Stochastic Structural Dynamics

Lecture-31

Monte Carlo simulation approach-7

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Probability of failure

$$P_f = \int_{g(x)<0} p_X(x)\,dx = \int_{-\infty}^{\infty} I\left[g(x)\right] p_X(x)\,dx = \left\langle I\left[g(X)\right]\right\rangle$$

$$\Theta = \sum_{i=1}^{n} \frac{1}{n} I\left[g\left(X_i\right)\right]$$

$$\mathrm{Var}\left(\Theta\right) = \sum_{i=1}^{n} \frac{1}{n^2} P_F(1-P_F) = \frac{P_F(1-P_F)}{n}$$

Variance reduction

$$P_F = \int_{-\infty}^{\infty} F(x) h_V(x) dx; F(x) = \frac{I\left\{g(x)\le 0\right\} p_X(x)}{h_V(x)}. \Rightarrow P_F = \left\langle F(X)\right\rangle_h$$

$$h_V(v) = \frac{I\left[g(v)\le 0\right] p_X(v)}{P_F}$$

Variance reduction

(a) Variance reduction can be viewed as a means to use known information about the problem.

(b) If nothing is known about the problem, variance reduction is not achievable.

(c) At the other extreme, that is, when everything about the problem is known, variance reduces to zero but then simulation itself is not needed.

(d) How do we get information about the problem?

- Perform a few cycles of brute force simulations and learn something about the problem.

Sub-set simulations using Markov Chain Monte Carlo (MCMC)

- S K Au and J L Beck, 2001, Estimation of small failure probabilities in high dimension by subset simulation, Probabilistic Engineering Mechanics, 16, 263-277
- J S Liu, 2001, Monte Carlo strategies in scientific computing, Springer, NY.

Basic idea

- Small failure probability can be expressed as a product of larger conditional failure probabilities.
- These larger conditional failure probabilities can be estimated with lesser computational effort.
- The method is applicable to a wide class of problems

Subset simulation : motivation

$m\ddot{y} + c\dot{y} + ky + f\left[y, \dot{y}, t\right] = q(t); y(0), \dot{y}(0)$ specified

$q(t)$: zero mean, stationary Gaussian random process.

$$q(t) = \sum_{n=1}^{N_0} a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

where $a_n, b_n \sim N\left(0, \sigma_n^2\right)$, $a_n \perp a_k \forall n \neq k, b_n \perp b_k \forall n \neq k, \&$

$a_n \perp b_k \forall n, k \in [1, N]$; $\int_{\omega_n}^{\omega_{n+1}} S_{qq}(\omega) d\omega = 2\pi\sigma_n^2$

Let $z(t) = h\left[y(t), \dot{y}(t), t\right]$ a metric of system performance.

We are interested in estimating $\mathrm{P}\left[z(t) \leq z^* \forall t \in [0, T]\right]$.

Note: The system parameters could also be random (θ).

$$1 - P_F = P\left[z(t) \le z^* \forall t \in [0, T] \right]$$

$$= P\left[\max_{t \in [0,T]} z(t) \le z^* \right]$$

$$= P\left[Z_m(X) - z^* \le 0 \right]$$

$$= P\left[g(X) > 0 \right]$$

$$Z_m(X) = \max_{t \in [0,T]} z(t)$$

$$g(X) = z^* - Z_m(X)$$

$$X = \left\{ (a_n, b_n)_{n=1}^{N_0}, \theta, z^* \right\}$$

$$P_F = \int_{-\infty}^{\infty} I\left[g(x) \le 0 \right] p_X(x)\, dx$$

$$P_F = \int_{-\infty}^{\infty} I\left[g(x) \le 0\right] p_X(x)\,dx$$

$$\hat{P}_F = \frac{1}{N}\sum_{i=1}^{N} I\left[g\left(X^{(i)}\right) \le 0\right]$$

