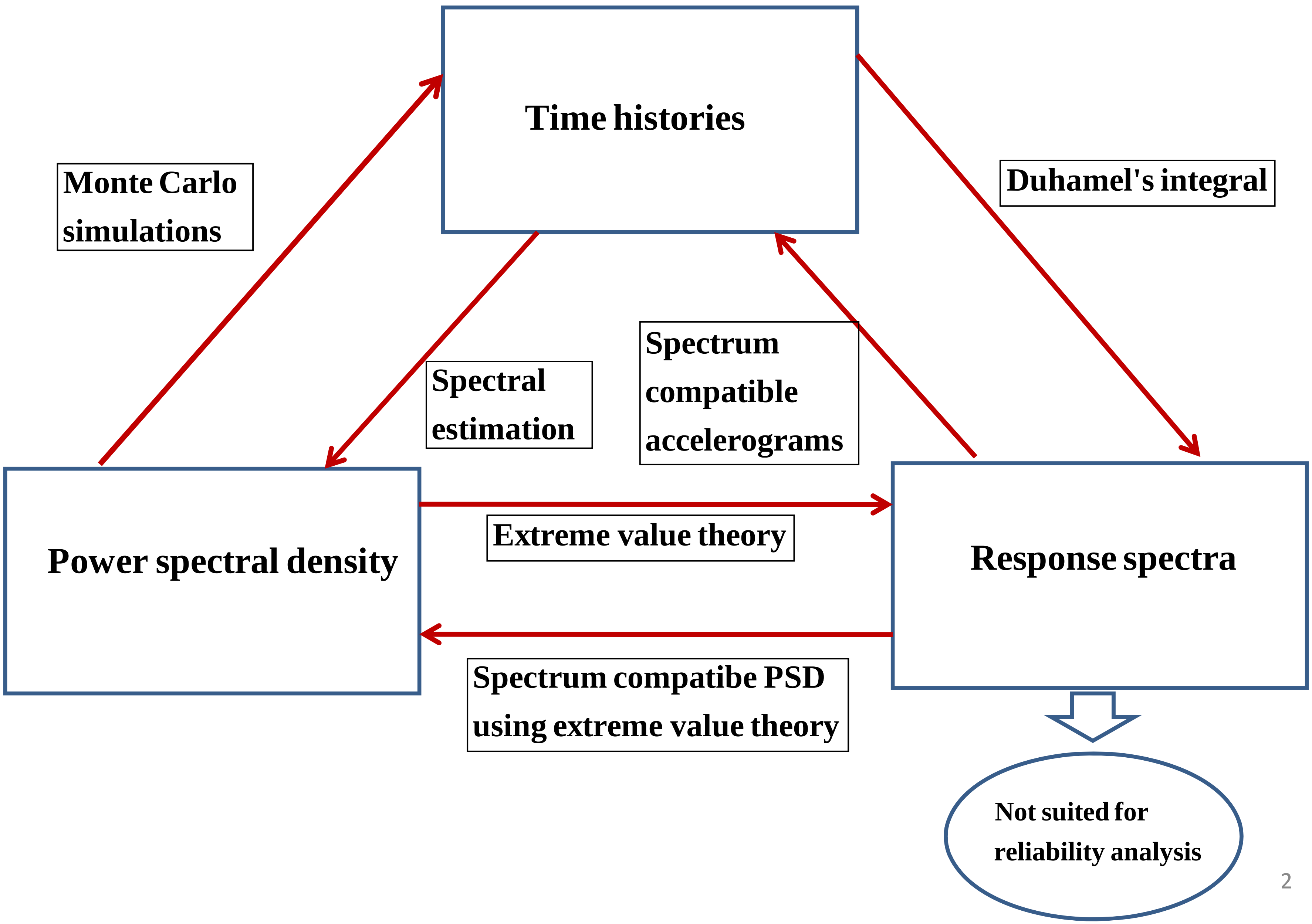
Alternatives for earthquake load specification



Modal combination rules : what is the basic problem?

$$EIz^{iv} + m\ddot{z} + c\dot{z} = -m\ddot{x}_g(t)$$

$$z(0,t) = 0; z'(0,t) = 0; EIz''(L,t) = 0; EIz'''(L,t) = 0$$

Eigenfunction expansion

$$z(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$$

with $\ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = \gamma_n \ddot{x}_g(t); n = 1, 2, \cdots, \infty$

What we know based on response spectrum based analysis?

We know $\max_{0<t<T} |a_n(t)|; n = 1, 2, \cdots, \infty.$

We wish to know: $\max_{0<t<T} |z(x,t)| = \max_{0<t<T} \left| \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \right|$

Difficulty

$$\max_{0<t<T}\left|\sum_{n=1}^{\infty} a_n(t)\phi_n(x)\right| \neq \sum_{n=1}^{\infty}\phi_n(x)\max_{0<t<T}\left|a_n(t)\right|$$

Remarks

- The extrema of $a_n(t)$ for $n=1,2,\cdots,\infty$ are likely to occur at different times and they may have different signs.
- Response spectra do not contain information on times at which extrema occur nor do they store the signs of the extrema.
- $\max_{0<t<T}\left|\sum_{n=1}^{\infty} a_n(t)\phi_n(x)\right|$ can occur at a time instant t^* at which none of $a_n(t)$; $n=1,2,\cdots,\infty$ need to attain their respective extremum values.

Modal combination rules

References

A Der Kiureghian, 1981, A response spectrum method for random vibration analysis of MDF systems, Earthquake Engineering and Structural Dynamics, 9, pp. 419-435

V K Gupta, 2002, Developments in response spectrum-based stochastic response of structural systems, ISET Journal of Earthquake Technology, 39(4), 347-365

Application of principles random vibration analysis in deriving modal combination rules

Consider a mdof system subject to single component of earthquake ground acceleration.

Consider a generic response quantity $R(t)$ and consider the modal representation

$$R(t) = \sum_{i=1}^{N} \Psi_i S_i(t)$$

$\Psi_i = i$-th mode particpation factor

$S_i(t) =$ contribution to $R(t)$ from the i-th mode.

Let the ground acceleration be modeled as a stationary random process and consider response in the steady state.

One sided PSD of $R(t)$ is given by

$$G_R(\omega) = \sum_{i=1}^{N}\sum_{j=1}^{N}\Psi_i\Psi_j G_F(\omega) H_i(\omega) H_j^*(\omega)$$

with $G_F(\omega)$=PSD of the ground acceleration and

$$H_j(\omega) = \frac{1}{\left(\omega_j^2 - \omega^2\right) + i2\eta_j\omega_j\omega}$$

The moments of the response PSD are given by

$$\lambda_m = \int_0^{\infty}\omega^m G_R(\omega)\,d\omega = \sum_{i=1}^{N}\sum_{j=1}^{N}\Psi_i\Psi_j\int_0^{\infty}\omega^m G_F(\omega) H_i(\omega) H_j^*(\omega)\,d\omega$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N}\Psi_i\Psi_j\lambda_{m,ij} \text{ with } \lambda_{m,ij} = \int_0^{\infty}\omega^m G_F(\omega) H_i(\omega) H_j^*(\omega)\,d\omega$$

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$$G_R(\omega) = \sum_{i=1}^{N}\sum_{j=1}^{N} \Psi_i \Psi_j G_F(\omega) H_i(\omega) H_j^*(\omega)$$

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$$\text{Let } \rho_{m,ij} = \frac{\lambda_{m,ij}}{\sqrt{\lambda_{m,ii}\lambda_{m,jj}}} \Rightarrow \lambda_m = \sum_{i=1}^{N}\sum_{j=1}^{N} \Psi_i \Psi_j \rho_{m,ij} \sqrt{\lambda_{m,ii}\lambda_{m,jj}}$$

