Stochastic Structural Dynamics

Lecture-30

Monte Carlo simulation approach-6

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Recall

$$dx(t) = a[x(t),t]dt + b[x(t),t]dB(t); x(t_0) = x_0$$

Sizes:

$$x(t) \sim d \times 1; dB(t) \sim m \times 1; a \sim d \times 1; b \sim d \times m$$

Time discretization:

$0 = t_0 < t_1 < \cdots < t_N = T$ with $\Delta = T / N$.

Notation: $Y_k(n) = x_k(t_n)$

$$\Delta x^2 \to 0 \text{ as } \Delta t \to 0$$

1.5 order Strong Taylor scheme

$$Y_k(n+1) = Y_k(n) + a_k(n)\Delta + b_k(n)\Delta W + \frac{1}{2}L^1 b_k(n)\left\{(\Delta W)^2 - \Delta\right\}$$

$$+L^1 a_k(n)\Delta Z + L^0 b_k(n)\left\{\Delta W \Delta - \Delta Z\right\} + \frac{1}{2}L^0 a_k(n)\Delta^2 + \frac{1}{2}L^1 L^1 b_k(n)\left\{\frac{1}{3}(\Delta W)^2 - \Delta\right\}\Delta W$$

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_k \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k=1}^{d}\sum_{l=1}^{d} b_k b_l \frac{\partial^2}{\partial x_k \partial x_l}; L^1 = \sum_{k=1}^{d} b_k \frac{\partial}{\partial x_k}$$

$$\begin{Bmatrix} \Delta W \\ \Delta Z \end{Bmatrix} = \begin{bmatrix} \sqrt{\Delta} & 0 \\ 0.5\Delta^{1.5} & \dfrac{0.5\Delta^{1.5}}{\sqrt{3}} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}; \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \equiv N\left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Remark

More general versions of the integration schemes are available: see P E Kloeden and E Platen, 1992, Numerical solution of stochastic differential equations, Springer – Verlag, Berlin

Bouc's oscillator under white noise

$$\ddot{x} + 2\eta\omega\dot{x} + \alpha x + (1-\alpha)z = f(t)$$

$$\dot{z} = -\gamma\,|\,\dot{x}\,|\,z\,|\,z\,|^{n-1} - \beta\dot{x}\,|\,z\,|^{n} + A\dot{x}$$

$$x(0) = x_0;\ \dot{x}(0) = \dot{x}_0;\ z(0) = z_0$$

$$< f(t) >= 0; < f(t_1) f(t_2) >= \sigma^2 \delta(t_1 - t_2)$$

$$dx_1(t) = x_2 dt$$

$$dx_2(t) = \left(-2\eta\omega x_2 - \alpha x_1 - (1-\alpha)x_3\right)dt + \sigma dw(t)$$

$$dx_3(t) = \left(-\gamma\,|\,x_2\,|\,x_3\,|\,x_3\,|^{n-1} - \beta x_2\,|\,x_3\,|^{n} + Ax_2\right)dt$$

$$a_1 = x_2$$

$$a_2 = \left(-2\eta\omega x_2 - \alpha x_1 - (1-\alpha)x_3\right)$$

$$a_3 = \left(-\gamma \,|\, x_2 \,|\, x_3 \,|\, x_3 \,|^{n-1} - \beta x_2 \,|\, x_3 \,|^{n} + A x_2\right)$$

$$b_1 = 0; b_2 = \sigma; b_3 = 0$$

$$L^1 a_1 = \sigma; L^1 a_2 = -2\eta\omega; L^1 a_3 = \sigma\left\{-\mathrm{sgn}(x_2) x_3 \,|\, x_3 \,|^{n-1} - \beta \,|\, x_3 \,|^{n} + A\right\}$$

$$L^0 a_1 = a_2; L^0 a_2 = -\alpha a_1 - a_2 2\eta\omega + a_3(1-\alpha)$$

$$L^0 a_3 = a_2\left\{-\gamma\,\mathrm{sgn}(x_2) x_3 \,|\, x_3 \,|^{n-1} - \beta \,|\, x_3 \,|^{n} + A\right\} +$$

$$a_3\left\{-\gamma \,|\, x_2 \,||\, x_3 \,|^{n-1} - \gamma \,|\, x_2 \,|\, x_3 (n-1) \,|\, x_3 \,|^{n-2} \,\mathrm{sgn}(x_3) - \beta x_2 n \,|\, x_3 \,|^{n-1} \,\mathrm{sgn}(x_3)\right\}$$