Remark

- $\hat{P}_F$ is an unbiased and consistent estimator of P_F with minimum variance. The optimal variance is given by

$$\sigma^2_{\hat{P}_F} = \frac{P_F(1-P_F)}{n}.$$

Subset simulations

$F = [g(X) \leq 0] =$ Failure event

Define

$F_1 \supset F_2 \supset \cdots \supset F_m = F$ such that

$$F_k = \bigcap_{i=1}^{k} F_i, k = 1, 2, \cdots, m$$

$$P_F = P(F_m) = P\left(\bigcap_{i=1}^{m} F_i\right)$$

$$= P\left(F_m \mid \bigcap_{i=1}^{m-1} F_i\right) P\left(\bigcap_{i=1}^{m-1} F_i\right)$$

$$= P(F_m \mid F_{m-1}) P\left(\bigcap_{i=1}^{m-1} F_i\right)$$

$$= P(F_1) \prod_{i=1}^{m-1} P(F_{i+1} \mid F_i)$$

Remarks

$$P_F = P\left(F_1\right)\prod_{i=1}^{m-1} P\left(F_{i+1} \mid F_i\right)$$

If F_i-s are configured such that $P\left(F_{i+1} \mid F_i\right)$ and $P\left(F_1\right)$ are much larger than P_F, then we will be able to estimate P_F in terms of product of "large" probabilities.

Suppose, $P_F \sim 10^{-6}$, then we could obtain an estimate of P_F as $10^{-6} \sim \left(10^{-1}\right)\times\left(10^{-1}\right)\times\left(10^{-1}\right)\times\left(10^{-1}\right)\times\left(10^{-1}\right)\times\left(10^{-1}\right)$.

Estimation of probability of failure of the order of 0.1 can be easily done using MCS because the failure events here are more frequent.

Remarks (continued)

$$P_F = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1} \mid F_i)$$

$P(F_1)$ can be estimated using a "brute force" Monte Carlo.

$P(F_{i+1} \mid F_i), i = 1, 2, \cdots, m-1$ can be estimated using MCMC.

Steps

1. Run a brute force Monte Carlo using, say, 200 samples. Evaluate the realization of the performance function at these 200 points. Rank order the these realizations and pick the 20^{th} ranked member and denote the performance function as g_1^*. Define a new performance function $g_1(X) = g(X) - g_1^*$.

   Define $F_1 = \left[g_1(X) \le 0 \right]$

   Clearly, $\hat{P}_{F_1} = \text{Estimate of } P\left[g_1(X) \le 0 \right] = 0.1.$

2. Store 20 members of X which lie in the failure region of $g_1(X)$.
3. Run 20 episodes of MCMC with each episode commencing from one of the 20 points in faiure region of $g_1(X)$. In each run continue with the simulaitons till 9 points are obtained in failure region of $g_1(X)$.

Steps (Continued)

4. This leads to 200 points in failure region of $g_1(X)$. Rank order the value of $g(X)$ at these 200 points and identify the 20$^{\text{th}}$ ranked member and denote it by g_2^*. Define a new performance function $g_2(X) = g(X) - g_2^*$.

Define $F_2 = \left[g_2(X) \leq 0 \right]$

Clearly, $\hat{P}_{F_2} = \text{Estimate of } P\left[g_2(X) \leq 0 \,|\, g_1(X) \leq 0 \right] = 0.1.$

5. Repeat this exercise till $F_m = F$ is reached.

6. Obtain the final probability of failure by using

$$P_F = P(F_1) \prod_{i=1}^{m-1} P(F_{i+1} \,|\, F_i)$$

Remarks

- The definition of F_i-s (as in the present illustrative explanation) ensures that P_{F_i}-s are all equal to 0.1.
- Estimates for sampling variance can be deduced.
- Choice of proposal density function:

In standard normal space, typically shifted normal pdf.

Example

Let $X_m = \max_{i \in [1,10]} |X_i|$

$\{X_i\}_{i=1}^{10}$: zero mean Gaussian random variables with covariance matrix given by

$\langle X_i^2 \rangle = 1 \forall i \in [1,10]$

$\langle X_i X_j \rangle = 0 \forall i, j \in [1,10]$ excepting

$\langle X_1 X_2 \rangle = 0.3; \langle X_4 X_5 \rangle = 0.4; \langle X_6 X_{10} \rangle = 0.2$