Remarks

- $\lambda_0 = \int_0^\infty G_R(\omega)\,d\omega = \sigma_R^2 \;\&\; \lambda_2 = \int_0^\infty \omega^2 G_R(\omega)\,d\omega = \sigma_{\dot{R}}^2$
- $\lambda_{0,ii} = \int_0^\infty G_{S_i}(\omega)\,d\omega = \sigma_{S_i}^2 \;\&\; \lambda_{2,ii} = \int_0^\infty \omega^2 G_{S_i}(\omega)\,d\omega = \sigma_{\dot{S}_i}^2$
- $\rho_{0,ij} = \frac{\lambda_{0,ij}}{\sqrt{\lambda_{0,ii}\lambda_{0,jj}}} = \frac{\sigma_{S_i S_j}}{\sqrt{\sigma_{S_i}^2 \sigma_{S_j}^2}} =$ cross correlation between $S_i(t)$ and $S_j(t)$
- $\rho_{2,ij} = \frac{\lambda_{2,ij}}{\sqrt{\lambda_{2,ii}\lambda_{2,jj}}} = \frac{\sigma_{\dot{S}_i \dot{S}_j}}{\sqrt{\sigma_{\dot{S}_i}^2 \sigma_{\dot{S}_j}^2}} =$ cross correlation between $\dot{S}_i(t)$ and $\dot{S}_j(t)$

For the case of $G_F(\omega) = G_0$ (white noise excitation) exact expressions for $\rho_{m,ij}$ for m=0,1,2 can be obtained and to a first order approximation these expressions are given by

$$\rho_{0,ij} = \frac{2\sqrt{\eta_i\eta_j}\left[\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)+\left(\omega_i^2-\omega_j^2\right)\left(\eta_i-\eta_j\right)\right]}{4\left(\omega_i-\omega_j\right)^2+\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)}$$

$$\rho_{1,ij} = \frac{2\sqrt{\eta_i\eta_j}\left[\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)-4\left(\omega_i-\omega_j\right)^2/\pi\right]}{4\left(\omega_i-\omega_j\right)^2+\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)}$$

$$\rho_{2,ij} = \frac{2\sqrt{\eta_i\eta_j}\left[\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)-\left(\omega_i^2-\omega_j^2\right)\left(\eta_i-\eta_j\right)\right]}{4\left(\omega_i-\omega_j\right)^2+\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)}$$

Remarks

- These approximations compare well with exact solutions (less than 1% error for frequency ratios between 0.8 to 1.0)
- These expressions can be used for the case when excitations are broad banded and the PSD function varies slowly in the neighbourhood of system natural frequencies.

Analysis of response peaks

Assume: excitation is Gaussian

$$R_\tau = \max_\tau |R(t)|$$

τ : time duration segmented from the steady state

$$P_{R_\tau}(r) = \left[1 - \exp\left(-\frac{s^2}{2}\right)\right] \exp\left[-\nu\tau \frac{1 - \exp\left(-\sqrt{0.5\pi}\delta_e s\right)}{\exp\left(s^2/2\right) - 1}\right]; r > 0$$

$$s = \frac{r}{\sigma_R} = \frac{r}{\sqrt{\lambda_0}} = \text{ normalized barrier}$$

$$\nu = \frac{\sigma_{\dot{R}}}{\pi\sigma_R} = \frac{1}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} = \text{ mean upcrossing rate}$$

$$\delta_e = \delta^{1.2}; \delta = \sqrt{\left(1 - \frac{\lambda_1^2}{\lambda_0 \lambda_2}\right)} = \text{shape factor}; 0 \le \delta \le 1$$

δ small $\Rightarrow$ narrow band process; δ close to unity $\Rightarrow$ broad band process

Peak factors

$\langle R_\tau \rangle = p\sigma_R$ & Std Dev $R_\tau = q\sigma_R$

$p, q =$ peak factors

For $10 < \nu\tau < 1000$ $0.11 < \delta < 1$,

$$p = \sqrt{2\ln \nu_e \tau} + \frac{0.5722}{\sqrt{2\ln \nu_e \tau}}; q = \frac{1.2}{\sqrt{2\ln \nu_e \tau}} - \frac{5.4}{13 + \left(2\ln \nu_e \tau\right)^{3.2}}$$

$$\nu_e = \begin{cases} \left(1.63\delta^{0.45} - 0.38\right)\nu & \text{for } \delta < 0.69 \\ \nu & \text{for } \delta \geq 0.69 \end{cases}$$

For $\nu\tau$ large, say ≥ 5000

$$p = \sqrt{2\ln \nu\tau} + \frac{0.5722}{\sqrt{2\ln \nu\tau}}$$

$$q = \frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2\ln \nu\tau}}$$

Response spectrum method

Let $\bar{S}_{\tau}(\omega,\eta)=$ mean value of the maximum absolute response of an oscillator over duration τ in the steady state.

ω=natural frequency of the oscillator

η=damping ratio of the oscillator

By definition $\bar{S}_{\tau}(\omega,\eta)=$ response spectrum of excitation $F(t)$.

Question

How to evaluate the response of a mdof system when $F(t)$ is specified in terms of $\bar{S}_{\tau}(\omega,\eta)$?

Recall

$$\nu_i = \frac{1}{\pi}\sqrt{\frac{\lambda_{2,ii}}{\lambda_{0,ii}}} = \text{ mean upcrossing rate\& } \delta_i = \sqrt{\left(1-\frac{\lambda_{1,ii}^2}{\lambda_{0,ii}\lambda_{2,ii}}\right)} = \text{shape factor}$$

For broad band excitations within the frequency range of inerest,

the above expressions can be approximated by results for the case of

excitation being white noise process. $\Rightarrow$

$$\nu_i = \frac{\omega_i}{\pi} \ \& \ \delta_i \approx 2\left(\frac{\eta_i}{\pi}\right)^{\frac{1}{2}}$$

Use this in

$$p = \sqrt{2\ln \nu_e \tau} + \frac{0.5722}{\sqrt{2\ln \nu_e \tau}}; q = \frac{1.2}{\sqrt{2\ln \nu_e \tau}} - \frac{5.4}{13+\left(2\ln \nu_e \tau\right)^{3.2}}$$

$$\nu_e = \begin{cases} \left(1.63\delta^{0.45} - 0.38\right)\nu & \text{for } \delta < 0.69 \\ \nu & \text{for } \delta \geq 0.69 \end{cases}$$

to get peakfactors p_i and q_i for each normal coordinate.

$$\bar{S}_\tau(\omega,\eta) = \left\langle \max_\tau \left| \mathrm{S}_i(t) \right| \right\rangle$$

Moments of the PSD of i - th mode response

$$\left.\begin{aligned}
\lambda_{0,ii} &= \frac{\bar{S}_\tau^2(\omega,\eta)}{p_i^2}; \\
\lambda_{1,ii} &= \frac{\omega_i \sqrt{\left(1-4\eta_i/\pi\right)}}{p_i^2} \bar{S}_\tau^2(\omega,\eta) \\
\lambda_{2,ii} &= \frac{\omega_i^2}{p_i^2} \bar{S}_\tau^2(\omega,\eta)
\end{aligned}\right\} \quad (*)$$

$$\left.\begin{aligned}
\rho_{0,ij} &= \frac{2\sqrt{\eta_i\eta_j}\left[\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)+\left(\omega_i^2-\omega_j^2\right)\left(\eta_i-\eta_j\right)\right]}{4\left(\omega_i-\omega_j\right)^2+\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)} \\
\rho_{1,ij} &= \frac{2\sqrt{\eta_i\eta_j}\left[\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)-4\left(\omega_i-\omega_j\right)^2/\pi\right]}{4\left(\omega_i-\omega_j\right)^2+\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)} \\
\rho_{2,ij} &= \frac{2\sqrt{\eta_i\eta_j}\left[\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)-\left(\omega_i^2-\omega_j^2\right)\left(\eta_i-\eta_j\right)\right]}{4\left(\omega_i-\omega_j\right)^2+\left(\omega_i+\omega_j\right)^2\left(\eta_i+\eta_j\right)}
\end{aligned}\right\}\quad (**)$$

Use (*) and (**) in

$$\lambda_m = \int_0^\infty \omega^m G_R(\omega)\, d\omega = \sum_{i=1}^{N}\sum_{j=1}^{N} \Psi_i \Psi_j \lambda_{m,ij}$$

with $\lambda_{m,ij} = \int_0^\infty \omega^m G_F(\omega) H_i(\omega) H_j^*(\omega) d\omega$

to get $\lambda_0, \lambda_1,$ and λ_2 in terms of response spectrum coordinates. Denote $\bar{R}_{i\tau} = \Psi_i \bar{S}(\omega_i, \eta_i). \Rightarrow$

$$\sigma_R = \left(\sum_i \sum_j \frac{1}{p_i p_j} \rho_{0,ij} \bar{R}_{i\tau} \bar{R}_{j\tau} \right)^{\frac{1}{2}}; \sigma_{\dot{R}} = \left(\sum_i \sum_j \frac{\omega_i \omega_j}{p_i p_j} \rho_{2,ij} \bar{R}_{i\tau} \bar{R}_{j\tau} \right)^{\frac{1}{2}}$$

Mean of the peak response

$$\bar{R}_\tau = p\sigma_R = \left(\sum_i \sum_j \frac{p^2}{p_i p_j} \rho_{0,ij} \bar{R}_{i\tau} \bar{R}_{j\tau} \right)^{\frac{1}{2}}$$

Standard deviation of the peak response

$$\sigma_{R_\tau} = q\sigma_R = \left(\sum_i \sum_j \frac{q^2}{p_i p_j} \rho_{0,ij} \bar{R}_{i\tau} \bar{R}_{j\tau} \right)^{\frac{1}{2}}$$

Here p and q are peak factors of the response.

