Stochastic Structural Dynamics

Lecture-32

Probabilistic methods in earthquake engineering-1

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Recall

We have developed methods to study systems governed by

$$M\ddot{X} + C\dot{X} + KX + F\left(X, \dot{X}\right) = G\left(t\right)$$

where $X\left(0\right) \& \dot{X}\left(0\right)$ are specified, and $G\left(t\right)$ is a vector random process.

The analysis has included characterization of response moments, pdf-s of system states, and reliability measures.

The equations governing the behavior of structures subjected to earthquake support motions have form similar to the above equation.

Therefore, what are the issues that we need to consider afresh?

Focus: uncertainties in vibratory response of structures during an earthquake

- Stochastic models for ground motions
- Response spectrum, PSD and time histories
- Modal combination rules
- Seismic risk analysis
- Performance based structural design (PBSD)

Aim:

- To introduce the basic ideas and facilitate future self-study

Uncertainties in earthquake engineering problems

Earthquakes
- When, where, and how earthquakes occur
- The details of ground motion
- The effect of ground motion on structure

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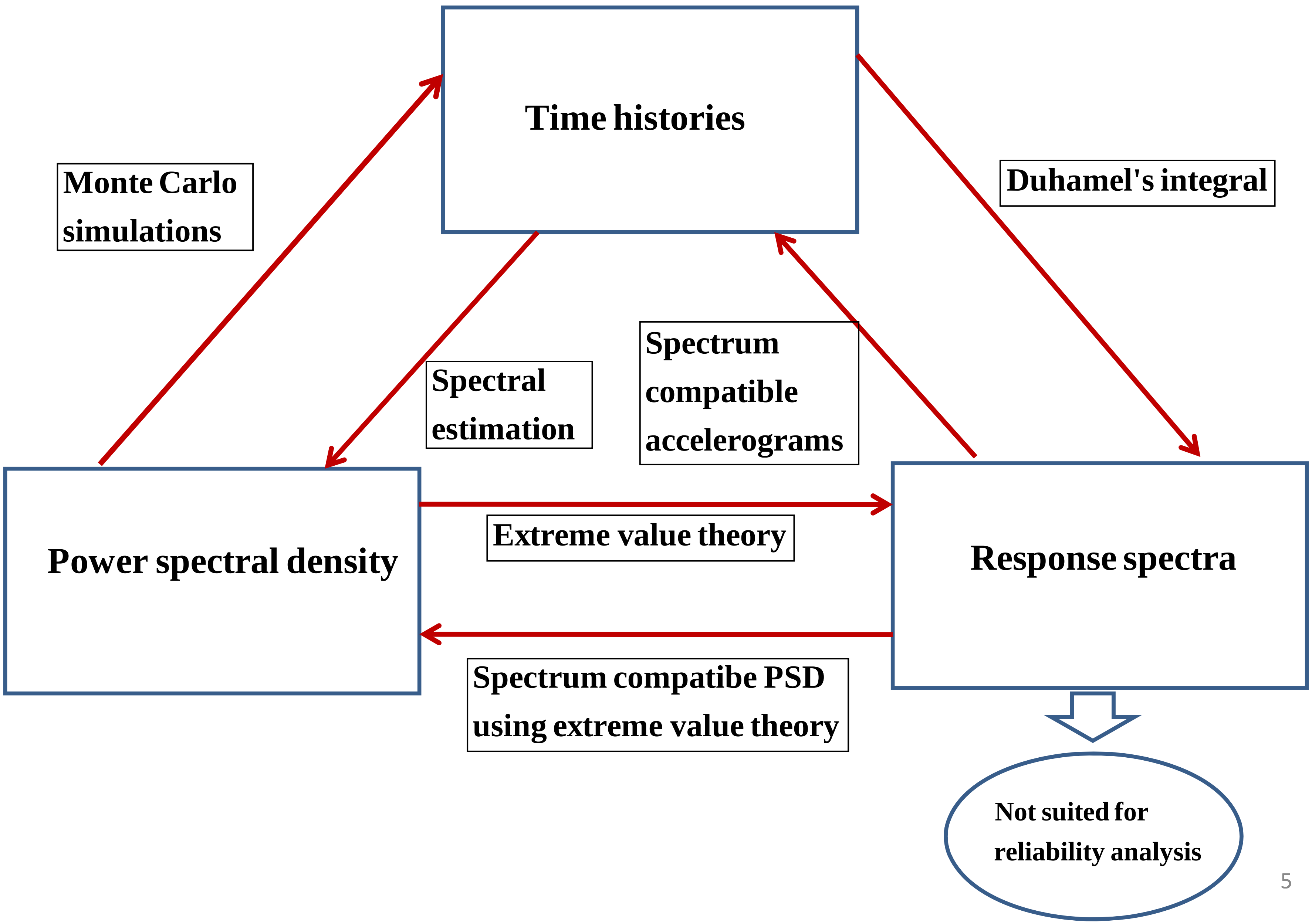
Dynamic response

Damage and loss

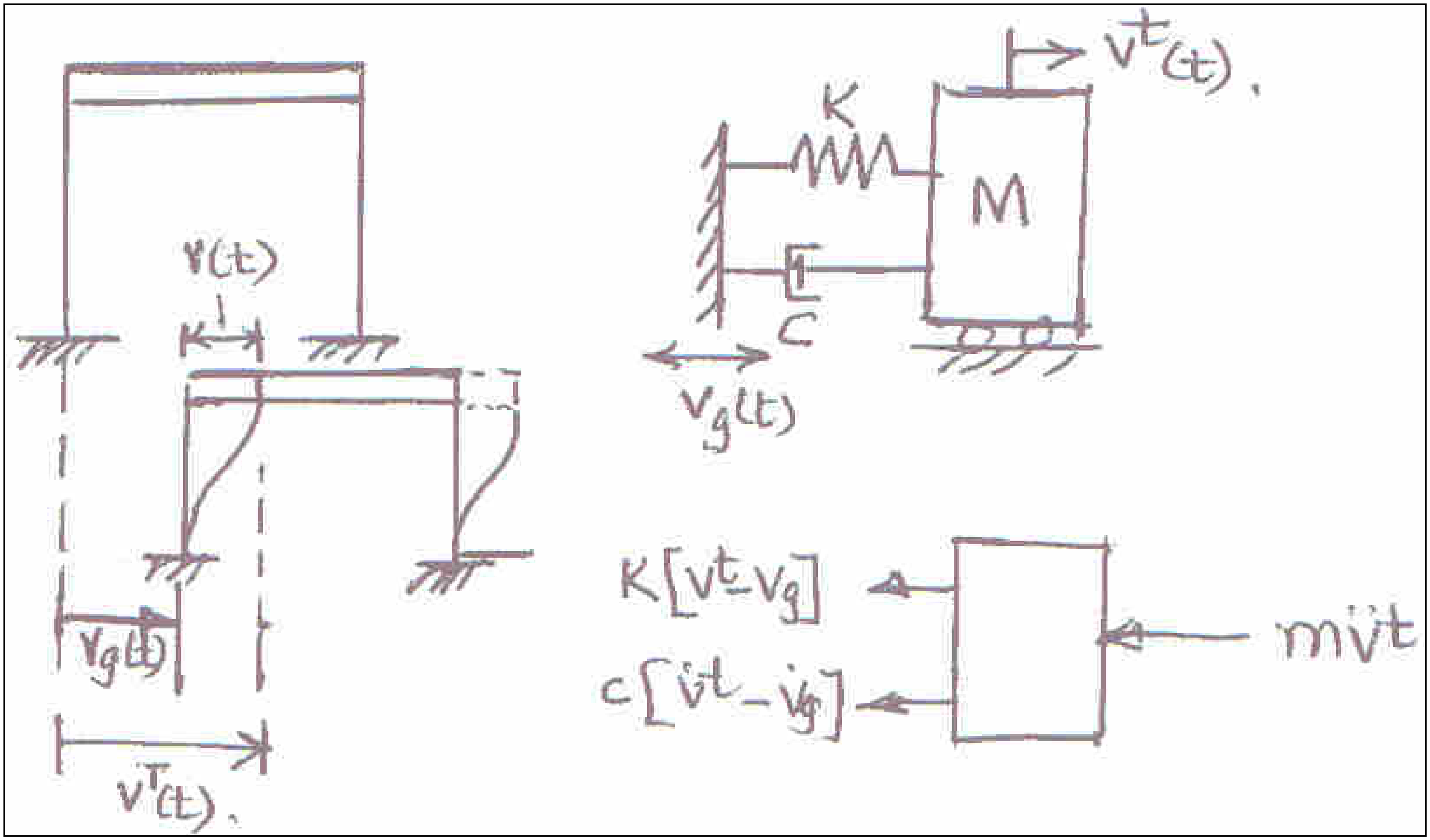
Alternatives for earthquake load specification



References

- R W Clough and J P Penzien, 1993, Dynamics of structures, McGraw Hill, NY
- N C Nigam and S Narayanan, 1994, Applications of random vibrations, Narosa, New Delhi

Response spectra



Response spectrum

•Effect based model for earthquake ground acceleration

Equation governing total displacement

$$m\ddot{v}^t + c(\dot{v}^t - \dot{v}_g) + k(v^t - v_g) = 0$$

Relative displacement $v(t) = v^t(t) - v_g(t)$

$$m\ddot{v} + c\dot{v} + kv = -m\ddot{v}_g$$

$$\Rightarrow \ddot{v} + 2\eta\omega\dot{v} + \omega^2 v = -\ddot{v}_g(t)$$

$$v(t)=\frac{1}{m\omega_d}\int_0^t \exp[-\eta\omega(t-\tau)]\sin[\omega_d(t-\tau)][-m\ddot{v}_g(\tau)]d\tau$$

$$=\frac{1}{\omega_d}\int_0^t \exp[-\eta\omega(t-\tau)]\sin[\omega_d(t-\tau)]\ddot{v}_g(\tau)d\tau$$

$$\eta<0.1\Rightarrow \omega_d\approx\omega\Rightarrow$$

$$v(t)=\frac{1}{\omega_n}\int_0^t \exp[-\eta\omega_n(t-\tau)]\sin[\omega_n(t-\tau)]\ddot{v}_g(\tau)d\tau$$

$$\dot{v}(t)=\int_0^t \ddot{v}_g(\tau)\exp[-\eta\omega(t-\tau)]\cos[\omega_n(t-\tau)]d\tau$$

$$-\eta\int_0^t \ddot{v}_g(\tau)\exp[-\eta\omega(t-\tau)]\sin[\omega_n(t-\tau)]d\tau$$

$$\ddot{v}^t(t) = -\frac{c}{m}(\dot{v}^t - \dot{v}_g) - \frac{k}{m}(v^t - v_g) = -2\eta\omega_n\dot{v} - \omega_n^2 v$$

$$\ddot{v}^t(t) = \omega_n(2\eta^2 - 1)\int_0^t \ddot{v}_g(\tau)\exp[-\eta\omega_n(t-\tau)]\sin[\omega_n(t-\tau)]d\tau -$$

$$-2\eta\omega_n\int_0^t \ddot{v}_g(\tau)\exp[-\eta\omega_n(t-\tau)]\cos[\omega_n(t-\tau)]d\tau$$

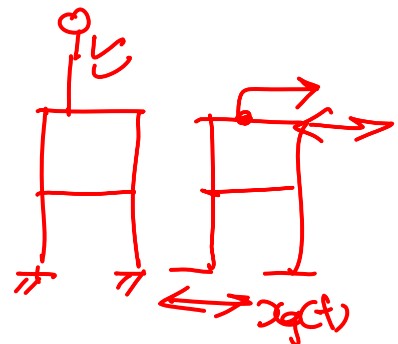
Remark

- Systems with the same η and ω_n respond identically to the same ground motion.