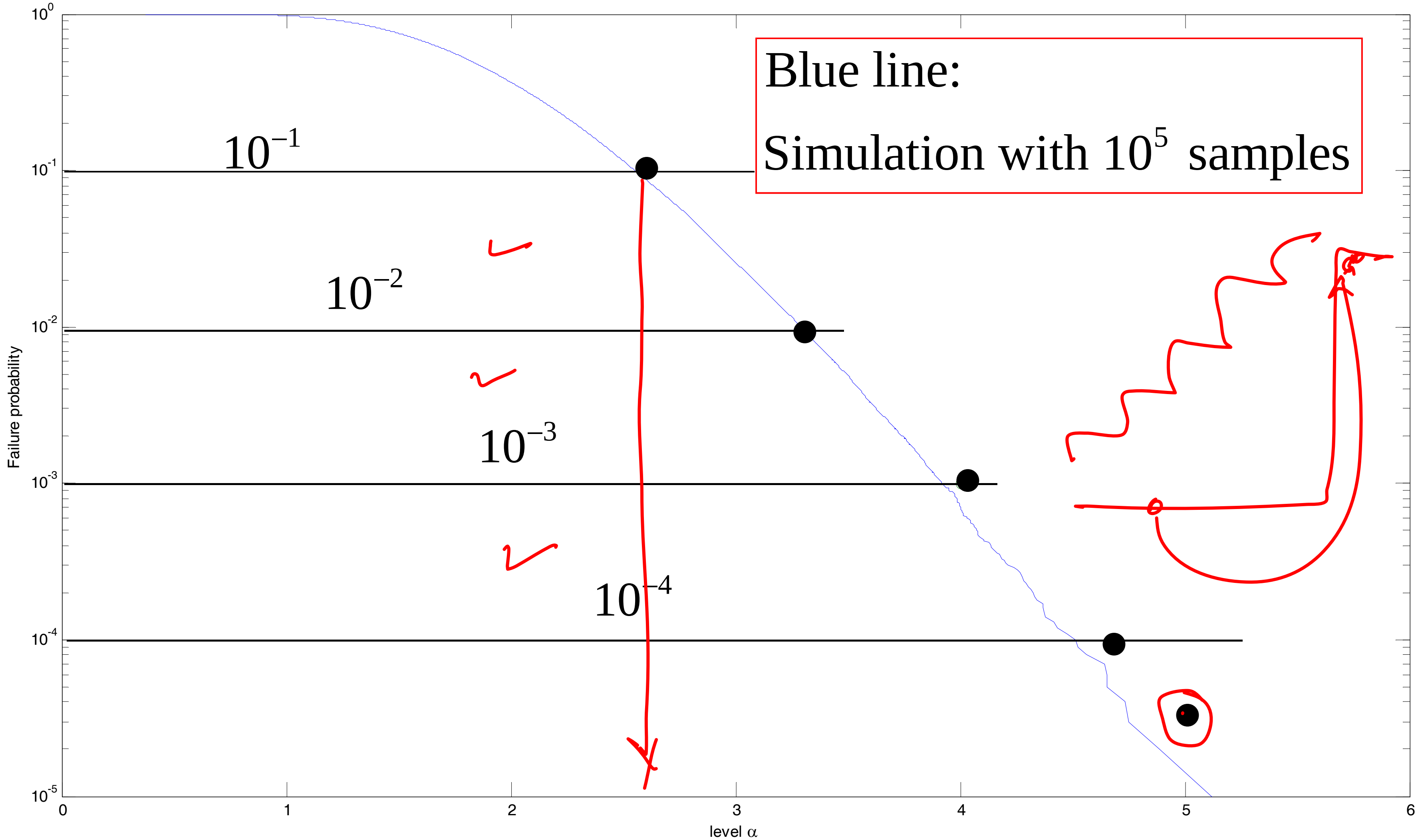
Question

Estimate $P_{X_m}(5)$ using subset simulations.

Number of samples: 200 at each subset

Proposal pdf $q\left(\bullet | X = \{x_i\}\right) \sim \mathrm{N}\left(\{x_i\}, I\right)$

$1-P_F$



Level α

Run	g_1^*	g_2^*	g_3^*	g_4^*	g_5^*	P_F
1	2.5388	1.5394	0.8291	0.1154	0.0	6.95E-05
2	2.4819	1.6062	0.8591	0.1662	0.0	5.75E-05
3	2.4454	1.4920	0.6616	0.0	-	1.00E-04
4	2.2659	1.2125	0.4420	0.0	-	2.65E-04