Recall

$$p = \sqrt{2\ln \nu_e \tau} + \frac{0.5722}{\sqrt{2\ln \nu_e \tau}}; q = \frac{1.2}{\sqrt{2\ln \nu_e \tau}} - \frac{5.4}{13 + \left(2\ln \nu_e \tau\right)^{3.2}}$$

Quantity of special interest: Mean of the peak response

$$\bar{R}_\tau = p\sigma_R = \left(\sum_i \sum_j \frac{p^2}{p_i p_j} \rho_{0,ij} \bar{R}_{i\tau} \bar{R}_{j\tau} \right)^{\frac{1}{2}}$$

It can be verified that the quantity $\frac{p}{p_i} \approx 1. \Rightarrow$

$$\bar{R}_\tau = \left(\sum_i \sum_j \rho_{0,ij} \bar{R}_{i\tau} \bar{R}_{j\tau} \right)^{\frac{1}{2}} = \left(\sum_i \bar{R}_{i\tau}^2 + \underbrace{\sum_i \sum_{\substack{j \\ i \neq j}} \rho_{0,ij} \bar{R}_{i\tau} \bar{R}_{j\tau}}_{\text{contributuon due to modal interactions}} \right)^{\frac{1}{2}}$$

$$\bar{R}_{\tau} \approx \left(\sum_i \bar{R}_{i\tau}^2 \right)^{\frac{1}{2}}$$: square root of sum of squares (SRSS) rule

$$\bar{R}_{\tau} = \left(\sum_i \sum_j \rho_{0,ij} \bar{R}_{i\tau} \bar{R}_{j\tau} \right)^{\frac{1}{2}} = \left(\sum_i \bar{R}_{i\tau}^2 + \underbrace{\sum_i \sum_{\substack{j \\ i \neq j}} \rho_{0,ij} \bar{R}_{i\tau} \bar{R}_{j\tau}}_{\text{contributuons due to modal interactions}} \right)^{\frac{1}{2}}$$

: complete quadratic combination (CQC) rule.

Remarks

- SRSS rule can be deemed satisfactory for systems in which the natural frequencies are well separated and modal damping is not very large. Excitation is broad banded and strong phase long enough.
- CQC rule allows for correction due to modal interactions and hence is suited for systems with closely spaced modes.
- CQC rule can be implemented without having to evaluate spectral moments.
- Mean peak response is not dependent explicitly on period τ

Recall : assumptions made

- Excitation has been taken to be stationary, Gaussian white noise. [Duration of the strong motion phase of the earthquake needs to be long and the excitation should be broad banded].
- The ratio of the response peak factor and the modal peak factor is taken to be unity.

Examples of stochastic models for earthquake ground motions

- Single component: stationary & nonstationary models
- Multi-component and spatially varying load models
- Gaussian and Poisson pulse process models

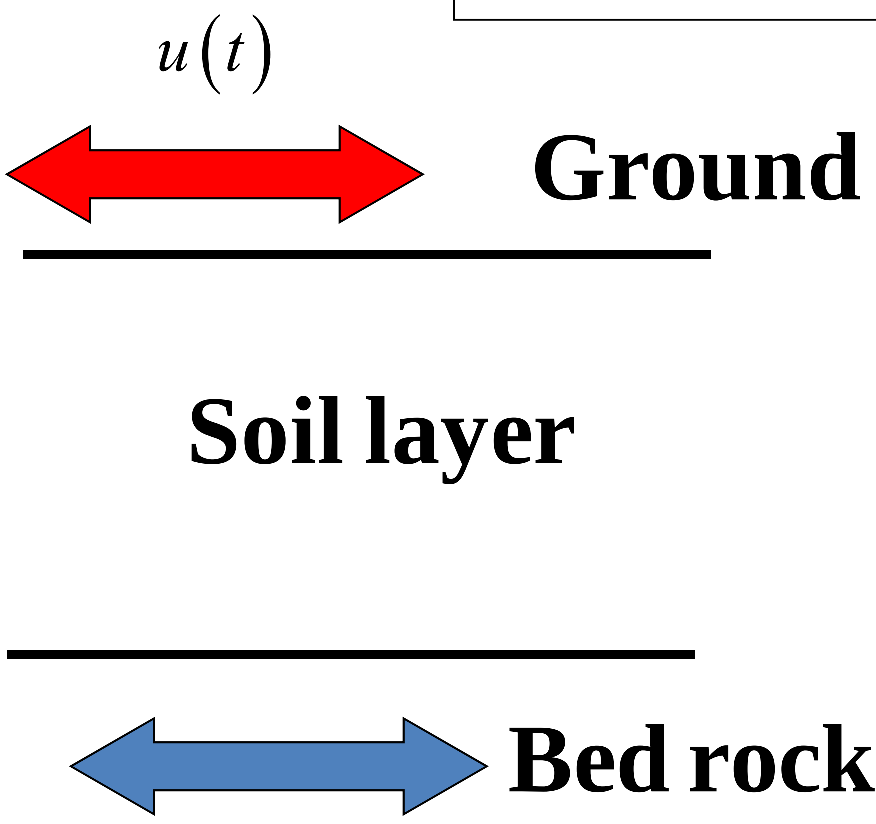
Main concerns

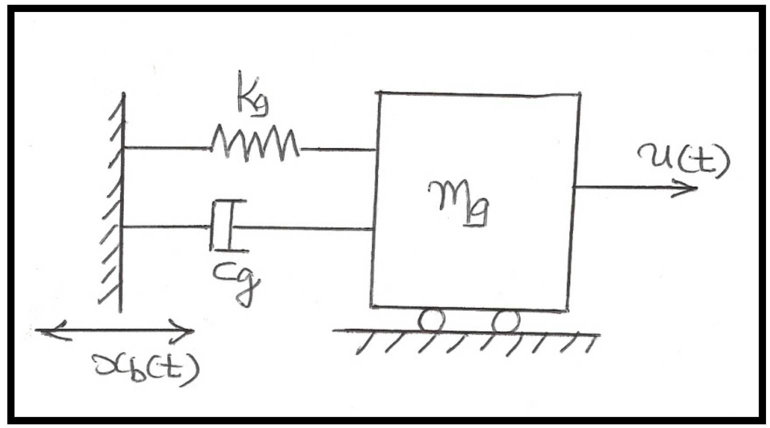
- frequency content
- transient nature and duration
- time dependent frequency content
- multi-component nature
- spatial variability
- translations and rotations
- models for displacement and velocity components
- seismological considerations