Response quantities of interest

$v(t)$:

- Maximum relative displacement
- Force in the spring (column) and hence the stresses in the column are proportional to $v(t)$



$v^t(t)$:

- Of interest in the study of secondary systems
- Pounding of adjacent buildings

Force in the spring

$$f_s(t) = k[v^t(t) - v_g(t)] = kv(t)$$

$$= m\omega_n^2 v(t)$$

$$= mA(t) = W\frac{A(t)}{g}$$

- $f_s(t)$ = base shear
- $A(t) = \omega_n^2 v(t)$: has units of acceleration (pseudo-acceleration)
- $\dfrac{A(t)}{g}$ = seismic coefficient
- Weight of the building $\times$ seismic coefficient = base shear
- Base shear $\times$ height of the building = base moment

$$U = \frac{1}{2}kv^2(t) = \frac{1}{2}m\omega_n^2 v^2(t)$$

$$= \frac{1}{2}m[\omega_n v(t)]^2 = \frac{m}{2}V_0^2(t)$$

- $U =$ strain energy stored
- $\omega_n v(t)$: has units of velocity (pseudo-velocity)

$x_g(t)$ $\dot{x}_g(t)$ $\ddot{x}_g(t)$

Let a family of sdof systems with damping *{η}* and natural frequencies *{ω_n}* be subjected to a given ground acceleration.

Let us determine the peak responses over time as a function of *{η}* and *{ω_n}*.

Definitions

$$S_d(\eta,\omega_n) = \max_{0<t<\infty} |v(t)| \quad \left(\text{Spectral relative displacement}\right)$$

$$S_v(\eta,\omega_n) = \max_{0<t<\infty} |\dot{v}(t)| \quad \left(\text{Spectral relative velocity}\right)$$

$$S_a(\eta,\omega_n) = \max_{0<t<\infty} |\ddot{v}^t(t)| \quad \left(\text{Spectral absolute acceleration}\right)$$

$$S_{pa}(\eta,\omega_n) = \omega_n^2 S_d(\eta,\omega_n) \quad \left(\text{Spectral pseudo acceleration}\right)$$

$$S_{pv}(\eta,\omega_n) = \omega_n S_d(\eta,\omega_n) \quad \left(\text{Spectral pseudo velocity}\right)$$

Definitions

$S_d(\omega_n, \eta)$
$S_v(\omega_n, \eta)$
$S_a(\omega_n, \eta)$
$S_{pa}(\omega_n, \eta)$
$S_{pv}(\omega_n, \eta)$

as a function of ω_n with η as a parameter is called the response spectrum for

Relative displacement
Relative velocity
Absolute acceleration
Pseudo acceleration
Pseudo velocity

Spectrum

- Connotes "frequency" on *x*-axis.
- "Frequency" often refers to the frequency parameter used in defining Fourier transform. In this context we talk of time and frequency domain representations of signals.
- In the context of response spectrum "frequency" is not the Fourier frequency but the natural frequencies of a family of sdof systems.
- Often period is plotted on the *x*-axis.

Limiting behavior of the spectra as $\omega_n \to \infty$

$$\ddot{v} + 2\eta\omega_n\dot{v} + \omega_n^2 v = -\ddot{v}_g(t)$$

$$\Rightarrow \lim_{\omega_n \to \infty} \omega_n^2 v \cong -\ddot{v}_g(t)$$

$$\Rightarrow \lim_{\omega_n \to \infty} \max_{0<t<\infty} \omega_n^2 |v(t)| \cong \max_{0<t<\infty} |\ddot{v}_g(t)|$$

$$\Rightarrow \lim_{\omega_n \to \infty} S_{pa}(\eta, \omega_n) \to \max_{0<t<\infty} |\ddot{v}_g(t)| \ \forall \eta$$

$$\max_{0<t<\infty} |\ddot{v}_g(t)| = ZPA \ or \ PGA$$

ZPA: zero period acceleration

PGA: peak ground acceleration

Limiting behavior of response spectra as $\omega_n \to 0$

$$v(t) = \frac{1}{\omega_d}\int_0^t \exp[-\eta\omega(t-\tau)]\sin[\omega_d(t-\tau)]\ddot{v}_g(\tau)d\tau$$

$$\lim_{\omega_n\to 0} v(t) = \int_0^t \lim_{\omega_n\to 0}\frac{\sin[\omega_d(t-\tau)]}{\omega_d}\ddot{v}_g(\tau)d\tau$$

$$= \int_0^t (t-\tau)\ddot{v}_g(\tau)d\tau = v_g(t)$$

$$\Rightarrow \lim_{\omega_n\to 0} v(t) \to v_g(t)$$

$$\Rightarrow \lim_{\omega_n\to 0}\max_{0<t<\infty}|v(t)| \to \max_{0<t<\infty}|v_g(t)|$$

$$\Rightarrow \lim_{\omega_n\to 0} S_d(\eta,\omega_n) \to \max_{0<t<\infty}|v_g(t)|\ \forall\eta$$

$$\eta = 0 \Rightarrow S_{pv}(0,\omega_n) = \max_{0<t<\infty}\left[\int_0^t \ddot{v}_g(\tau)\sin\omega_n(t-\tau)d\tau\right]$$

$$S_v(0,\omega_n) = \max_{0<t<\infty}\left[\int_0^t \ddot{v}_g(\tau)\cos\omega_n(t-\tau)d\tau\right]$$

$$\Rightarrow S_{pv}(0,\omega_n) \approx S_v(0,\omega_n) \quad \text{except} \quad \text{for very small } \omega_n.$$

$$\eta = 0 \Rightarrow S_a(0,\omega_n) = \max_{0<t<\infty}\left[\omega_n\int_0^t \ddot{v}_g(\tau)\sin\omega_n(t-\tau)d\tau\right]$$

$$S_{pa}(0,\omega_n) = \max_{0<t<\infty}\left[\omega_n\int_0^t \ddot{v}_g(\tau)\sin\omega_n(t-\tau)d\tau\right]$$

$$\Rightarrow S_a(0,\omega_n) = S_{pa}(0,\omega_n) = \omega S_{pv}(0,\omega_n)$$

$$\eta \neq 0 \;\&\; 0 < \eta < 0.2 \Rightarrow S_a(0,\omega_n) \cong \omega S_{pv}(0,\omega_n)$$

Tripartite plots

$S_d(\eta, \omega_n), S_{pv}(\eta, \omega_n)$ & $S_{pa}(\eta, \omega_n)$

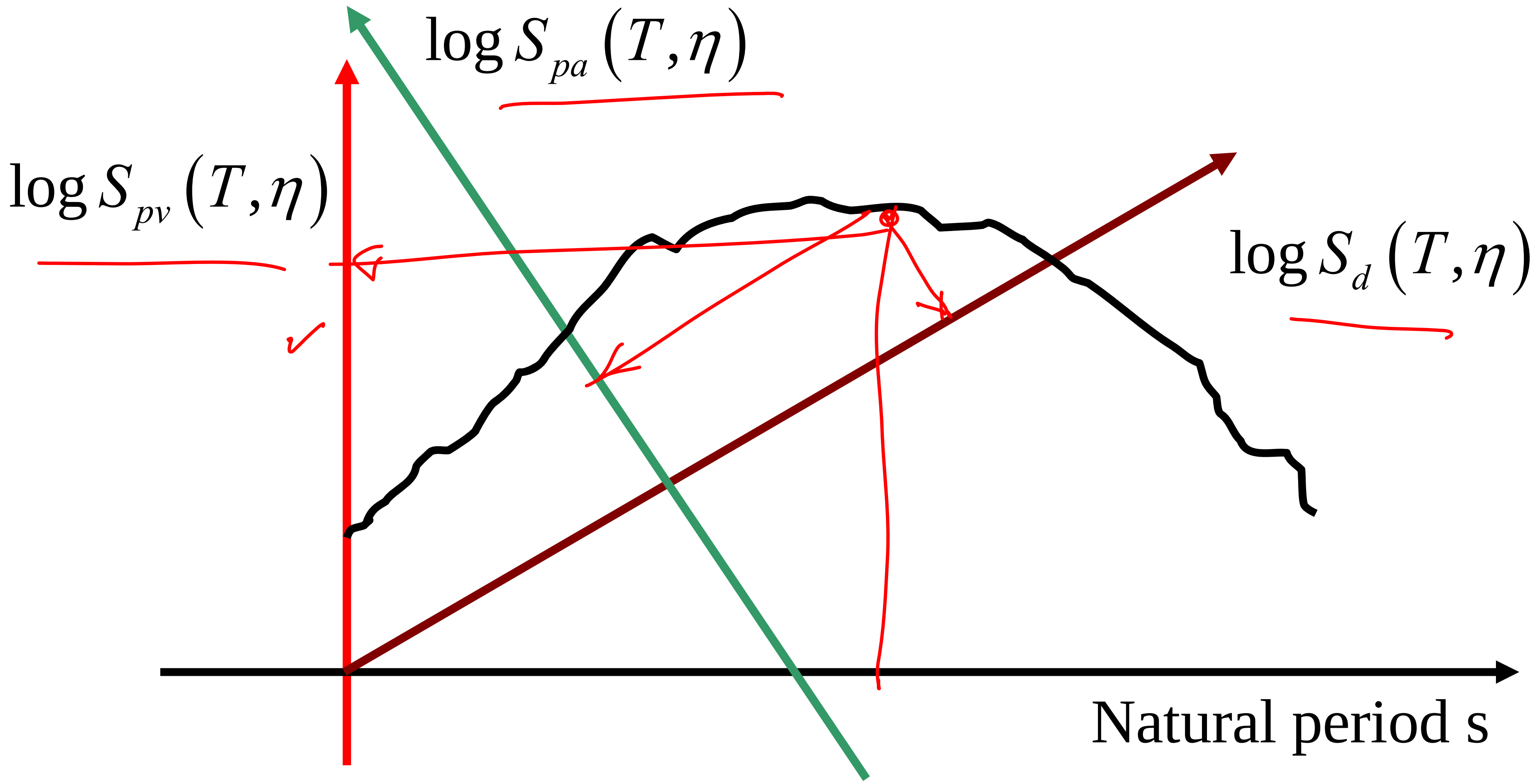
$$S_{pv}(\eta, \omega_n) = \omega_n S_d(\eta, \omega_n)$$

$$S_d(\eta, \omega_n) = \frac{1}{\omega_n} S_{pv}(\eta, \omega_n)$$

$$S_a(\eta, \omega_n) = \omega_n S_{pv}(\eta, \omega_n)$$

$$\log S_d(\eta, \omega_n) = \log S_{pv}(\eta, \omega_n) - \log \omega_n$$

$$\log S_a(\eta, \omega_n) = \log S_{pv}(\eta, \omega_n) + \log \omega_n$$



$\log S_{pa}(T,\eta)$
$\log S_{pv}(T,\eta)$
$\log S_{d}(T,\eta)$
Natural period s

Why so many response spectra?