Example

$$X(t) = \sum_{n=1}^{25} a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

$$a_n \sim \text{iid N}\left[0, \sqrt{\frac{1}{2\pi}}\right];\ b_n \sim \text{iid N}\left[0, \sqrt{\frac{1}{2\pi}}\right]$$

$$a_n \perp b_k \forall n, k \in [1, 25]$$

$$\omega_n = 2\pi n$$

$$X_m = \max_{0<t<10} |X(t)|$$

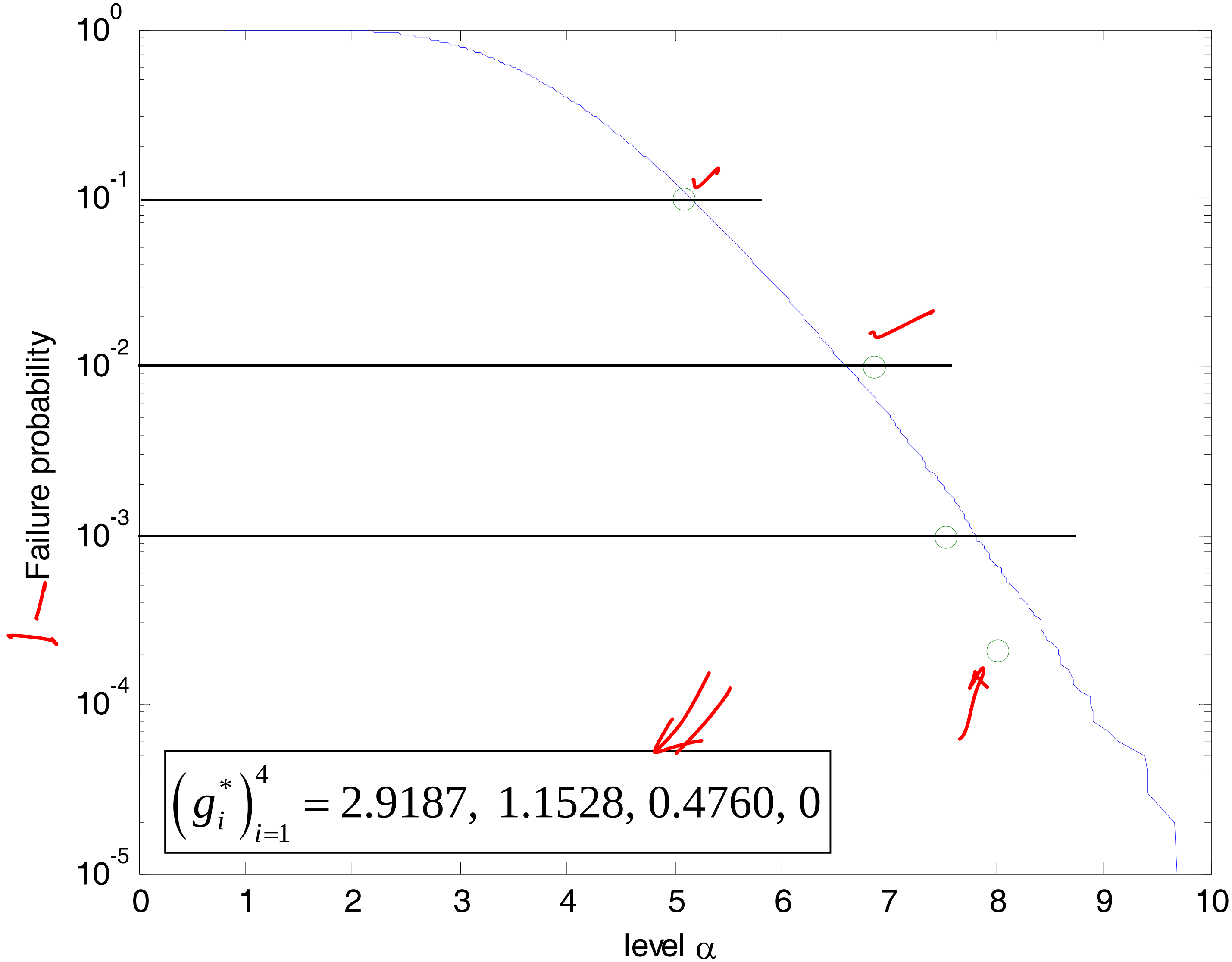
Question

What is $\mathrm{P}[X_m \leq 8]$?