Kanai – Tajimi & Clough and Penzien

Power spectral density function models

for free field earthquake ground acceleration





$\ddot{x}_b(t)$: White noise

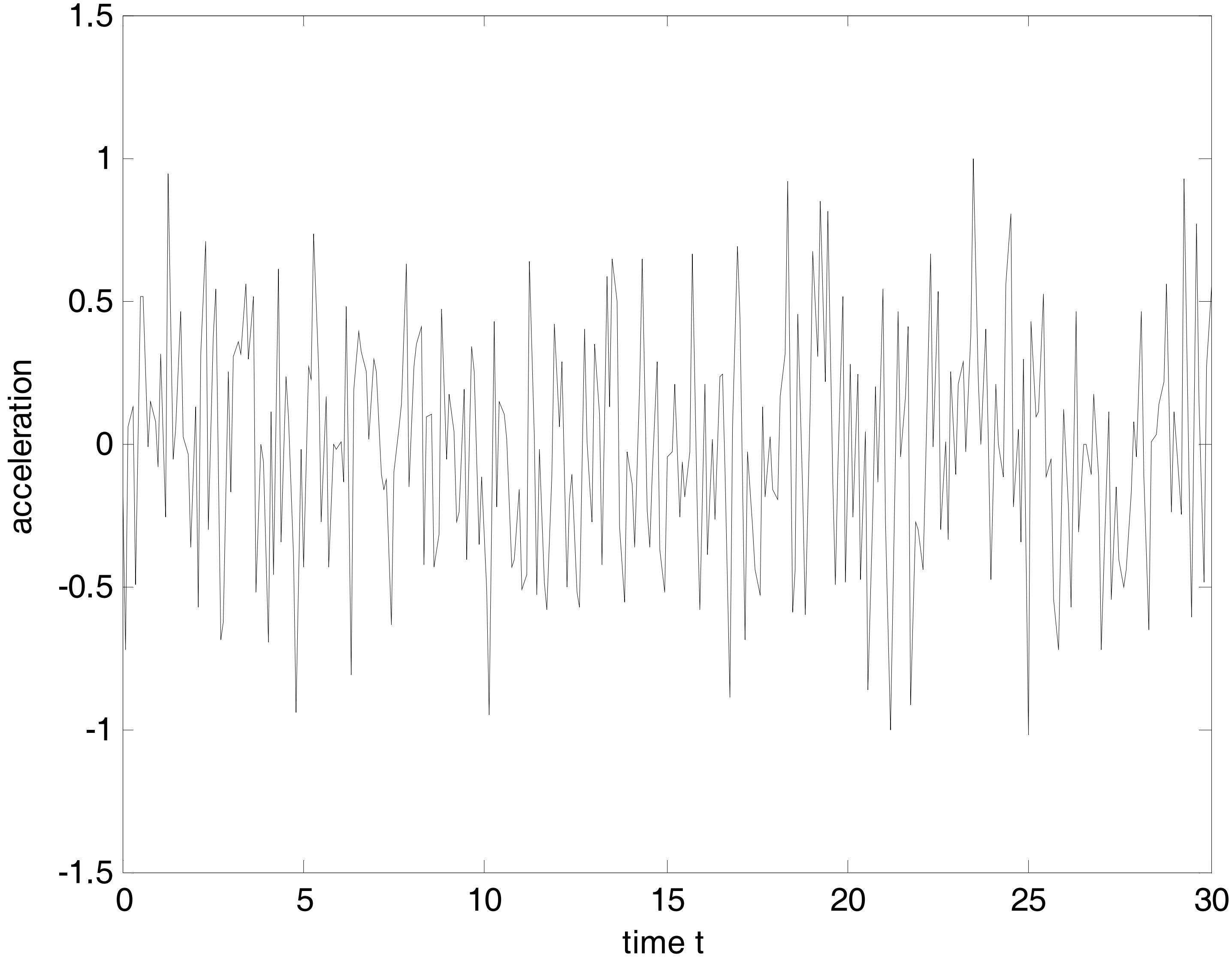
Local site condtions are accounted for

$$S(\omega) = I\,\frac{\left(\omega_g^4 + 4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\eta_g^2\omega_g^2\omega^2}$$

Clough and Penzien model

$$S(\omega) = I\,\frac{\left(\omega_g^4 + 4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\eta_g^2\omega_g^2\omega^2}\underbrace{|H_f(\omega)|^2}_{\text{High pass filter}}$$

$$= I\,\frac{\left(\omega_g^4 + 4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\eta_g^2\omega_g^2\omega^2}\underbrace{\frac{\left(\omega/\omega_f\right)^4}{\left[1-\left(\omega/\omega_f\right)^2\right]^2 + 4\varsigma_f^2\left(\omega/\omega_f\right)^2}}_{\text{High pass filter}}$$



1.5
1
0.5
0
-0.5
-1
-1.5
acceleration
0
5
10
15
20
25
30
time t

How to allow for nonstationary nature of ground accelerations?

Nonstationarity : in amplitude modulation & frequency content.

Strategy: Use a deterministic modulaitng function.

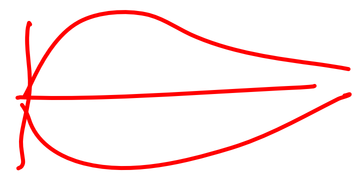
$$\ddot{X}_g(t) = e(t)S(t)$$

$e(t) =$ deterministic envelope function

$S(t)$=zero mean stationary Gaussian random process

(with PSD given by Kanai-Tajimi or

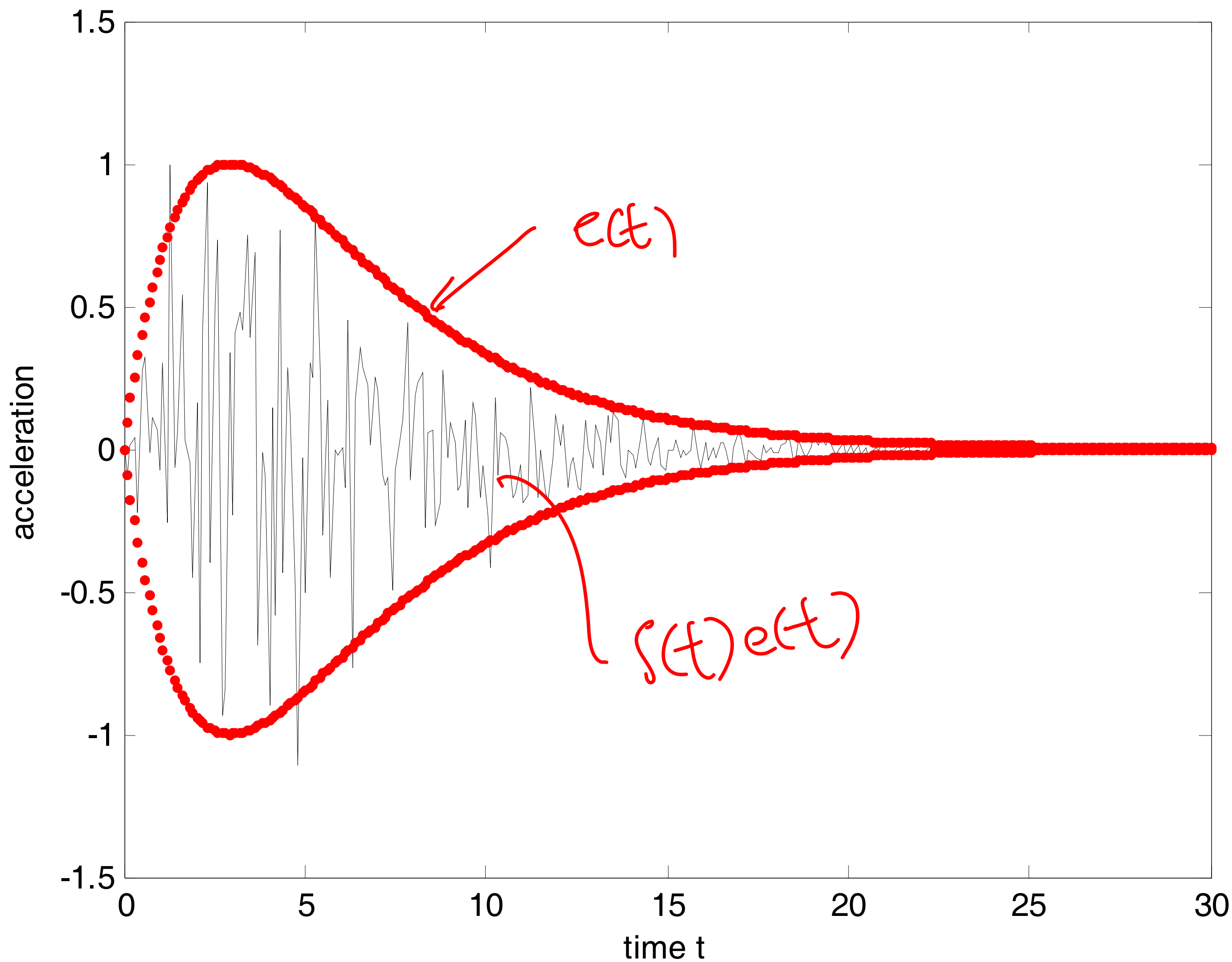
Clough and Penzien models)