1.5 order Strong Taylor scheme

$$Y_k(n+1) = Y_k(n) + a_k(n)\Delta + b_k(n)\Delta W + \frac{1}{2}L^1 b_k(n)\left\{(\Delta W)^2 - \Delta\right\}$$

$$+L^1 a_k(n)\Delta Z + L^0 b_k(n)\{\Delta W\Delta - \Delta Z\} + \frac{1}{2}L^0 a_k(n)\Delta^2 + \frac{1}{2}L^1 L^1 b_k(n)\left\{\frac{1}{3}(\Delta W)^2 - \Delta\right\}\Delta W$$

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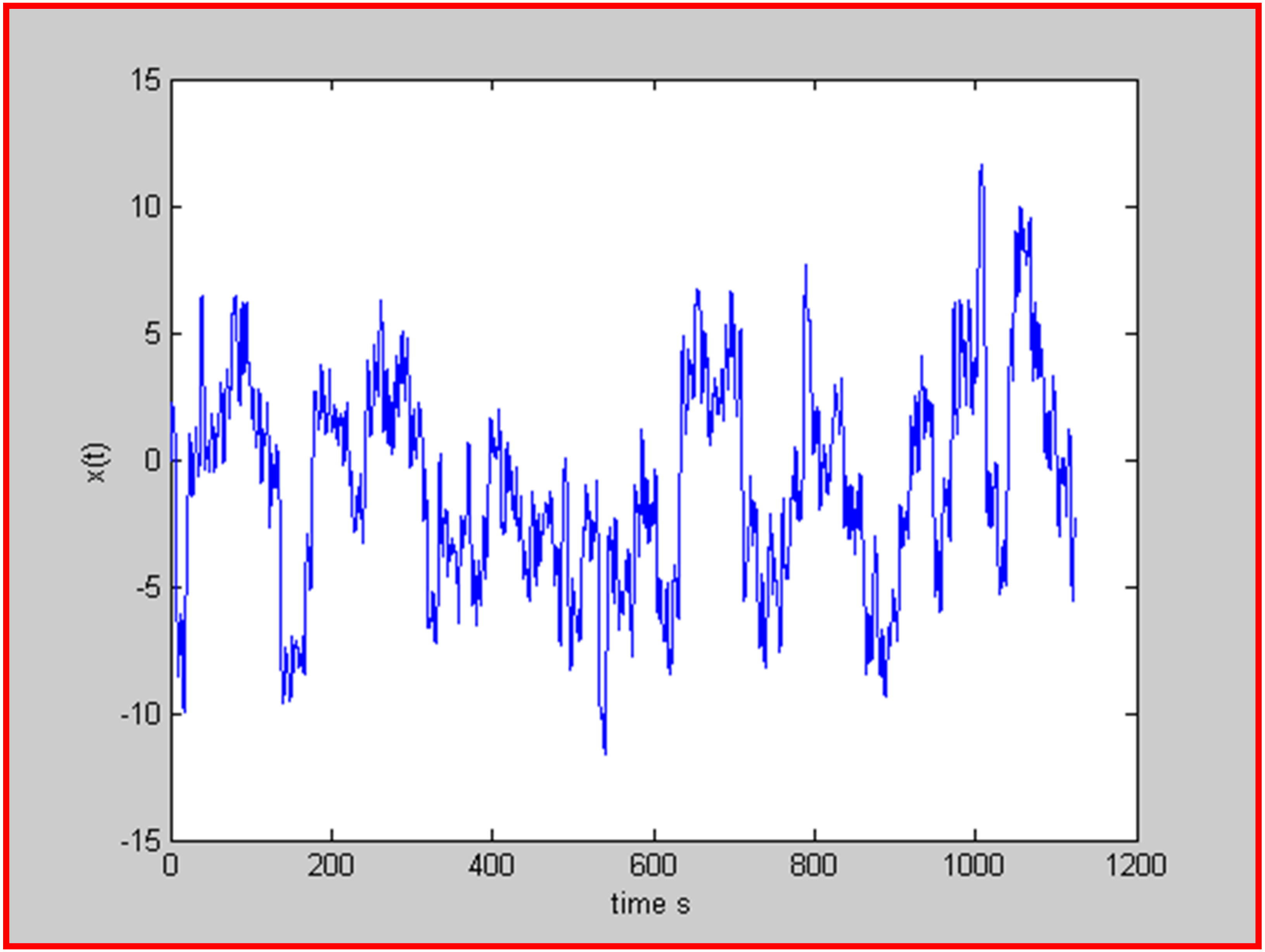
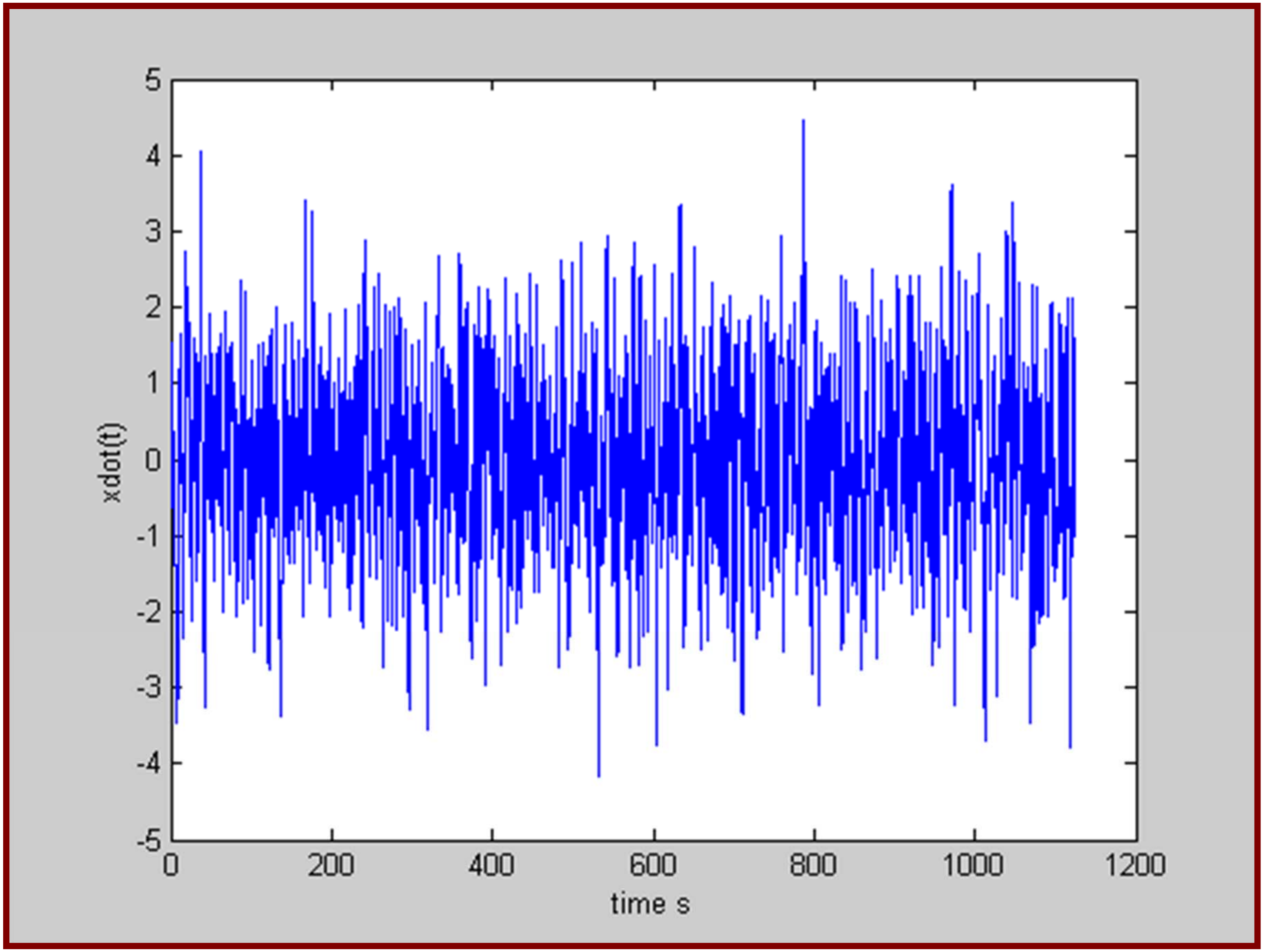
Numerical values