Number of samples: 200 per subset

Proposal pdf $q\left(\bullet | X = \{x_i\}\right) \sim \mathrm{N}\left(\{x_i\}, 0.4I\right)$

Brute force Monte Carlo with 10^5 samples



10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
Failure probability
0 1 2 3 4 5 6 7 8 9 10
level α
$\left(g_i^*\right)_{i=1}^{4} = 2.9187,\ 1.1528,\ 0.4760,\ 0$

Series representation for random processes : revisited

Karhunen - Loeve expansion

Preliminaries

Let $f(t)$ be a deterministic function defined over $|t| \le \frac{T}{2}$.

Let us assume that $f(t)$ is well behaved in a suitable sense.

Consider a sequence of functions $\{\phi_n(t)\}_{n=1}^{\infty}$ which satisfy completeness requirements and the orthogonality conditions

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n(t)\phi_k(t)\,dt = \delta_{nk}$$

$\delta_{nk} = 1 \quad n = k$

$= 0 \quad n \neq k$

$f(t)$ can be expressed in terms of the convergent series

$$f(t) = \sum_{n=1}^{\infty} b_n \phi_n(t)$$

with a measure of error of representation given by

total meansquare error given by

$$\varepsilon = \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[f(t) - \sum_{n=1}^{\infty} b_n \phi_n(t) \right]^2 dt.$$

The constants b_n can be determined using the conditions

$$\frac{\partial \varepsilon}{\partial b_k} = 0; k = 1, 2, \cdots, \infty$$

$$\frac{\partial \varepsilon}{\partial b_k} = 0; \quad k = 1, 2, \cdots, \infty \Rightarrow$$

$$b_k = \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_k(t) f(t) dt; \quad k = 1, 2, \cdots, \infty$$

Question

Can similar formulation be developed for representing random process $x(t)$?

Reference :

H K Van Trees, 2001, Detection, estimation, and modulation theory, Vol. I, John Wiley, NY pp. 178-198.

Recall

Fourier representation of a Gaussian random process

Let $X(t)$ be a zero mean, stationary, Gaussian random process defined as

$$X(t)=\sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t;\ \ \omega_n = n\omega_0$$

$$a_n \sim N(0,\sigma_n), b_n \sim N(0,\sigma_n), \langle a_n a_k \rangle = 0 \forall n \neq k, \langle b_n b_k \rangle = 0 \forall n \neq k,$$
$$\langle a_n b_k \rangle = 0 \forall n,k = 1,2,\cdots,\infty$$

$$\Rightarrow \langle X(t) \rangle = \sum_{n=1}^{\infty} \left\{ \langle a_n \rangle \cos\omega_n t + \langle b_n \rangle \sin\omega_n t \right\} = 0$$

$$R_{XX}(\tau)=\sum_{n=1}^{\infty} \sigma_n^2 \cos\omega_n \tau;\ S_{XX}(\omega)=\sum_{n=1}^{\infty} S(\omega_n)\Delta\omega_n \delta(\omega-\omega_n)$$

$$\sigma_n^2 = \frac{S(\omega_n)\Delta\omega_n}{2\pi}$$

If $X(t)$ is mean square periodic we can use the Fourier representation with uncorrelated coefficients.

$$X(t) = \sum_{n=1}^{\infty} a_n \cos \omega_n t + b_n \sin \omega_n t; \;\; \omega_n = n\omega_0$$

Can we obtain series representations with uncorrelated coeffcients when $X(t)$ is not mean square periodic?

Or, more generally, when $X(t)$ is not even stationary?

How can we proceed if $X(t)$ is non-Gaussian?

Consider $x(t)$ to be a zero mean Gaussian random process

-not necessarily stationary

-not necessarily mean square periodic

Consider the series

$$x(t)=\sum_{n=1}^{\infty} a_n \phi_n(t); \quad |t|<\frac{T}{2}$$

Here $\{a_n\}_{n=1}^{\infty}$ are a set of random variables and $\{\phi_n(t)\}_{n=1}^{\infty}$ are a set of deterministic functions such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n(t)\phi_m(t)\,dt=\delta_{nm} \Rightarrow a_k=\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_k(t)\,x(t)\,dt$$

We would like to select $\{\phi_n(t)\}_{n=1}^{\infty}$ such that $\langle a_n a_k \rangle = \lambda_n \delta_{nk}$.