Examples

$$e(t) = A_0\left[\exp(-\alpha t) - \exp(-\beta t)\right]; \alpha > \beta > 0$$

$$e(t) = \left(A_0 + A_1 t\right)\exp(-\alpha t)$$



1.5
1
0.5
0
-0.5
-1
-1.5
acceleration
0
5
10
15
20
25
30
time t
e(t)
s(t)e(t)

$$\ddot{y}_1 + 2\eta_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = e(t)s(t)$$

$$\ddot{y}_2 + 2\eta_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = 2\eta_1\omega_1\dot{y}_1 + \omega_1^2 y_1$$

$$\begin{Bmatrix} \text{Ground displacement} \\ \text{Ground velocity} \\ \text{Ground acceleration} \end{Bmatrix} = \begin{Bmatrix} y_2(t) \\ \dot{y}_2(t) \\ \ddot{y}_2(t) \end{Bmatrix}$$

Introduce

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\eta_1\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_1^2 & 2\eta_1\omega_1 & -\omega_2^2 & -2\eta_2\omega_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} e(t)s(t)$$

Examples for envelope function

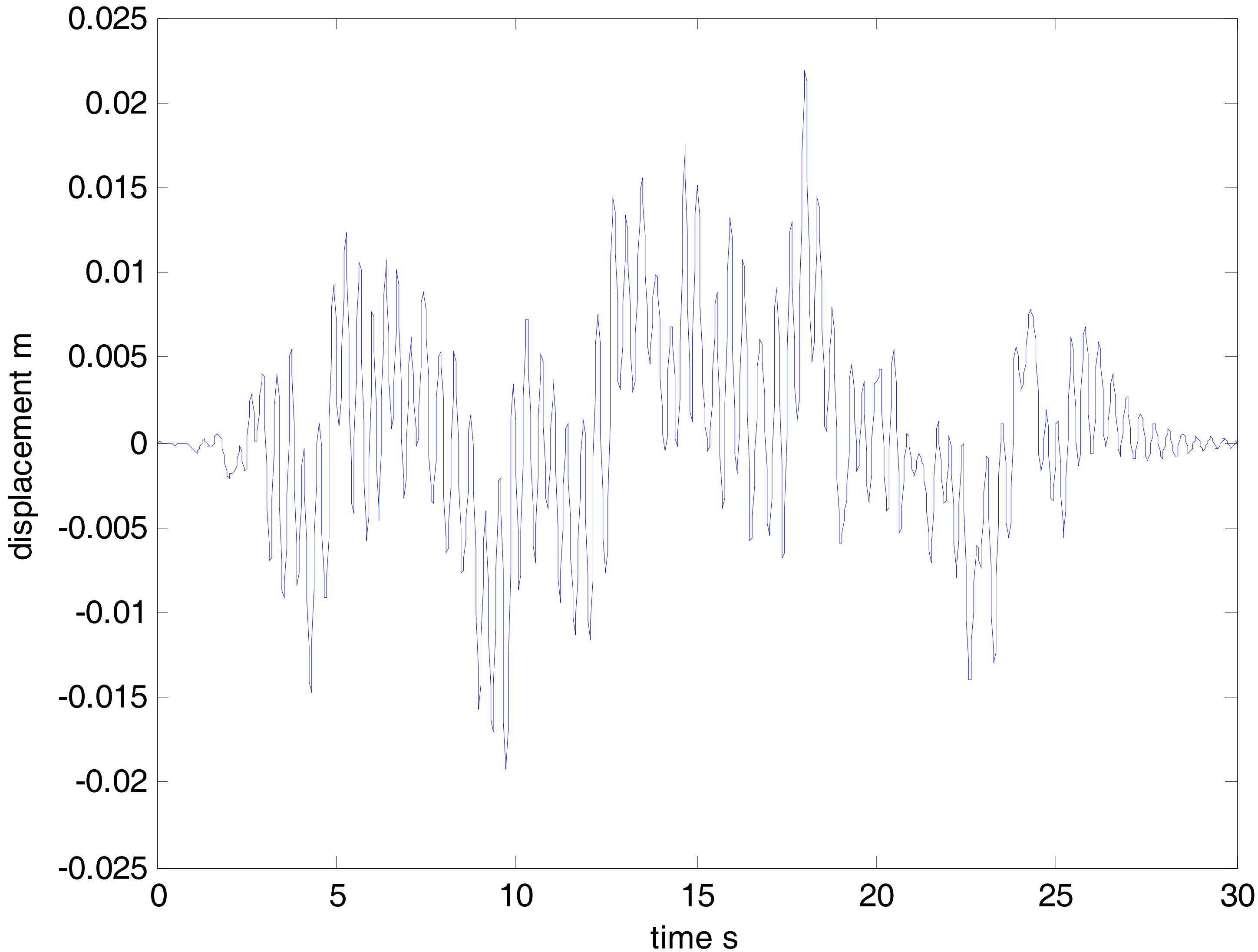
$$e(t) = \left(\frac{t}{4}\right)^2 \text{ for } 0 < t < 4\text{s}$$

$$=1 \text{ for } 4 < t < 24\text{s}$$

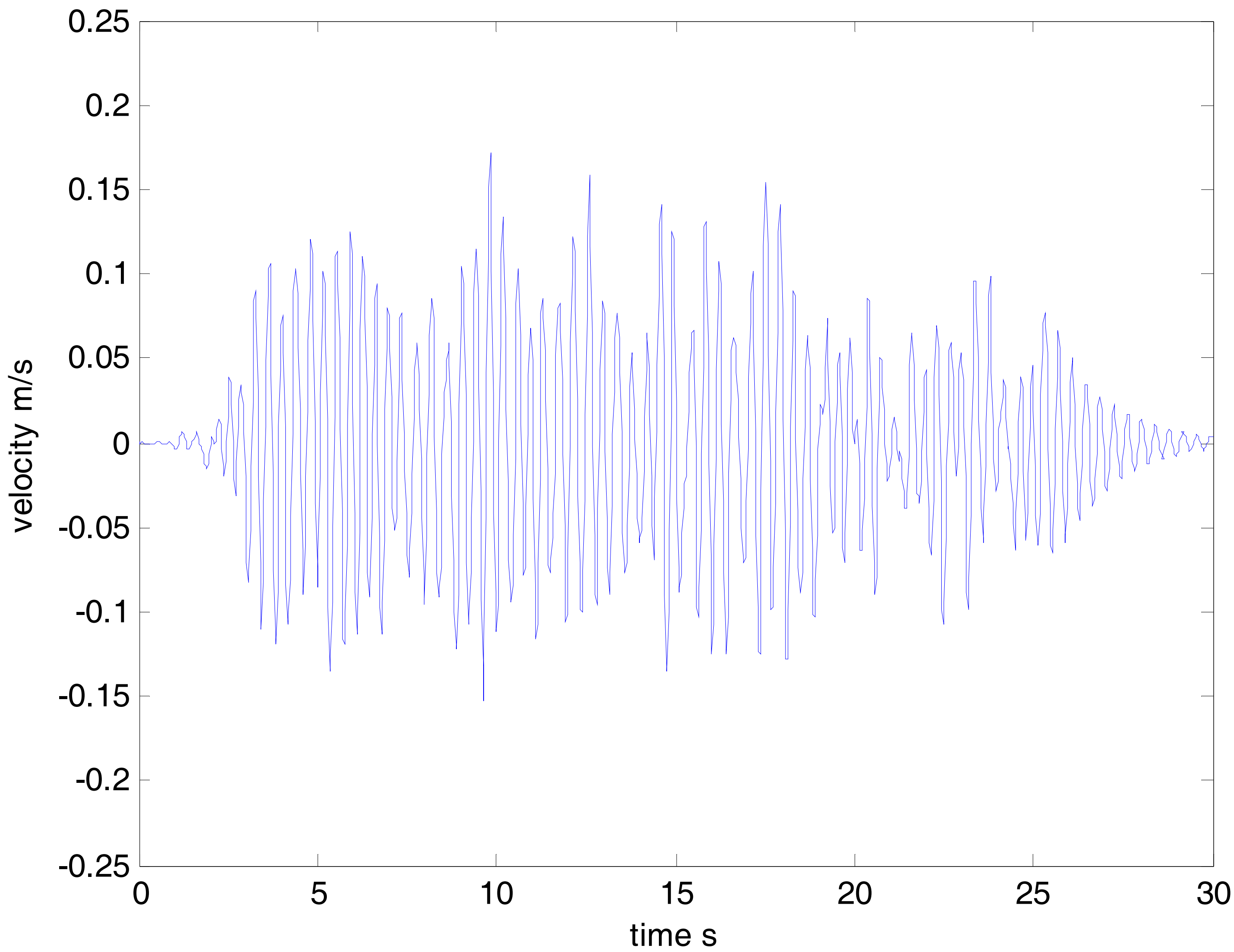
$$=\exp\left[-\frac{1}{2}(t-24)^2\right] \text{ for } t > 24 \text{ s}$$