- **Each response spectrum provides a physically meaningful quantity**

$S_d(\eta, \omega_n)$	**Peak deformation**
$S_{pv}(\eta, \omega_n)$	**Peak strain energy**
$S_{pa}(\eta, \omega_n)$	**Peak force in the spring** **Base shear** **Base moment**

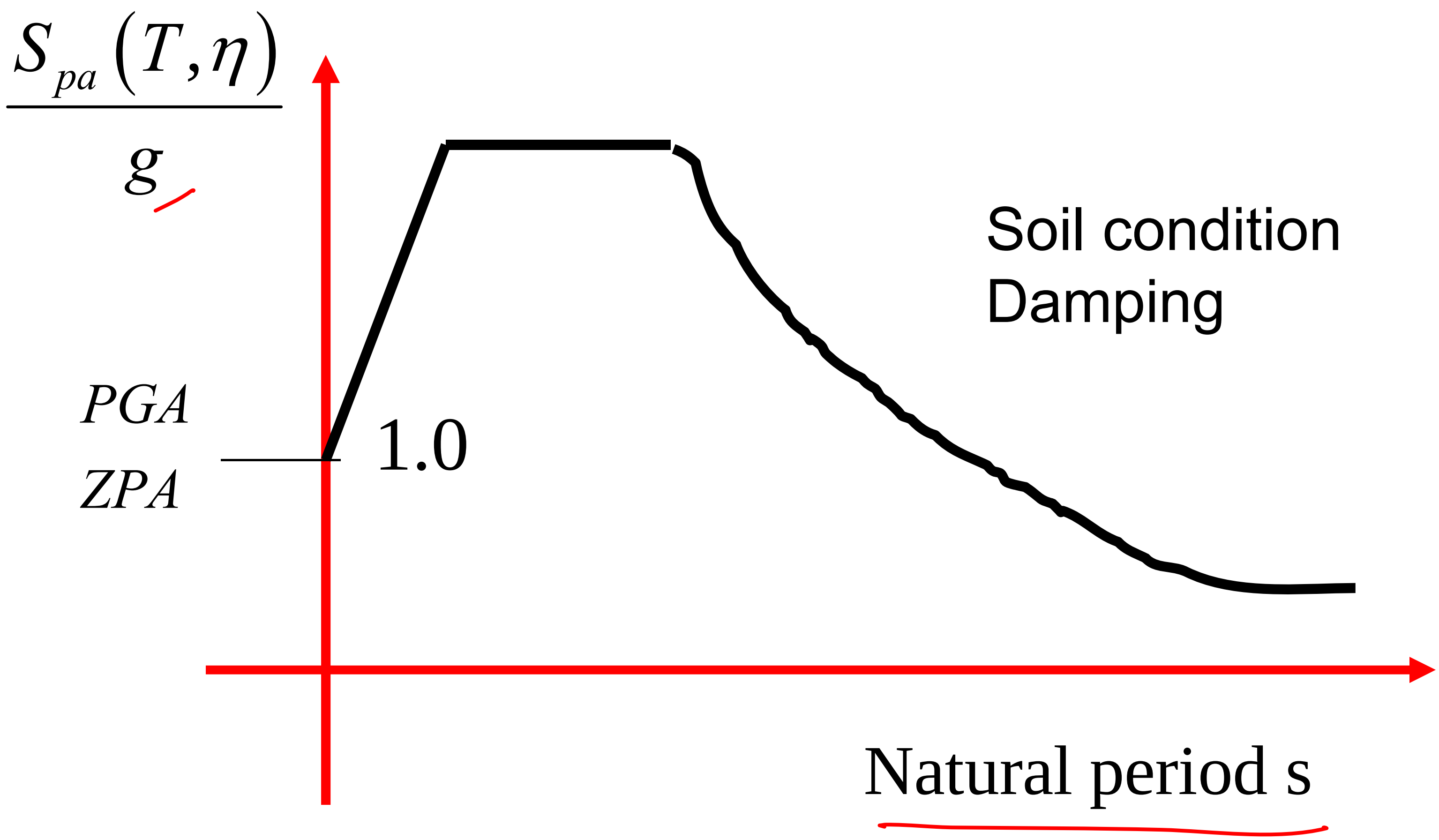
The shape of the response spectrum can be approximated more readily for design purposes with the aid of all three spectral quantities than any one of them taken alone.

Helps in understanding characteristics of response spectra.

Helps in constructing response spectra.

Helps in relating structural dynamics concepts to building codes.

Smooth design response spectra

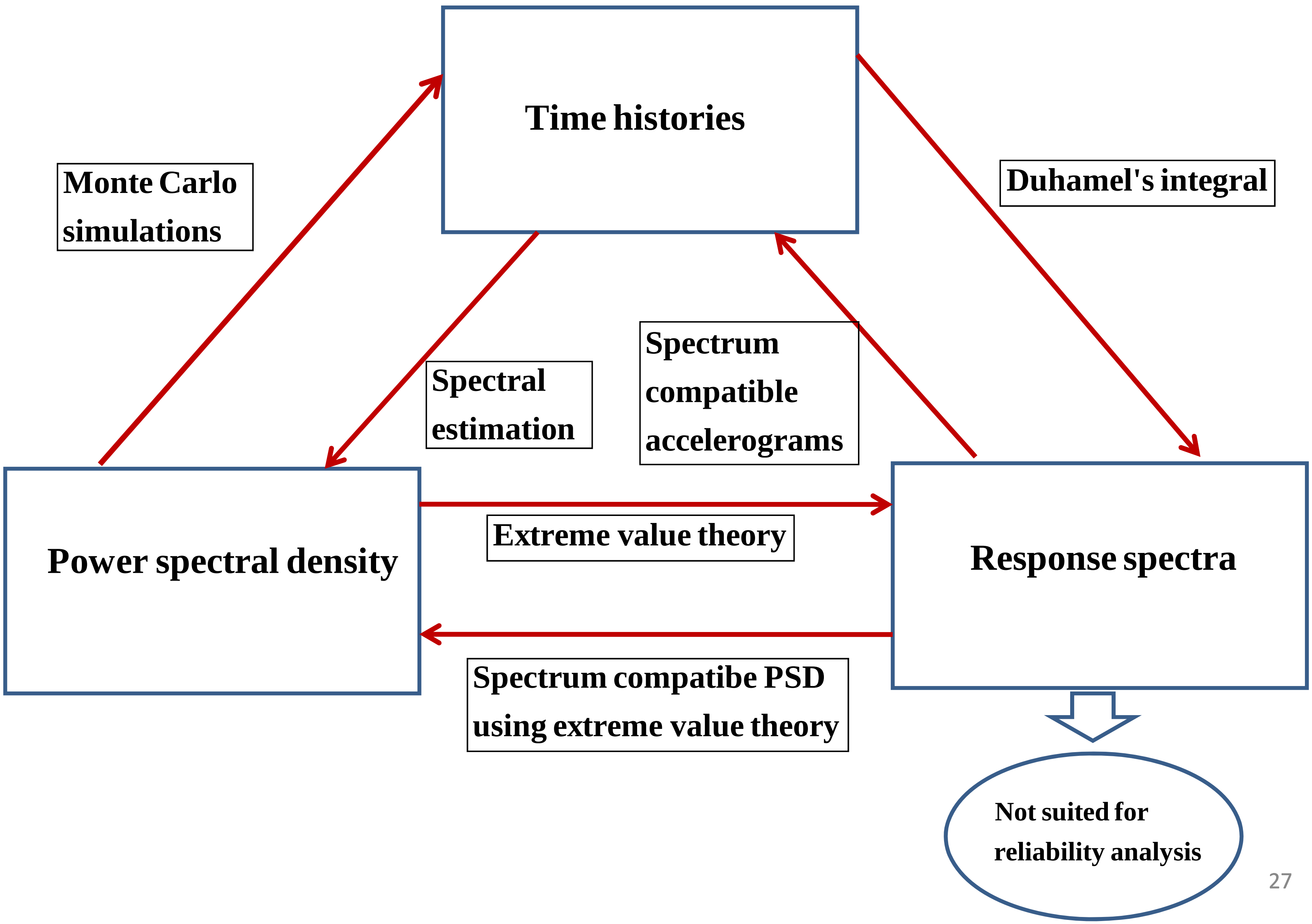


PGA : arrived at based on seismic hazard analysis

Factors that influence response spectrum at a given site

- Source mechanism
- Epicentral distance
- Focal depth
- Geological conditions
- Richter’s magnitude
- Soil condition
- Damping and stiffness of the system

Alternatives for earthquake load specification



How to generate a response spectrum compatible with a given PSD?

$$\ddot{x}+2\eta_n\omega_n\dot{x}+\omega_n^2 x=-\ddot{x}_g$$

$\ddot{x}_g(t)=$ zero mean, stationary, Gaussian random process;

$$\ddot{x}_g(t)\sim N\left[0,S_{gg}(\omega)\right]$$

$$X_m=\max_{0<t<T}\left|x(t)\right|$$

$$P_{X_m}(\alpha)=\exp\left[-\nu^+(\alpha)T\right]$$

$$\nu^+(\alpha)=\frac{\sigma_{\dot{x}}}{2\pi\sigma_x}\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right)$$

with $\sigma_x^2=\int_{-\infty}^{\infty}\left|H(\omega)\right|^2 S_{gg}(\omega)d\omega$ &

$$\sigma_{\dot{x}}^2=\int_{-\infty}^{\infty}\left|H(\omega)\right|^2\omega^2 S_{gg}(\omega)d\omega$$

For a given probability p, the corresponding α is given by

$$p = \exp\left[-\frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \exp\left(-\frac{\alpha^2}{2\sigma_x^2} \right) T \right]$$

$$\Rightarrow \alpha = \left\{ -2\sigma_x^2 \ln\left[-\frac{2\pi\sigma_x}{\sigma_{\dot{x}} T} \ln(p) \right] \right\}^{\frac{1}{2}}$$

Let $R(\omega_n, \eta_n)$ be the given pseudo-acceleration response spectrum.