$$\langle a_n \rangle = 0 \Rightarrow \langle x(t) \rangle = \sum_{n=1}^{\infty} \langle a_n \rangle \phi_n(t) = 0$$

$$x(t)=\sum_{n=1}^{\infty}a_n\phi_n(t);\ \ |t|<\frac{T}{2}$$

$$\Rightarrow x(t_1)=\sum_{n=1}^{\infty}a_n\phi_n(t_1)$$

$$\Rightarrow \langle a_k x(t_1)\rangle=\sum_{n=1}^{\infty}\langle a_k a_n\rangle\phi_n(t_1)$$

If we impose the requirement $\langle a_k a_n\rangle=\lambda_k\delta_{nk}$ we get

$$\left\langle x(t_1)\int_{-\frac{T}{2}}^{\frac{T}{2}}\phi_k(t)x(t)\,dt\right\rangle=\lambda_k\phi_k(t_1)$$

$$\Rightarrow \int_{-\frac{T}{2}}^{\frac{T}{2}}\phi_k(t)\langle x(t_1)x(t)\rangle dt=\lambda_k\phi_k(t_1)$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} R_{xx}(\tau,t)\phi(\tau)\,d\tau = \lambda\phi(t);\ |t| < \frac{T}{2}$$

Kernel

$A\phi = \lambda\phi$

operator

Remarks

- This is an integral eigenvalue problem.
- The kernel $R_{xx}(\tau,t)$ is nonnegative definite.
- $\phi(t)$ = eigenfunction; λ=eigenvalue
- Exact solutions are available for a few cases.
- Numerical solutions can be obtained by using Galerkin's method

Example

$$R_{xx}(\tau) = P\exp(-\alpha|\tau|) \Leftrightarrow S_{xx}(\omega) = \frac{2\alpha P}{\omega^2 + \alpha^2}; -\infty < \omega < \infty$$

$$\int_{-T}^{T} P\exp(-\alpha|t-u|)\phi(u)du = \lambda\phi(t)$$

$$\int_{-T}^{t} P\exp[-\alpha(t-u)]\phi(u)du + \int_{t}^{T} P\exp[-\alpha(u-t)]\phi(u)du = \lambda\phi(t)$$

Differentiate with respect to t

$$\int_{-T}^{t} P(-\alpha)\exp[-\alpha(t-u)]\phi(u)du + P$$

$$+\int_{t}^{T} P\alpha\exp[-\alpha(u-t)]\phi(u)du - P = \lambda\dot{\phi}(t)$$

$$\int_{-T}^{t} P(-\alpha)\exp\left[-\alpha(t-u)\right]\phi(u)\,du + \int_{t}^{T} P\alpha\exp\left[-\alpha(u-t)\right]\phi(u)\,du = \lambda\dot{\phi}(t)$$

$$\lambda\dot{\phi}(t) = -P\alpha\exp(-\alpha t)\int_{-T}^{t}\exp(\alpha u)\phi(u)\,du + P\alpha\exp(\alpha t)\int_{t}^{T}\exp(-\alpha u)\phi(u)\,du$$

Differentiate with respect to *t*

$$\lambda\ddot{\phi}(t) = P\alpha^2\exp(-\alpha t)\int_{-T}^{t}\exp(\alpha u)\phi(u)\,du - P\alpha\exp(-\alpha t)\exp(\alpha t)\phi(t)$$

$$+P\alpha^2\exp(\alpha t)\int_{t}^{T}\exp(-\alpha u)\phi(u)\,du - P\alpha\exp(\alpha t)\exp(-\alpha t)\phi(t)$$

$$= -2P\alpha + P\alpha^2\int_{-T}^{T}\exp\left(-\beta\left|t-u\right|\right)\phi(u)\,du$$

$$= -2P\alpha\phi(t) + \alpha^2\lambda\phi(t)$$

$$\lambda\ddot{\phi}(t) = -2P\alpha\phi(t) + \alpha^2\lambda\phi(t)$$

$$\Rightarrow \ddot{\phi}(t) - \frac{\alpha^2\left(\lambda - \frac{2P}{\alpha}\right)}{\lambda}\phi(t) = 0$$

$$\Rightarrow \ddot{\phi}(t) + b^2\phi(t) = 0 \text{ with } b^2 = \frac{\alpha^2\left(\lambda - \frac{2P}{\alpha}\right)}{\lambda}$$

$$\phi(t) = c_1 \exp(ibt) + c_2 \exp(ibt)$$

It can be shown that (Exercise) b-s are roots of the equation

$$\left(\tan bT + \frac{b}{\alpha}\right)\left(\tan bT - \frac{\alpha}{b}\right) = 0$$

$$\lambda_i = \frac{2P\alpha}{\alpha^2 + b_i^2}; i = 1, 2, \cdots, \infty$$

$$\phi_i(t) = \frac{\cos(b_i t)}{\sqrt{T}\left(1 + \frac{\sin 2b_i T}{2b_i T}\right)^{0.5}} \quad (i \text{ odd})$$