$$e(t) = a\left[\exp(-\alpha t) - \exp(-\beta t)\right]; \alpha > \beta > 0$$

$$e(t) = (A_0 + A_1 t)\exp(-\alpha t)$$



0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
displacement m
0
5
10
15
20
25
30
time s



0.25
0.2
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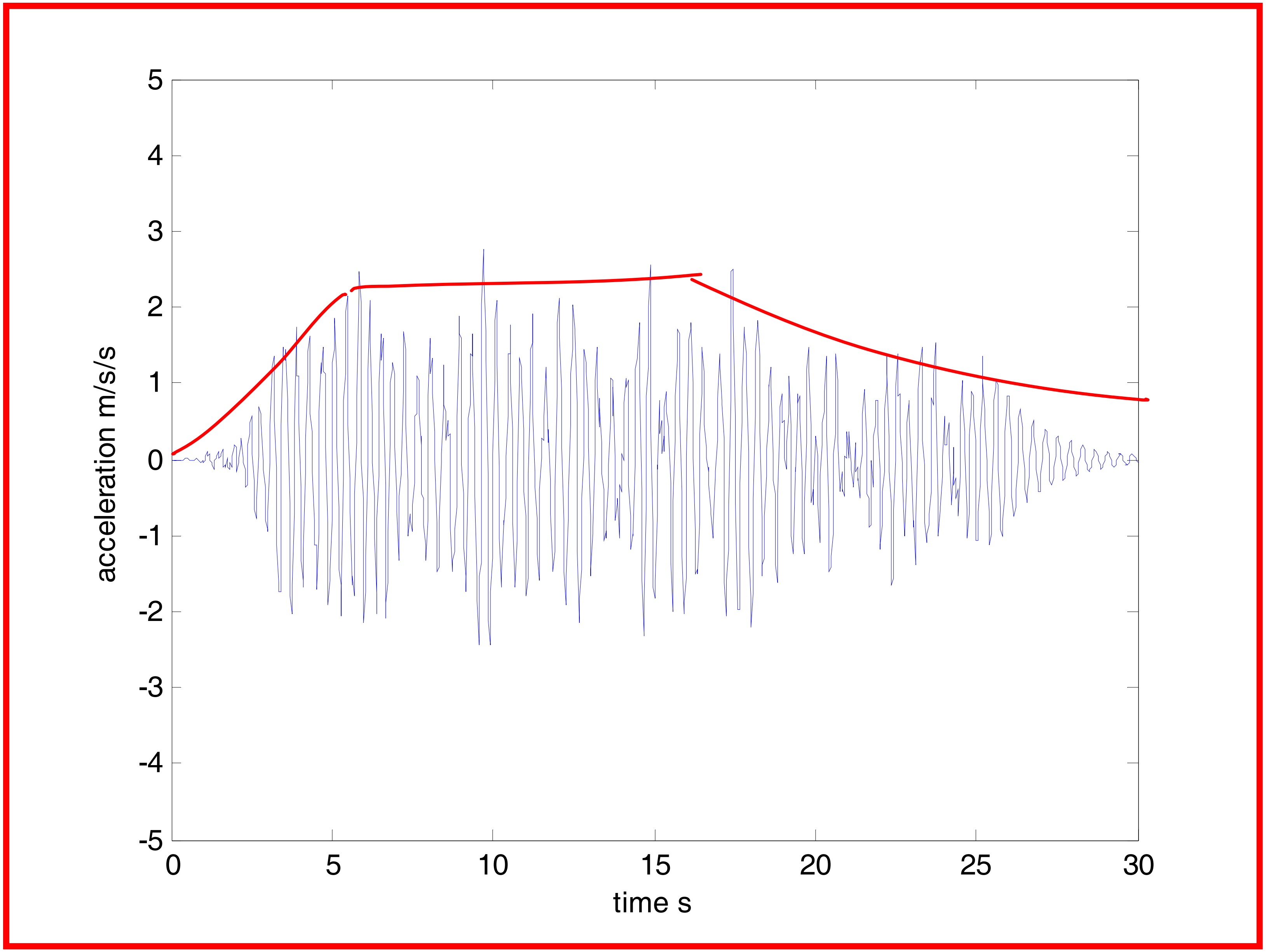
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
velocity m/s
0
5
10
15
20
25
30
time s



acceleration m/s/s
time s

Nonstationarity in frequency content

- Random pulse processes
- Evolutionary psd functions

RECALL

Characteristic function and characteristic functional

• Let X be a random variable.

$$M_X(\theta) = \langle \exp[i\theta X] \rangle = \int_{-\infty}^{\infty} \exp[i\theta x] p_X(x) dx \Rightarrow$$

$$M_X(\theta) = 1 + \sum_{n=1}^{\infty} \frac{i^n \theta^n}{n!} \langle X^n \rangle \Rightarrow \langle X^n \rangle = \frac{1}{i^n} \frac{d^n M_X(\theta)}{d\theta^n}\bigg|_{\theta=0}$$

Example: Let $X \sim N(m,\sigma) \Rightarrow M_X(\theta) = \exp\left(im\theta - \frac{1}{2}\sigma^2\theta^2 \right)$

• Log characteristic function: $\ln M_X(\theta) = \ln\left\{ 1 + \sum_{n=1}^{\infty} \frac{i^n \theta^n}{n!} \langle X^n \rangle \right\}$

For $X \sim N(m,\sigma)$, $\ln M_X(\theta) = im\theta - \frac{1}{2}\sigma^2\theta^2$

•Cumulants: $\ln M_X(\theta) = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \kappa_n(X)$

$$\kappa_n = \frac{1}{i^n} \frac{d^n}{d\theta^n} \ln M_X(\theta) \Big|_{\theta=0} = n^{\text{th}} \text{ order cumulant}$$

•Let $(X_i)_{i=1}^{m}$ be a set of random variables

$$M_X(\theta_1, \theta_2, \cdots, \theta_m) = \left\langle \exp\left(i \sum_{n=1}^{m} \theta_n X_n \right) \right\rangle = \int_{-\infty}^{\infty} \exp\left(i \tilde{\theta}^t \tilde{x} \right) p_{\tilde{X}}(\tilde{x}) \, d\tilde{x}$$

$= m$-dimensional joint characteristic function

$$\left\langle X_1^{m_1} X_2^{m_2} \cdots X_m^{m_m} \right\rangle =$$

$$\frac{1}{i^{m_1+m_2+\cdots+m_m}} \left(\frac{\partial^{m_1+m_2+\cdots+m_m}}{\partial x_1^{m_1} \partial x_2^{m_2} \cdots \partial x_m^{m_m}} M_X(\theta_1, \theta_2, \cdots, \theta_m) \right) \Bigg|_{\theta_1=0, \theta_2=0, \cdots, \theta_m=0}$$

$$M_X(\theta_1,\theta_2,\cdots,\theta_m) = 1 + (i\theta_j)\langle X_j \rangle + \frac{1}{2!}(i\theta_j)(i\theta_k)\langle X_j X_k \rangle + \cdots$$

$$\ln M_X(\theta_1,\theta_2,\cdots,\theta_m) = (i\theta_j)\kappa_1(X_j) + \frac{1}{2!}(i\theta_j)(i\theta_k)\kappa_2(X_j X_k) + \cdots$$

$$\kappa_{m_1+m_2+\cdots+m_m}(X_1, X_2, \cdots, X_m) =$$

$$\frac{1}{i^{m_1+m_2+\cdots+m_m}}\left(\frac{\partial^{m_1+m_2+\cdots+m_m}}{\partial x_1^{m_1}\partial x_2^{m_2}\cdots\partial x_m^{m_m}} \ln M_X(\theta_1,\theta_2,\cdots,\theta_m)\right)\Bigg|_{\theta_1=0,\theta_2=0,\cdots,\theta_m=0}$$

$$\kappa_1(X_j) = \langle X_j \rangle$$

$$\kappa_2(X_j X_k) = \left\langle \left(X_i - \mu_{X_i}\right)\left(X_j - \mu_{X_j}\right)\right\rangle$$

$$\vdots$$

For a vector of Gaussian random variables it can be shown that all cumulants of order ≥ 3 are zero.