We interpret $R(\omega_n, \eta_n)$ as the p - th percentile point.

$$\Rightarrow R(\omega_n, \eta_n) = \omega_n^2 \left\{ -2\sigma_x^2 \ln\left[-\frac{2\pi\sigma_x}{\sigma_{\dot{x}} T} \ln(p) \right] \right\}^{\frac{1}{2}} \quad \text{(typically } p\text{=84\%)}$$

How to generate a PSD compatible with a given response spectrum?

$$\ddot{x} + 2\eta_n \omega_n \dot{x} + \omega_n^2 x = -\ddot{x}_g$$

$\ddot{x}_g(t) =$ zero mean, stationary, Gaussian random process;

$$\ddot{x}_g(t) \sim N\left[0, S_{gg}(\omega)\right]$$

To a first approximation we assume

$$\sigma_x^2 = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{gg}(\omega) d\omega \approx$$

$$(2\eta_n \omega_n)\left|H(\omega_n)\right|^2 S_{gg}(\omega_n) = (2\eta_n \omega_n)\frac{1}{\left(2\eta_n \omega_n^2\right)^2} S_{gg}(\omega_n)$$

$$\& \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \approx \omega_n$$

$$\Rightarrow R^2\left(\omega_n,\eta_n\right)=\omega_n^4\left\{-2\frac{S_{gg}\left(\omega_n\right)}{2\eta_n\omega_n^3}\ln\left[-\frac{2\pi}{\omega_n T}\ln\left(p\right)\right]\right\}$$

To a first approximation we thus get

$$S_{gg}\left(\omega_n\right)=\frac{\eta_n R^2\left(\omega_n,\eta_n\right)}{\omega_n\left\{-\ln\left[-\frac{1}{\omega_n T}\ln\left(p\right)\right]\right\}}.$$

Steps

(1) Set iteration N=1

(2) Start with the initial guess on the PSD given by

$$S^N(\omega_n) = \frac{\eta_n R^2(\omega_n, \eta_n)}{\omega_n \left\{-\ln\left[-\frac{1}{\omega_n T}\ln(p)\right]\right\}}$$

(3)Evaluate σ_x^2 and $\sigma_{\dot{x}}^2$ using

$$\sigma_x^2 = \int_{-\infty}^{\infty} |H(\omega)|^2 S_{gg}^N(\omega) d\omega \,\&$$

$$\sigma_{\dot{x}}^2 = \int_{-\infty}^{\infty} |H(\omega)|^2 \omega^2 S_{gg}^N(\omega) d\omega$$

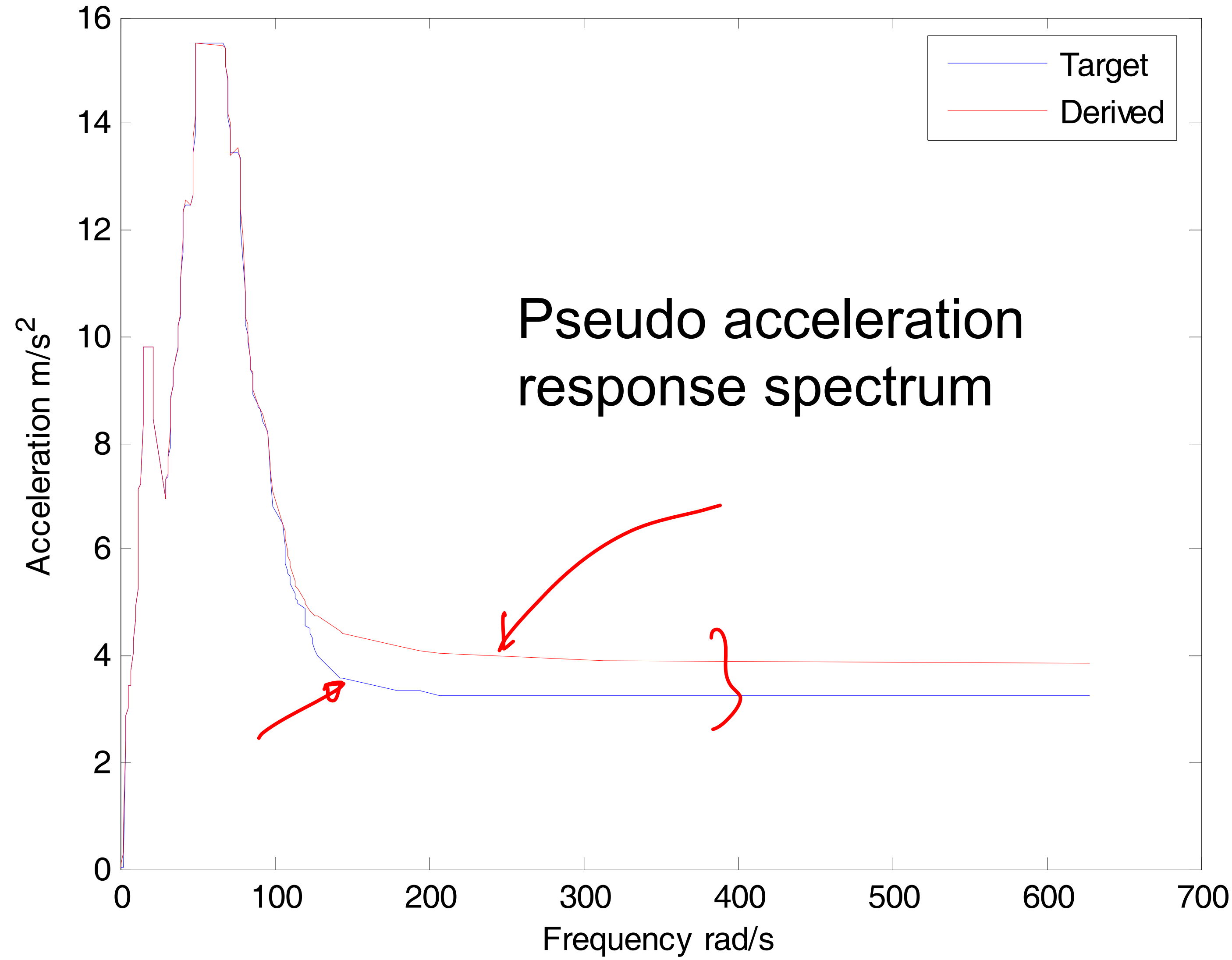
(4)Evaluate $R^N(\omega_n, \eta_n) = \omega_n^2 \left\{ -2\sigma_x^2 \ln\left[-\frac{2\pi\sigma_x}{\sigma_{\dot{x}} T} \ln(p) \right] \right\}^{\frac{1}{2}}$.

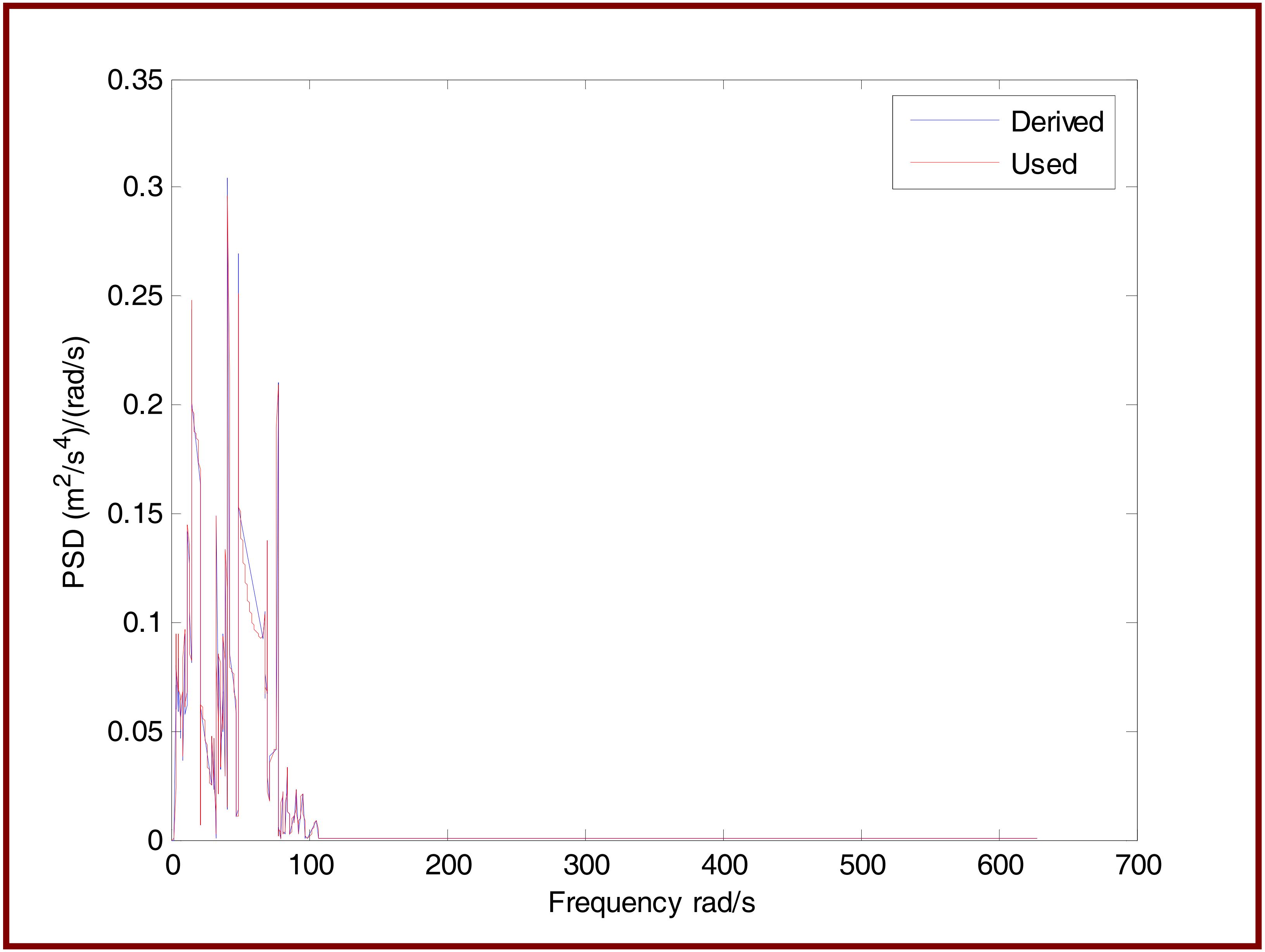
(5)Obtain an improved estimate of PSD using