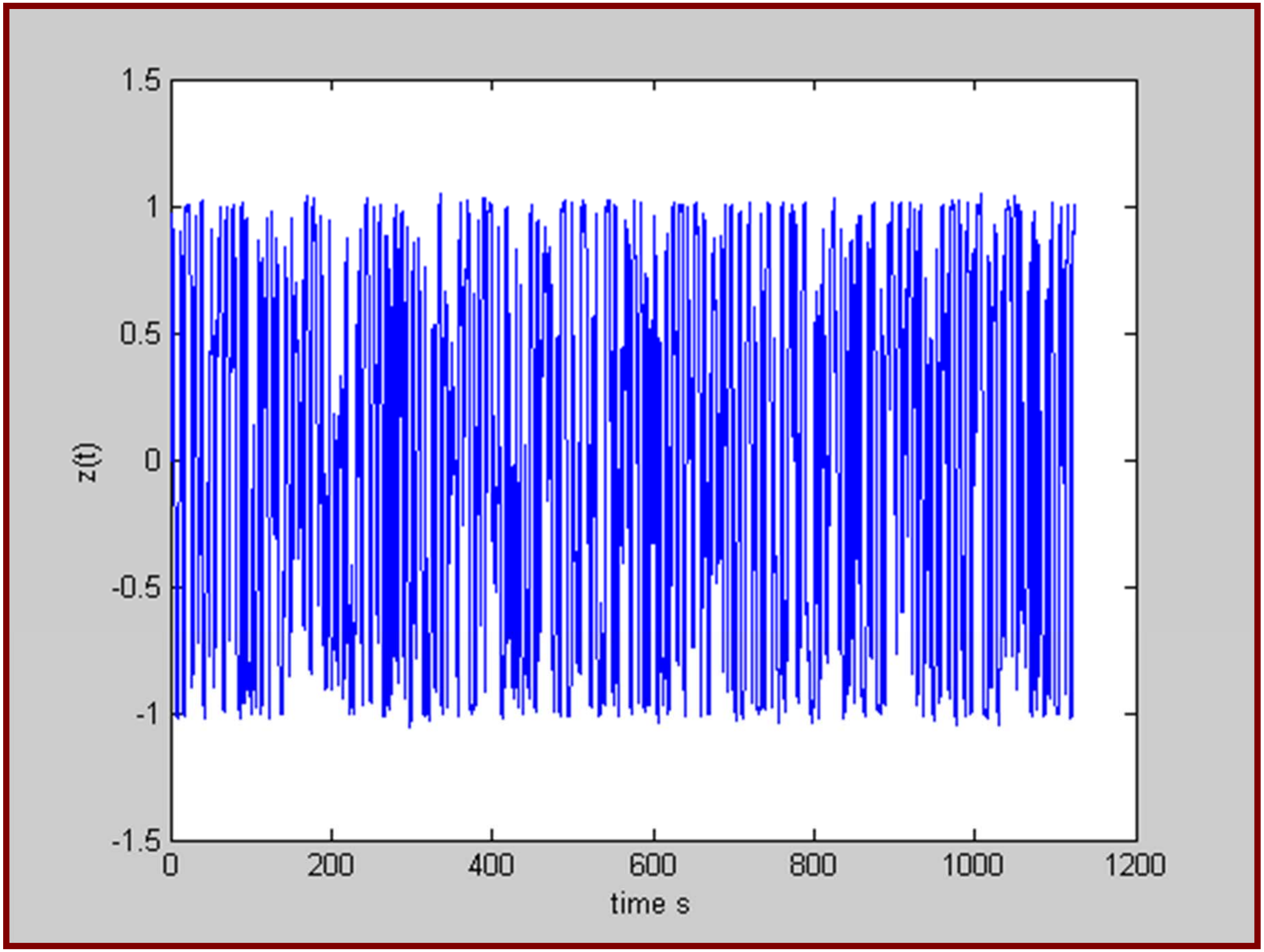
$$\eta = 0.05, \alpha = 0.05, \beta = 0.5, \eta\omega = 0.02, A = 1, n = 2$$