Characteristic functional

Let $X(t)$ be a random process.

$$M_X\left[\theta(t)\right]=\left\langle \exp\left(\int_T i\theta(t)X(t)\,dt\right)\right\rangle$$

We could select $\theta(t)=\sum_{j=1}^{m}\theta_j\delta\left(t-t_j\right)$ to characterize

m random variables $\left\{X\left(t_j\right)\right\}_{j=1}^{m}$.

$$M_X\left[\theta(t)\right]=1+i\int\theta(t)\left\langle X(t)\right\rangle dt+\frac{i^2}{2}\int\int\theta(t_1)\theta(t_2)\left\langle X(t_1)X(t_2)\right\rangle dt_1dt_2+\cdots$$

$$\ln M_X\left[\theta(t)\right]=i\int\theta(t)\kappa_1\left[X(t)\right]dt+$$

$$\frac{i^2}{2}\int\int\theta(t_1)\theta(t_2)\kappa_2\left[X(t_1)X(t_2)\right]dt_1dt_2+\cdots$$

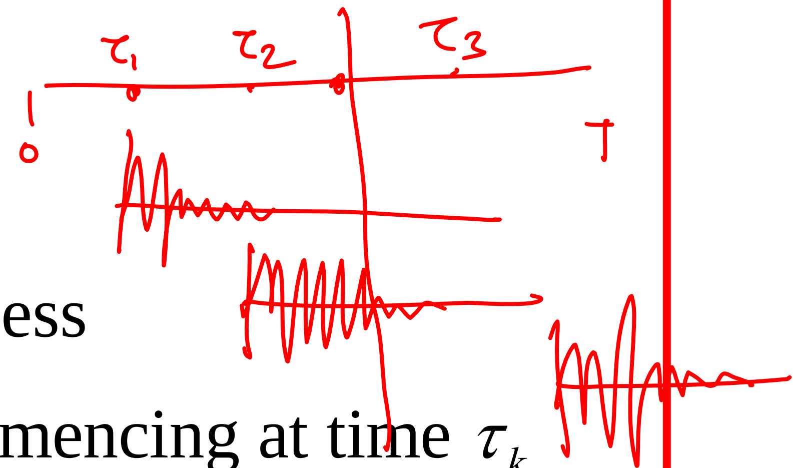
Let $X(t)$ be a Gaussian random process.

$$M_X\left[\theta(t)\right]=\left\langle \exp\left(\int_T i\theta(t)X(t)\,dt\right)\right\rangle$$

$$=\exp\left[i\int_T \mu_X(t)\theta(t)\,dt-\frac{1}{2}\int_T\int_T C_{XX}(t_1,t_2)\theta(t_1)\theta(t_2)\,dt_1dt_2\right]$$

$$\ln M_X\left[\theta(t)\right]=i\int_T \mu_X(t)\theta(t)\,dt-\frac{1}{2}\int_T\int_T C_{XX}(t_1,t_2)\theta(t_1)\theta(t_2)\,dt_1dt_2$$

Poisson pulse process



$$X(t) = \sum_{k=1}^{N(t)} W_k(t, \tau_k); 0 < t < T$$

$N(t)$ = Poisson counting process

$W_k(t, \tau_k)$ = random pulse commencing at time τ_k

τ_k = random points distributed uniformly in 0 to T

Simplified version

$$X(t) = \sum_{k=1}^{N(t)} Y_K w(t, \tau_k); 0 < t < T$$

Y_K = random amplitude of the k-th pulse (iid rvs).

$w(t, \tau_k)$ = a deterministic pulse commencing at $t = \tau_k$

such that $w(t, \tau_k) = 0$ for $t < \tau_k$

$$X(t) = \sum_{k=1}^{N(T)} Y_K w(t, \tau_k); 0 < t < T$$

$$M_X[\theta(t)] = \left\langle \exp\left[i\int_0^T \theta(t) X(t) dt \right] \right\rangle$$

$$= \left\langle \exp\left[i\int_0^T \theta(t) \sum_{k=1}^{N(T)} Y_K w(t, \tau_k) dt \right] \right\rangle$$

$$= E\left\{ \left\langle \exp\left[i\int_0^T \theta(t) \sum_{k=1}^{n} Y_K w(t, \tau_k) dt \mid N(T) = n \right] \right\rangle P[N(T) = n] \right\}$$

$$= \sum_{n=0}^{\infty} P[N(T) = n] \left\langle \exp\left[i\int_0^T \theta(t) \sum_{k=1}^{n} Y_K w(t, \tau_k) dt \right] \right\rangle$$

$$M_X\left[\theta(t)\right]=\sum_{n=0}^{\infty}P\left[N(T)=n\right]\left\langle \exp\left[i\int_0^T\theta(t)\sum_{k=1}^{n}Y_K w(t,\tau_k)dt\right]\right\rangle$$

$$\left\langle \exp\left[i\int_0^T\theta(t)\sum_{k=1}^{n}Y_K w(t,\tau_k)dt\right]\right\rangle=\left\langle \prod_{k=1}^{n}\exp\left\{i\int_0^T\theta(t)Y_k w(t,\tau_k)dt\right\}\right\rangle$$

$$=(1+\alpha)^n$$

$$\alpha=\left\langle \sum_{m=1}^{\infty}\frac{i^m}{m!}\left[\int_0^T\theta(t)Y_k w(t,\tau_k)dt\right]^m\right\rangle$$

$$P\left[N(T)=n\right]=\frac{e^{-\int_0^T\lambda(\tau)d\tau}\left(\int_0^T\lambda(\tau)d\tau\right)^n}{K!}$$

$$\alpha = \left\langle \sum_{m=1}^{\infty} \frac{i^m}{m!} \left[\int_0^T \theta(t) Y_k w(t, \tau_k) dt \right]^m \right\rangle$$

$$= \sum_{m=1}^{\infty} \frac{i^m}{m!} \left\{ \int_{-\infty}^{\infty} y^m p_Y(y) dy \right\} \int_0^T \int_0^T \cdots \int_0^T \theta(t_1) \theta(t_2) \cdots \theta(t_m) \frac{\int_0^T w(t_1, \tau) w(t_2, \tau) \cdots w(t_m, \tau) \lambda(\tau) d\tau}{\int_0^T \lambda(\tau) d\tau} dt_1 dt_2 \cdots dt_m$$

$$M_X\left[\theta(t)\right] = \sum_{n=0}^{\infty} P\left[N(T)=n\right](1+\alpha)^n$$

$$= \sum_{n=0}^{\infty} \exp\left(-\int_0^T \lambda(\tau)\,d\tau\right)\frac{1}{n!}\left[\int_0^T \lambda(\tau)\,d\tau\right]^n (1+\alpha)^n$$

$$= \exp\left(-\int_0^T \lambda(\tau)\,d\tau\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left[(1+\alpha)\int_0^T \lambda(\tau)\,d\tau\right]^n$$

$$= \exp\left(-\int_0^T \lambda(\tau)\,d\tau\right)\exp\left[(1+\alpha)\int_0^T \lambda(\tau)\,d\tau\right] = \exp\left(\alpha\int_0^T \lambda(\tau)\,d\tau\right)$$

$$\ln M_X\left[\theta(t)\right] = \alpha\int_0^T \lambda(\tau)\,d\tau$$

$$e^a \quad \frac{a^k}{k!}$$

$$\ln M_X\left[\theta(t)\right] = \alpha\int_0^T \lambda(\tau)\,d\tau$$

$$= \sum_{m=1}^{\infty} \frac{i^m \left\langle Y^m \right\rangle}{m!} \int_0^T \int_0^T \cdots \int_0^T \theta(t_1)\theta(t_2)\cdots\theta(t_m)$$

$$\left[\int_0^T w(t_1,\tau)\,w(t_2,\tau)\cdots w(t_m,\tau)\,\lambda(\tau)\,d\tau\right] dt_1 dt_2 \cdots dt_m$$