$$S^{N+1}(\omega) = S^N(\omega)\left[\frac{R(\omega)}{R^N(\omega)}\right]^2$$

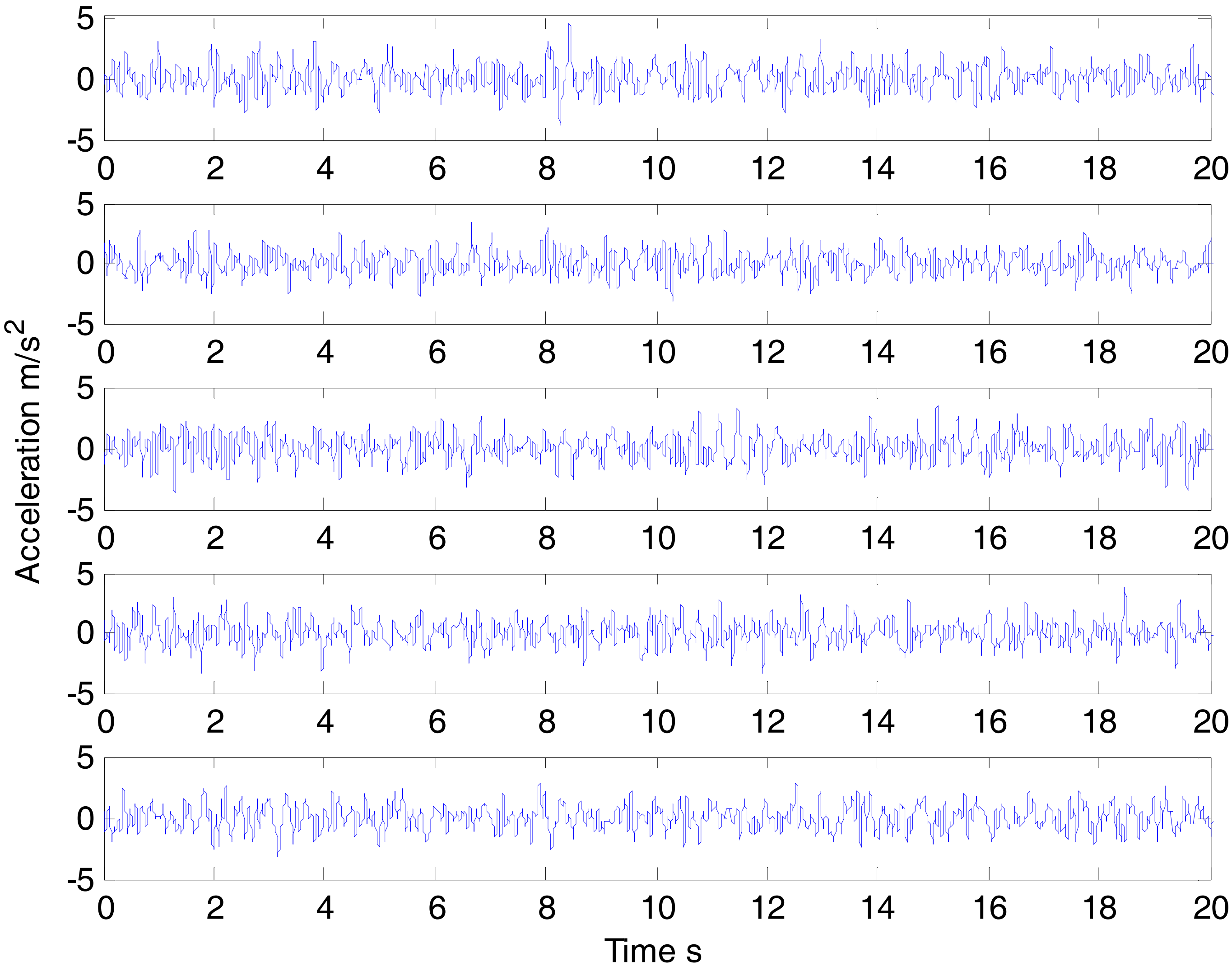
(6)Stop iterations if the PSD function has converged;

if not go to step 3.

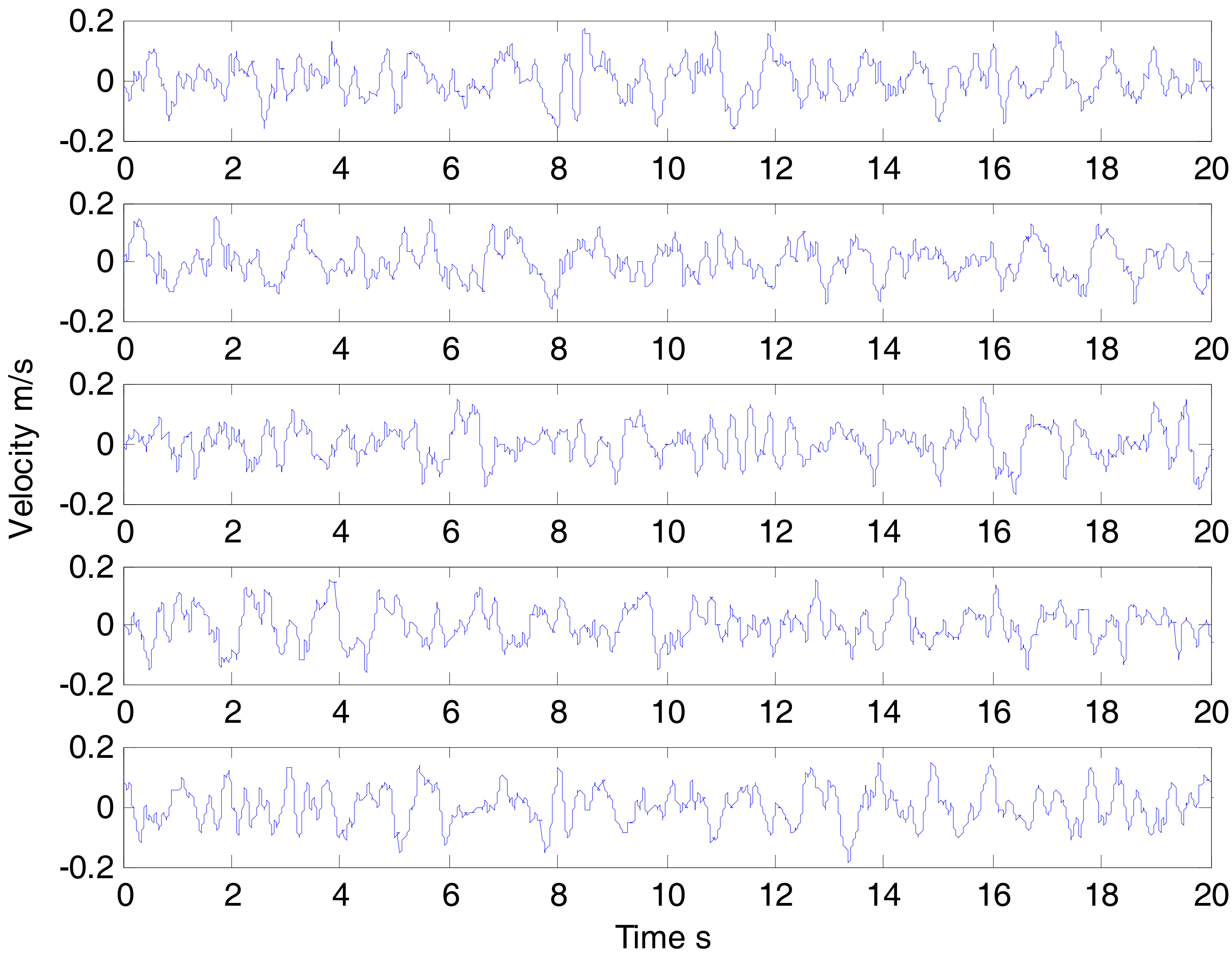

16
14
12
10
8
6
4
2
0
Acceleration m/s^2
0
100
200
300
400
500
600
700
Frequency rad/s
Target
Derived
Pseudo acceleration
response spectrum