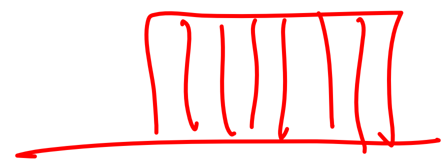
$$\phi_i(t) = \frac{\sin(b_i t)}{\sqrt{T}\left(1 - \frac{\sin 2b_i T}{2b_i T}\right)^{0.5}} \quad (i \text{ even}); \ |t| < T$$

Remark

The eigenfuncitons are sinusoids (as in Fourier series) but the frequencies are not uniformly spaced.

Example : Bandlimited white noise process

$$R_{xx}(t,u) = P\frac{\sin\alpha(t-u)}{\alpha(t-u)} \Leftrightarrow S_{xx}(\omega) = \frac{\pi P}{\alpha} \text{ for } |\omega| \le \alpha$$



$$\lambda\phi(t) = \int_{-\frac{T}{2}}^{\frac{T}{2}} P\frac{\sin\alpha(t-u)}{\alpha(t-u)}\phi(u)\,du$$

$$\Rightarrow$$

$$(1-t^2)\ddot{f}(t) - 2t\dot{f}(t) + (\mu - c^2t^2)f(t) = 0; |t| \le 1$$

$$c = \frac{\alpha T}{2}; \mu = \text{eigenvalue}$$

Eigenfunctions: angular prolate spheroidal functions

Example

Consider $x(t)$ to be the Brownian motion process

defined over $0 < t < T$.

$$\langle x(t) \rangle = 0; R_{xx}(t,u) = \sigma^2 \min(t,u)$$

$$\lambda\phi(t) = \int_0^T \sigma^2 \min(t,u)\phi(u)\,du = \sigma^2 \int_0^t u\phi(u)\,du + \sigma^2 t \int_t^T \phi(u)\,du$$

$$\lambda\dot{\phi}(t) = \sigma^2 t\phi(t) + \sigma^2 \int_t^T \phi(u)\,du - \sigma^2 t\phi(t) = \sigma^2 \int_t^T \phi(u)\,du$$

$$\lambda\ddot{\phi}(t) = -\sigma^2\phi(u) \Rightarrow \ddot{\phi} + \frac{\sigma^2}{\lambda}\phi = 0$$

$$\lambda_n = \frac{\sigma^2 T^2}{(n-0.5)^2 \pi^2}; \phi_n(t) = \left(\frac{2}{T}\right)^{\frac{1}{2}} \sin\left[(n-0.5)\frac{\pi t}{T}\right]; 0 < t < T$$

$n = 1, 2, \cdots, \infty$

Series represetation of partially specified non - Gaussian random processes using Nataf's transformation

Let $X(t)$ be a random process whose first order pdf and the ACF functions are available. No further informaiton about the process is available.

$X(t)$ need not be stationary.

How to represent $X(t)$ in a series?

Define $Y(t) = \dfrac{X(t) - m_X(t)}{\sigma_X(t)}$ so that

$\langle Y(t) \rangle = 0 \;\&\; \langle Y^2(t) \rangle = 1.$

Introduce a new random process Z(t) through the transformation

$\Phi\left[Z(t)\right] = P_Y\left[Y(t)\right]$

Here $\Phi[\bullet] =$ PDF of $\mathrm{N}(0,1)$ random variable.