Compare this with

$$\ln M_X\left[\theta(t)\right] = i\int \theta(t)\,\kappa_1\left[X(t)\right] dt +$$

$$\frac{i^2}{2}\int\int \theta(t_1)\theta(t_2)\,\kappa_2\left[X(t_1)X(t_2)\right] dt_1 dt_2 + \cdots$$

$$\kappa_m\left[X(t_1)X(t_2)\cdots X(t_m)\right]=$$

$$\langle Y^m\rangle\int_0^{\min(t_1,t_2,\cdots,t_m)} w(t_1,\tau)w(t_2,\tau)\cdots w(t_m,\tau)\lambda(\tau)d\tau$$

Note: $w(t,\tau)=0\forall t<\tau.$

$$\Rightarrow$$

$$\mu_X(t)=\mu_Y\int_0^t w(t,\tau)\lambda(\tau)d\tau$$

$$\kappa_{XX}(t_1,t_2)=\langle Y^2\rangle\int_0^{\min(t_1,t_2)} w(t_1,\tau)w(t_2,\tau)\lambda(\tau)d\tau$$

$$\sigma_X^2(t)=\langle Y^2\rangle\int_0^t w^2(t,\tau)\lambda(\tau)d\tau$$

Special case

$$w(t,\tau) = w(t-\tau) \,\&\, \lambda(\tau) = \lambda$$

$$\Rightarrow$$

$$\mu_X = \mu_Y \lambda \int_{-\infty}^{\infty} w(u)\,du$$

$$\kappa_{XX}(t_1,t_2) = \langle Y^2 \rangle \lambda \int_{-\infty}^{\infty} w(u)\,w(t_2 - t_1, u)\,du$$

$$\sigma_X^2 = \langle Y^2 \rangle \lambda \int_{-\infty}^{\infty} w^2(u)\,du$$

Evolutionary random process (Intutive explanation)

Consider $\{X_i(t)\}_{i=1}^{N}$ to be N zero mean, stationary random processes with PSD $S_i(\omega)$.

Consider the time interval 0 to T and divide into N segments.

Define a random process $X(t)$ as

$$X(t) = \begin{cases} X_1(t) \text{ for } 0 < t < t_1 \\ X_2(t) \text{ for } t_1 < t < t_2 \\ \vdots \\ X_N(t) \text{ for } t_{N-1} < t < t_N \end{cases}$$

$X(t)$ is a nonstationary random process

The PSD function of $X(t)$ can be written as

$$S_{XX}(\omega,t)=\begin{cases} S_1(\omega) \text{ for } 0<t<t_1 \\ S_2(\omega) \text{ for } t_1<t<t_2 \\ \vdots \\ S_N(\omega) \text{ for } t_{N-1}<t<t_N \end{cases}$$

$X(t)$ is called a evolutionary random process.

Spectral representaiton of an evolutionary random process

Consider the representation

$$X(t) = \int_{-\infty}^{\infty} a(t,\omega)\exp(i\omega t)\,dZ(\omega)$$

$a(t,\omega)$ = deterministic function (in general, complex valued)

$Z(\omega)$ = orthogonal increment random process (complex valued)

with $\langle dZ(\omega)\rangle = 0$ & $\langle dZ(\omega_1)\,dZ^*(\omega_2)\rangle = \delta(\omega_1-\omega_2)\,d\Psi(\omega)$

$$\langle X(t)\rangle = \int_{-\infty}^{\infty} a(t,\omega)\exp(i\omega t)\langle dZ(\omega)\rangle = 0$$

$$\langle X(t_1)X^*(t_2)\rangle =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t_1,\omega_1)\,a^*(t_2,\omega_2)\exp\left[i(\omega_1 t_1-\omega_2 t_2)\right]\langle dZ(\omega_1)\,dZ^*(\omega_2)\rangle$$

$$\left\langle X(t_1) X^*(t_2) \right\rangle =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t_1,\omega_1) a^*(t_2,\omega_2) \exp\left[i(\omega_1 t_1 - \omega_2 t_2)\right] \left\langle dZ(\omega_1) dZ^*(\omega_2) \right\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(t_1,\omega_1) a^*(t_2,\omega_2) \exp\left[i(\omega_1 t_1 - \omega_2 t_2)\right] \delta(\omega_2 - \omega_1) d\Psi(\omega_1,\omega_2)$$

$$= \int_{-\infty}^{\infty} a(t_1,\omega) a^*(t_2,\omega) \exp\left[i\omega(t_1 - t_2)\right] d\Psi(\omega)$$

$$\sigma_X^2(t) = \int_{-\infty}^{\infty} \left|a(t,\omega)\right|^2 d\Psi(\omega)$$

$$d\Psi(\omega) = \Phi(\omega) d\omega$$

If $\Psi(\omega)$ is differentiable, the above integral can be interpreted as the Riemann integral and we get

$$\sigma_X^2(t) = \int_{-\infty}^{\infty} \left|a(t,\omega)\right|^2 \Phi(\omega) d\omega$$

$$\sigma_X^2(t) = \int_{-\infty}^{\infty} \left|a(t,\omega)\right|^2 \Phi(\omega)\, d\omega$$

We interpret $S_{XX}(\omega) = \left|a(t,\omega)\right|^2 \Phi(\omega)$ as the nonstationary (evolutionary) PSD function of $X(t)$.

Remark

If $X(t) = e(t)Y(t)$ where $e(t)$ is deterministic and $Y(t)$ is a zero mean stationary random process $\Rightarrow$

$$\sigma_X^2(t) = e^2(t) \int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega$$

$X(t) =$ Uniformly modulated nonstationary random process.

This does not take into account the variation of frequency content with respect to time.

Filtered Poisson Process models for earthquake ground motions

Rationale

During earthquakes slips occur along fault lines in an intermittent manner. This sends out a train of stress waves in the earth crust. This eventually results in ground shaking.

Recall

$$X(t) = \sum_{j=1}^{N(T)} Y_j w(t, \tau_j); 0 < t \le T$$

$N(T)$ = counting process, Poisson; arrival rate $= \lambda(t)$

τ_j = arrival times; random

$w(t, \tau_j)$ = Deterministic pulse shape $(= 0 \, \forall t \le \tau_j)$.

Y_j = random magnitude of the j-th pulse.

$$m_X(t) = m_Y \int_0^t w(t,\tau)\lambda(\tau)\,d\tau,$$

$$C_{XX}(t_1,t_2) = E(Y^2) \int_0^{\min(t_1,t_2)} w(t_1,\tau)\,w(t_2,\tau)\,\lambda(\tau)\,d\tau,$$

$$\sigma_X^2(t) = E(Y^2) \int_0^t w^2(t,\tau)\,\lambda(\tau)\,d\tau$$

Reference

Y K Lin and G C Cai, 1995, McGraw Hill, NY.

Selection of the shape of the pulse

Model -1

As in Kanai Tajimi model, the soil layer is modeled as an elastic half-space which can be represented as a sdof system.

$$\ddot{u} + 2\eta_g\omega_g\dot{u} + \omega_g^2 u = 2\eta_g\omega_g\dot{R} + \omega_g^2 R$$

$$H_1(\omega) = \frac{\omega_g^2 + i2\eta_g\omega_g\omega}{\left(\omega_g^2 - \omega^2\right)^2 + \left(2\eta_g\omega_g\omega\right)^2}$$

$$h_1(t) = \omega_g \exp\left(-\eta_g\omega_g t\right)\left\{\frac{1-2\eta_g^2}{\sqrt{1-\eta_g^2}}\sin\omega_{gd}t + 2\eta_g\cos\omega_{gd}t\right\}; t > 0$$

$$G(t) = \sum_{j=1}^{N(T)} Y_j h_1\left(t-\tau_j\right)$$