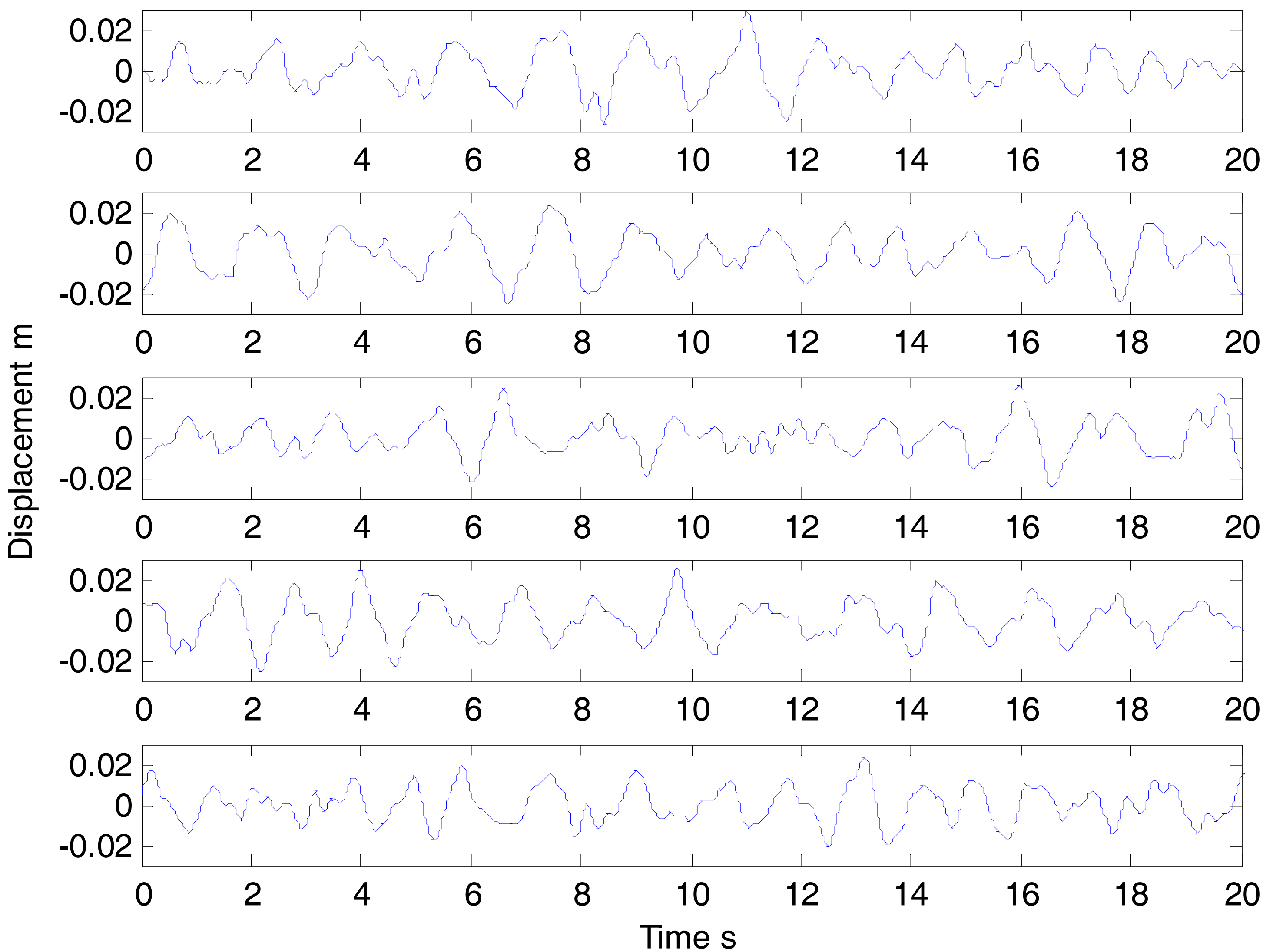


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
PSD (m²/s⁴)/(rad/s)
0
100
200
300
400
500
600
700
Frequency rad/s
Derived
Used