$Z(t)$ is a zero mean Gaussian random process with an unknown covariance function.

$$\Phi\left[Z(t)\right] = P_Y\left[Y(t)\right]$$

$$Y(t) = P_Y^{-1}\left\{\Phi\left[Z(t)\right]\right\}$$

$$\left\langle Y(t_1)Y(t_2)\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_Y^{-1}\left\{\Phi\left[z_1\right]\right\}P_Y^{-1}\left\{\Phi\left[z_2\right]\right\}\phi\left(z_1, z_2; 0, \rho^*\right)dz_1dz_2$$

$$\rho_{XX}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_Y^{-1}\left\{\Phi\left[z_1\right]\right\}P_Y^{-1}\left\{\Phi\left[z_2\right]\right\}\phi\left[z_1, z_2; 0, \rho^*(t_1, t_2)\right]dz_1dz_2$$

Remarks

- RHS is known and $\rho^*(t_1, t_2)$ is not known
- $\left|\rho_{XX}(t_1, t_2)\right| \le 1 \& \left|\rho^*(t_1, t_2)\right| \le 1$
- $\phi\left[z_1, z_2; 0, \rho^*(t_1, t_2)\right] = 2-$dimensional Gaussian pdf

Solve the eigenvalue problem

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} R_{zz}(\tau,t)\phi(\tau)d\tau = \lambda\phi(t); |t| < \frac{T}{2}$$

by using numerical methods.

$$\Rightarrow Z(t) = \sum_{n=1}^{\infty} a_n\phi_n(t)$$

$$\Rightarrow$$

$$X(t) = m_X(t) + \sigma_X(t) P_Y^{-1}\Phi\left[\sum_{n=1}^{\infty} a_n\phi_n(t)\right]$$

Monte Carlo simulation of response of systems with spatially distributed random parameters

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]+P(t)\frac{\partial^2 y}{\partial x^2}+m(x)\frac{\partial^2 y}{\partial t^2}+c(x)\frac{\partial y}{\partial t}=f(x,t)+\xi(x,t)$$

$$y(x,0)=y_0(x)$$

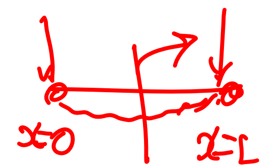
$$\dot{y}(x,0)=\dot{y}_0(x)$$

$$\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]_{x=0}=k_2\left[\frac{\partial y}{\partial x}\right]_{x=0};\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]_{x=l}=-k_4\left[\frac{\partial y}{\partial x}\right]_{x=l}$$

$$\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]_{x=0}=-k_1[y]_{x=0};\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]_{x=l}=-k_2[y]_{x=0}$$

Remarks

- 4^{th} order, 2-point stochastic boundary value problem
- Evolution of randomness in space and time
- Markovian properties in space is not possible



- Discretization of random fields is also essential
- Natural frequencies, modeshapes, Green's functions are all stochastic in nature.
- $EI(x)$, $m(x)$, and $c(x)$ cannot take negative values

  $\Rightarrow$ Gaussian models are not valid

  (especially when considering problem of reliability evaluation)

Approach: employ KL expansions for

$EI(x),\ m(x),$ and $c(x)$.

Note: These processes are non-Gaussian in nature.

Assume that they are independent.

Discretization using KL-expansion and Nataf's transformation

$$EI(x) = m_{EI}(x) + \sigma_{EI}(x) P_{Y_1}^{-1} \Phi \underbrace{\left[\sum_{n=1}^{N_1} a_n \phi_n(x) \right]}_{\text{KL expansion}}$$

$$m(x) = m_m(x) + \sigma_m(x) P_{Y_2}^{-1} \Phi \left[\sum_{n=1}^{N_2} b_n \varphi_n(x) \right]$$

$$c(x) = m_c(x) + \sigma_c(x) P_{Y_3}^{-1} \Phi \left[\sum_{n=1}^{N_3} d_n \psi_n(x) \right]$$

$$y(x,t)=\sum_{n=1}^{N}\alpha_n(t)\Psi_n(x)$$

$\{\Psi_n(x)\}_{n=1}^{N}$ = modeshapes of the system with deterministic properties

$\Rightarrow$ Use method of wieghted residues (e.g., Galerkin's method) to get

$\mathrm{M}(\theta)\ddot{\alpha}+C(\theta)\dot{\alpha}+K(\theta)\alpha=F(t)$ along with associated ics.

- M, C, K = random matrices (fully populated)
- Starting point for applicaiton of methods such as the subset simulations

Summary

- Simulations of random variables and random processes
- Fourier and KL expansions
- Introduction to statistical inference and estimation theory
- Introduction to calculus of Brownian motion and implications on numerical simulations
- Estimation of low probability of failure
- Variance reduction: adaptive procedures
- Discretization of spatially varying random quantities.