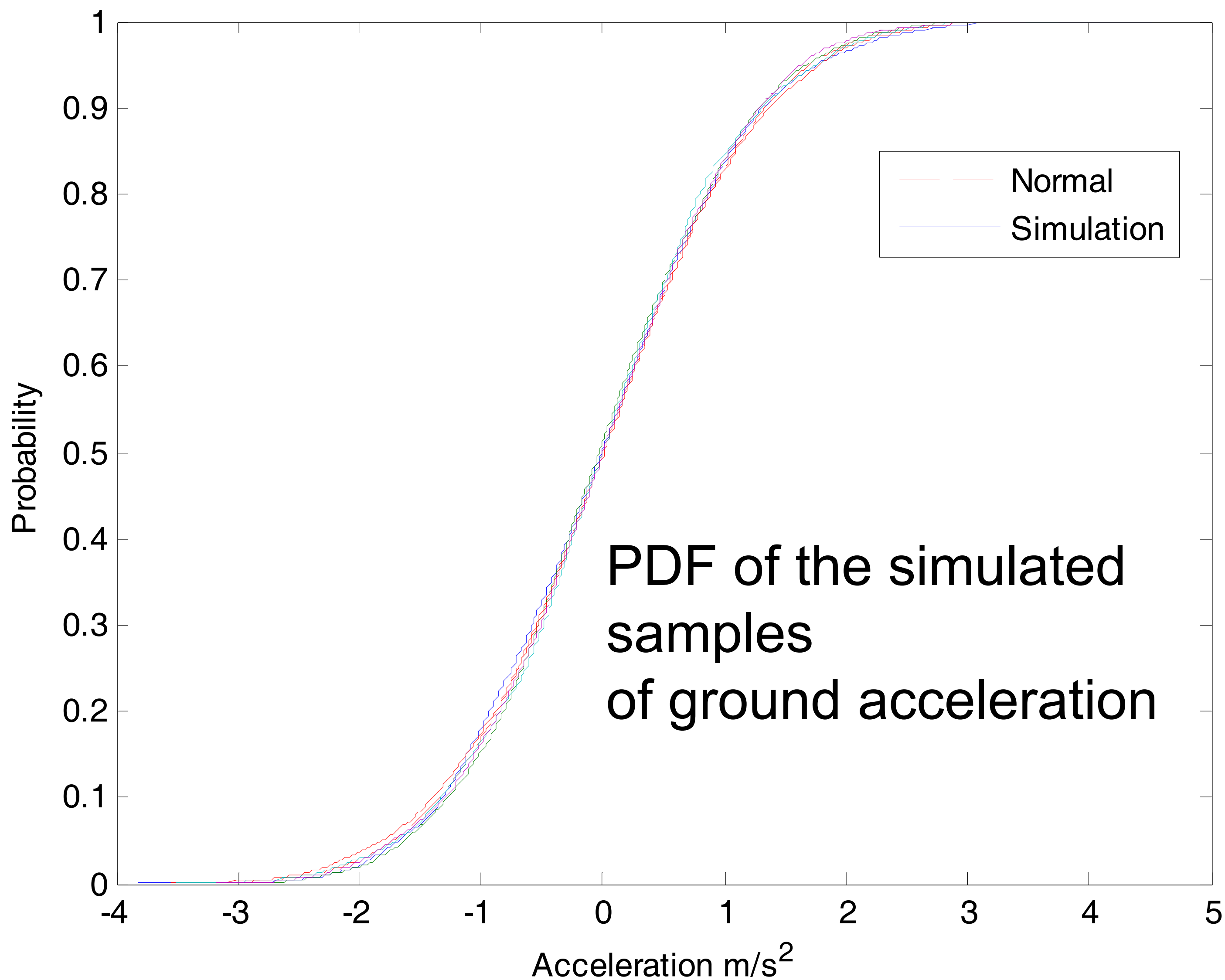
Acceleration m/s^2
Time s

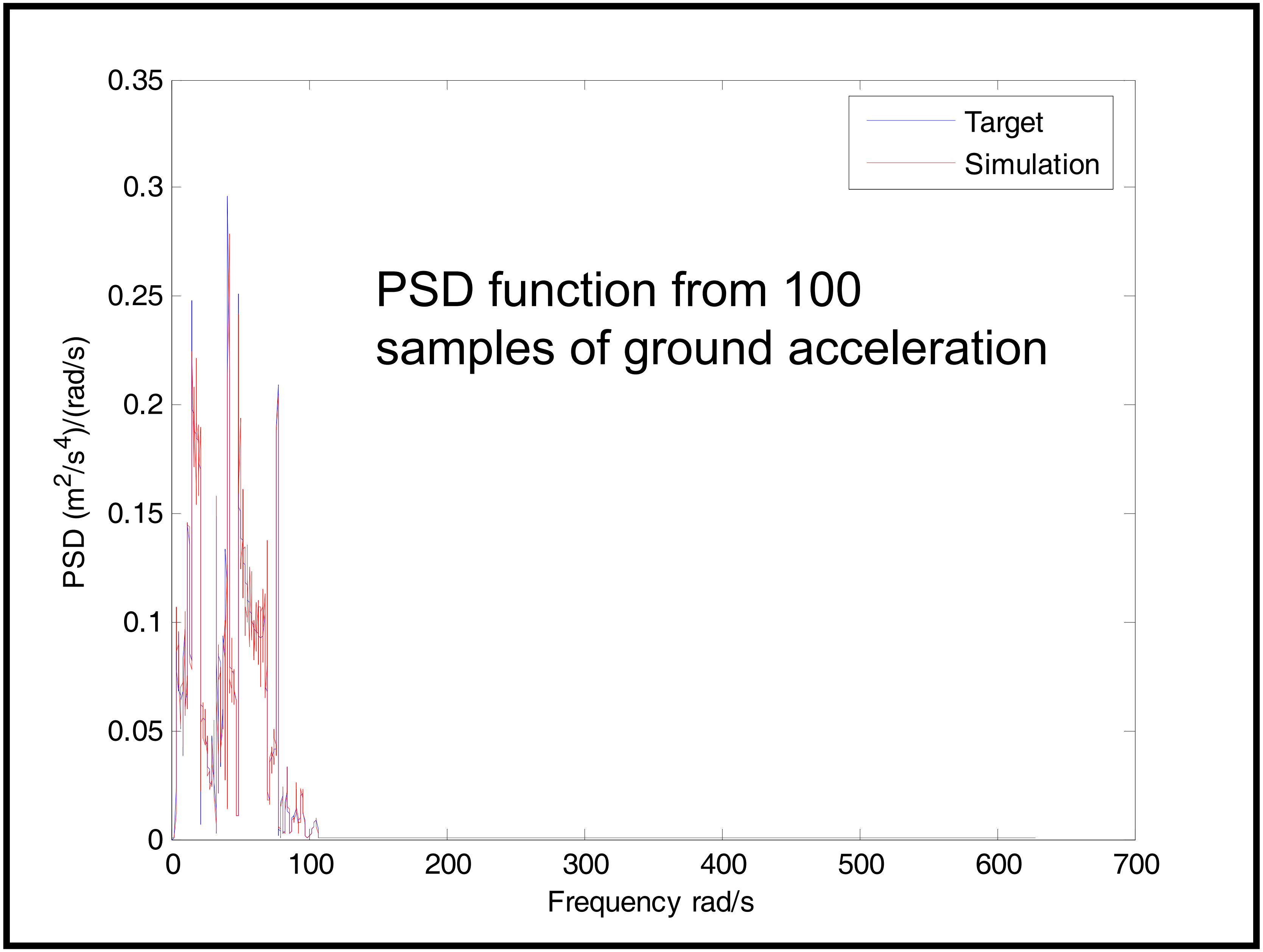

Velocity m/s
Time s
0.2
0
-0.2
0 2 4 6 8 10 12 14 16 18 20


0.02
0
-0.02
0 2 4 6 8 10 12 14 16 18 20
Displacement m
Time s

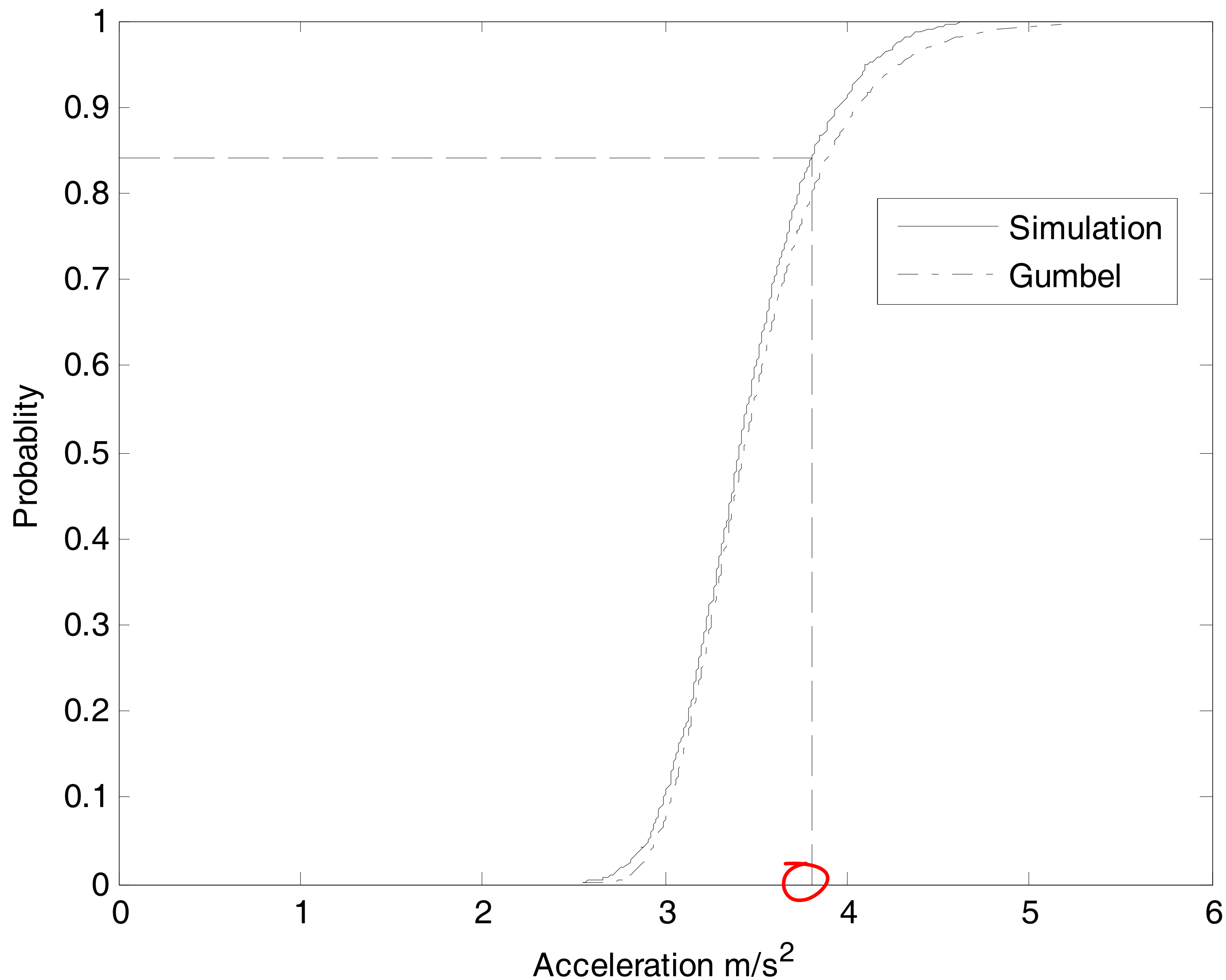
Variance(σ^2) = 1.0893 m^2/s^4 Std. deviation (σ) = 1.0437 m/s^2 3σ = 3.1311 m/s^2		
Time history no.	Maximum (m/s^2)	Minimum (m/s^2)
1.	4.5063	-3.8202
2.	3.4130	-3.4098
3.	3.3419	-3.5512
4.	3.7498	-3.5094
5.	3.4756	-3.1706

Zero Peak Acceleration (ZPA) =3.2530 m/s2


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Probability
-4
-3
-2
-1
0
1
2
3
4
5
Acceleration m/s2
Normal
Simulation
PDF of the simulated samples of ground acceleration



PSD function from 100
samples of ground acceleration
Target
Simulation
PSD (m^2/s^4)/(rad/s)
Frequency rad/s
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
100
200
300
400
500
600
700



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Probablity
0
1
2
3
4
5
6
Acceleration m/s2
Simulation
Gumbel

Response spectrum method provides maximum response of sdof systems as a function of natural frequencies and damping ratios.

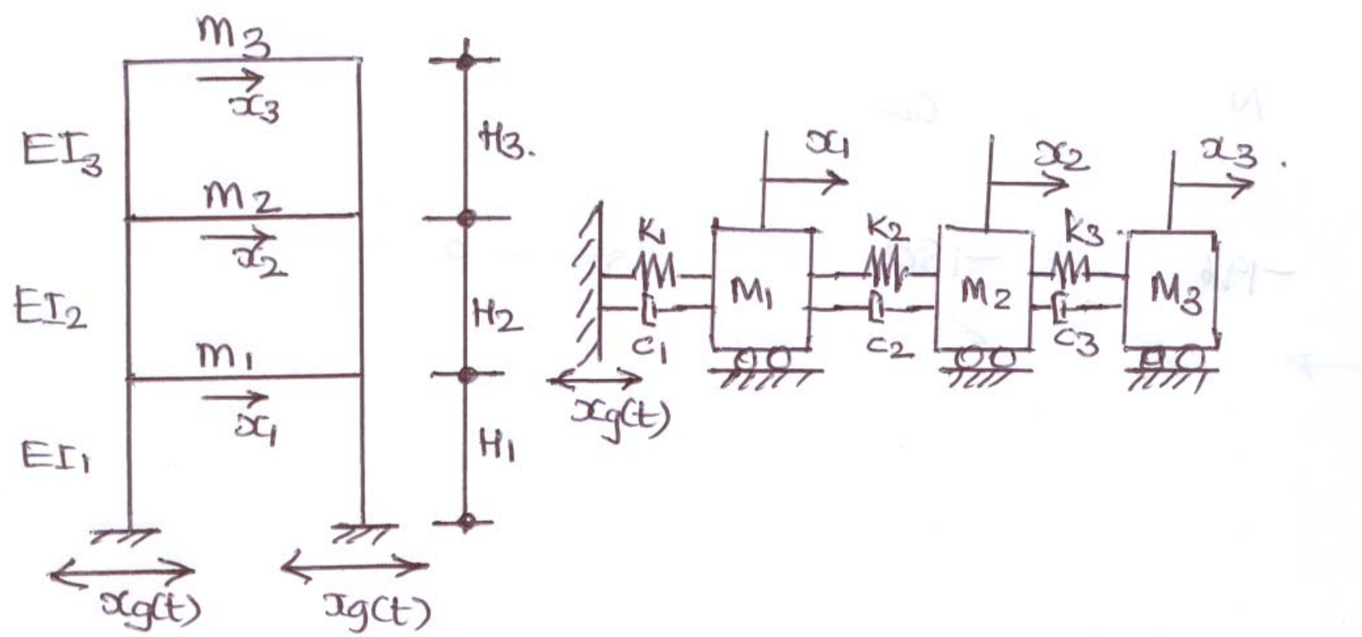
How to deal with mdof systems?

Hope :

MDOF systems can be decomposed into a set of uncoupled sdof oscillators.

How can this be taken advantage of?

Building frame under earthquake support motion



Equation of motion for total displacement

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} k_1 x_g(t) + c_1 \dot{x}_g(t) \\ 0 \\ 0 \end{Bmatrix}$$

Relative displacement $y_1 = x_1 - x_g \quad y_2 = x_2 - x_g \quad y_3 = x_3 - x_g$

Equation of motion for relative displacement

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{Bmatrix} \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = -\begin{Bmatrix} m_1 \ddot{x}_g \\ m_2 \ddot{x}_g \\ m_3 \ddot{x}_g \end{Bmatrix}$$

$$M\ddot{Y} + C\dot{Y} + KY = -M\Gamma \ddot{x}_g(t)$$

$$\Gamma = \{1 \quad 1 \quad 1\}^T$$

$$M\ddot{Y} + C\dot{Y} + KY = -M\Gamma\ddot{x}_g(t)$$

$$\Gamma = \{1 \quad 1 \quad 1\}^T$$

$$Y = \Phi Z$$

$$\Phi^T M \Phi = I \quad \Phi^T K \Phi = Diag\left[\omega_n^2\right] \quad \Phi^T C \Phi = Diag\left[2\eta_n\omega_n\right]$$

$$\Phi^T M \Phi \ddot{Z} + \Phi^T C \Phi \dot{Z} + \Phi^T K \Phi Z = -\Phi^T M \Gamma \ddot{x}_g$$

$$\Rightarrow \ddot{z}_n + 2\eta_n\omega_n\dot{z}_n + \omega_n^2 z_n = \Xi_n\ddot{x}_g \quad n = 1, 2, \cdots, n$$

$$\{\Xi\} = -\Phi^T M \Gamma$$

$$z_n(t) = \exp(-\eta_n\omega_n t)[A_n \cos\omega_{dn}t + B_n \sin\omega_{dn}t] + \int_0^t \Xi_n\ddot{x}_g(\tau)h_n(t-\tau)d\tau$$

Displacement: $y_k(t) = \sum_{n=1}^{N} \Phi_{kn} z_n(t)$

$$= \sum_{n=1}^{N} \Phi_{kn} \{\exp(-\eta_n \omega_n t)[A_n \cos \omega_{dn} t + B_n \sin \omega_{dn} t] + \int_0^t \Xi_n \ddot{x}_g(\tau) h_n(t-\tau) d\tau\}$$

Elastic forces: $F_s = KY = K\Phi Z = \omega^2 M \Phi Z$

Base shear: $V_0 = \sum_{n=1}^{N} F_{sn}$

Overturing moment: $M_o = \sum_{n=1}^{N} x_n F_{sn}$

$$M\ddot{V} + C\dot{V} + KV = -M\{1\}\ddot{v}_g(t)$$

$$V = \Phi Z \quad \Phi^t M \Phi = \bar{M} \quad \Phi^t K \Phi = \bar{K}$$

$$M_n \ddot{z}_n + C_n \dot{z}_n + K_n z_n = -\Phi^t M\{1\}\ddot{v}_g(t) = \Gamma_n \ddot{v}_g(t)$$

$$\{\Gamma_n\} = -\Phi^t M\{1\}$$

$$\Rightarrow \ddot{z}_n + 2\eta_n \omega_n \dot{z}_n + \omega_n^2 z_n = \frac{\Gamma_n}{M_n}\ddot{v}_g(t)$$

Γ_n = modal participation factor

or modal excitation factor

$$\ddot{v} + 2\eta\omega_n\dot{v} + \omega_n^2 v = \ddot{v}_g(t)$$

$$v(t) \cong \frac{1}{\omega_n}\int_0^t \exp[-\eta\omega_n(t-\tau)]\sin[\omega_n(t-\tau)]\ddot{v}_g(\tau)d\tau$$

$$z_n(t) = \frac{\Gamma_n}{M_n} v_n(t)$$

$$V = \Phi Z \Rightarrow V_k(t) = \sum_{n=1}^{p} \Phi_{kn} z_n(t) \Rightarrow V_k(t) = \sum_{n=1}^{p} \Phi_{kn} \frac{\Gamma_n}{M_n} v_n(t)$$

$$\text{Put} \quad \gamma_{kn} = \Phi_{kn}\frac{\Gamma_n}{M_n} \Rightarrow V_k(t) = \sum_{n=1}^{p} \gamma_{kn} v_n(t)$$

$$\max_{0<t<T} \gamma_{kn} v_n(t) = \gamma_{kn} S_D(\eta_n, \omega_n) \qquad \max_{0<t<T} |V_k(t)| = ?$$

Spring force

$$F_s(t) = KV(t) = K\Phi Z$$

Recall $K\varphi_n = \omega_n^2 M\varphi_n$

$$\Rightarrow F_s(t) = [\omega_n^2]M\Phi Z$$

$$\Rightarrow F_s(t) = [\omega_n^2]M\Phi\{\frac{\Gamma_n}{M_n}v_n(t)\}$$

$$\Rightarrow F_s(t) = M\Phi\{\frac{\Gamma_n}{M_n}a_n(t)\}$$

$$\max_{0<t<T}|F_s(t)| = ?$$

Base shear

$$V_B = \sum_{i=1}^{N} F_{si}(t) = \langle 1 \rangle F_s(t)$$

$$V_B = \langle 1 \rangle M \Phi \left\{ \frac{\Gamma_n}{M_n} a_n(t) \right\}$$

Recall $\Gamma = M\Phi^t\{1\} \Rightarrow \Gamma^t = \langle 1 \rangle M\Phi$

$$\Rightarrow V_B = \Gamma^t \left\{ \frac{\Gamma_n}{M_n} a_n(t) \right\} = \left\{ \Gamma_1 \; \Gamma_2 \; \cdots \; \Gamma_N \right\} \begin{Bmatrix} \frac{\Gamma_1}{M_1} a_1(t) \\ \frac{\Gamma_2}{M_2} a_2(t) \\ \vdots \\ \frac{\Gamma_N}{M_N} a_N(t) \end{Bmatrix}$$

$R(t) = \sum_{i=1}^{N} \psi_i S_i(t)$

$$\Rightarrow V_B(t) = \sum_{n=1}^{N} \frac{\Gamma_n^2}{M_n} a_n(t); \qquad \max_{0<t<T} \left| V_B(t) \right| = ?$$

Note : The quantity $\frac{\Gamma_n^2}{M_n}$ has units of mass and is called the effective modal mass.

Prove that $\sum_{n=1}^{N} \frac{\Gamma_n^2}{M_n} =$ total mass.

Total mass: $M_T = <1> M \{1\}$

$$\{1\} = \Phi Z \Rightarrow M\{1\} = M \Phi Z$$

$$\Rightarrow \Phi^t M \{1\} = \Phi^t M \Phi Z = \{M_n Z_n\}$$

$$\Rightarrow \{\Gamma_n\} = \{M_n Z_n\} \Rightarrow Z_n = \frac{\Gamma_n}{M_n} \Rightarrow \{1\} = \Phi \left\{ \frac{\Gamma_n}{M_n} \right\}$$

$$\Rightarrow M_T = <1> M \{1\} = <1> M \Phi \left\{ \frac{\Gamma_n}{M_n} \right\}$$

$$\Rightarrow M_T = < \Gamma_1 \Gamma_2 \cdots \Gamma_N > \left\{ \frac{\Gamma_n}{M_n} \right\} = \sum_{n=1}^{N} \frac{\Gamma_n^2}{M_n} \quad QED$$

Modal combination rules : what is the basic problem?

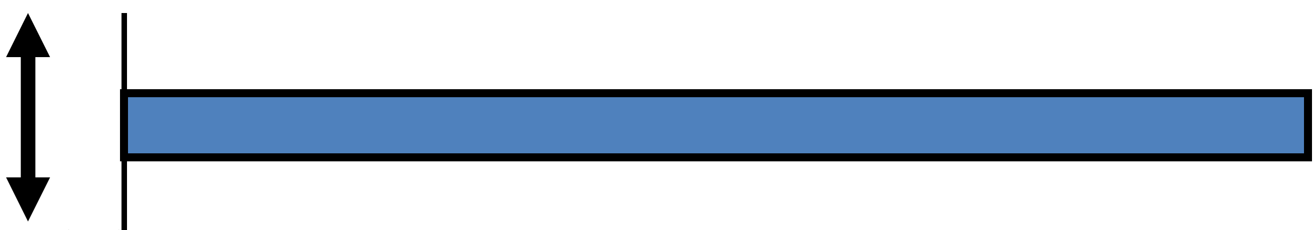
$$EIy^{iv} + m\ddot{y} + c\dot{y} = 0$$

$$y(0,t) = x_g(t); y'(0,t) = 0; EIy''(L,t) = 0; EIy'''(L,t) = 0$$

$$y(x,t) = z(x,t) + x_g(t)$$

$$\Rightarrow EIz^{iv} + m\ddot{z} + c\dot{z} = -m\ddot{x}_g(t)$$

$$z(0,t) = 0; z'(0,t) = 0; EIz''(L,t) = 0; EIz'''(L,t) = 0$$



$x_g(t)$

Eigenfunction expansion

$$z(x,t)=\sum_{n=1}^{\infty}a_n(t)\phi_n(x)$$

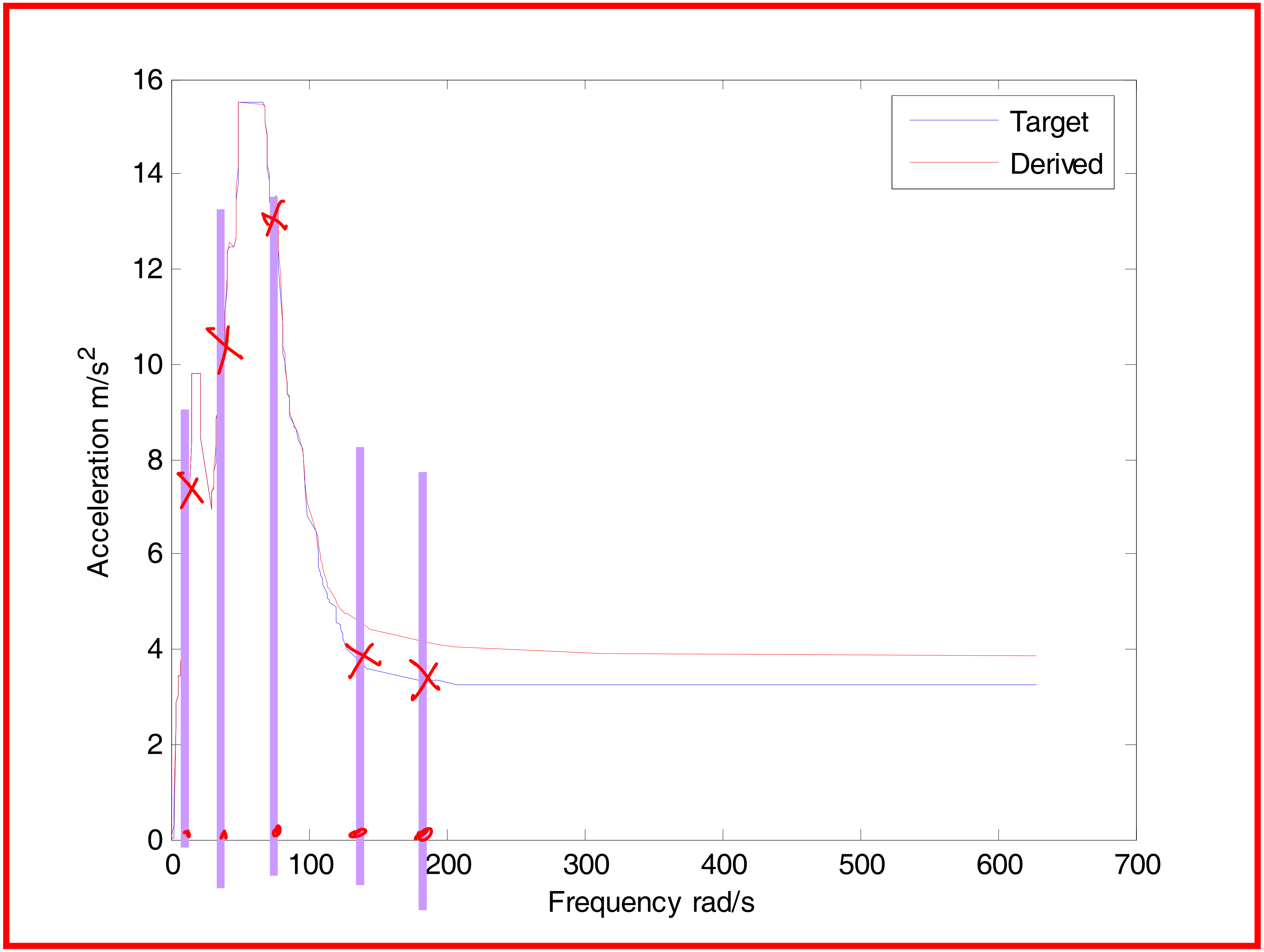
with $\ddot{a}_n+2\eta_n\omega_n\dot{a}_n+\omega_n^2 a_n=\gamma_n\ddot{x}_g(t); n=1,2,\cdots,\infty$

What we know based on response spectrum based analysis?

We know $\max_{0<t<T}\left|a_n(t)\right|; n=1,2,\cdots,\infty.$

What we wish to know?

$$\max_{0<t<T}\left|z(x,t)\right|=\max_{0<t<T}\left|\sum_{n=1}^{\infty}a_n(t)\phi_n(x)\right|$$



16
14
12
10
8
6
4
2
0
Acceleration m/s^2
0
100
200
300
400
500
600
700
Frequency rad/s
Target
Derived

Difficulty

$$\max_{0<t<T}\left|\sum_{n=1}^{\infty} a_n(t)\phi_n(x)\right| \neq \sum_{n=1}^{\infty}\phi_n(x)\max_{0<t<T}\left|a_n(t)\right|$$

Remarks

- The extrema of $a_n(t)$ for $n=1,2,\cdots,\infty$ are likely to occur at different times and they may have different signs.
- Response spectra do not contain information on times at which extrema occur nor do they store the signs of the extrema.
- $\max_{0<t<T}\left|\sum_{n=1}^{\infty} a_n(t)\phi_n(x)\right|$ can occur at a time instant t^* at which none of $a_n(t)$; $n=1,2,\cdots,\infty$ need to attain their respective extremum values.

Problem of modal combination rules

How best to obtain $\max_{0<t<T} \left| \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \right|$ in terms of $\max_{0<t<T} \left| a_n(t) \right|$, and the modal characteristics of the vibrating system?

Modal characteristics: natural frequencies, mode shapes, modal damping ratios, and participation factors.

These rules are formulated based on methods of random vibration analysis.