$$\gamma = 0.5, \sigma = 1.0, T = 35 \text{ s}, 5000 \text{ samples}$$

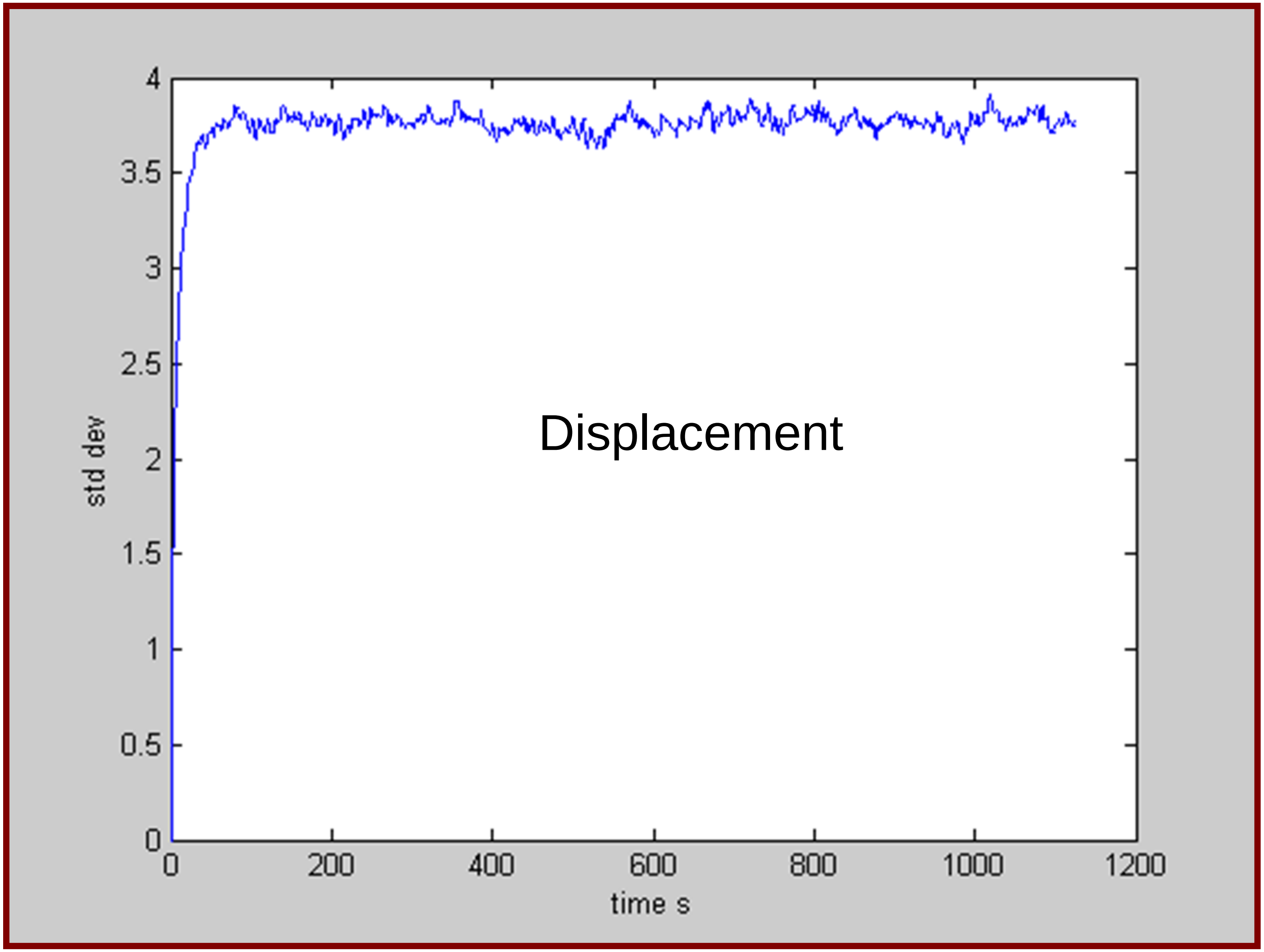

15
10
5
0
-5
-10
-15
x(t)
0
200
400
600
800
1000
1200
time s

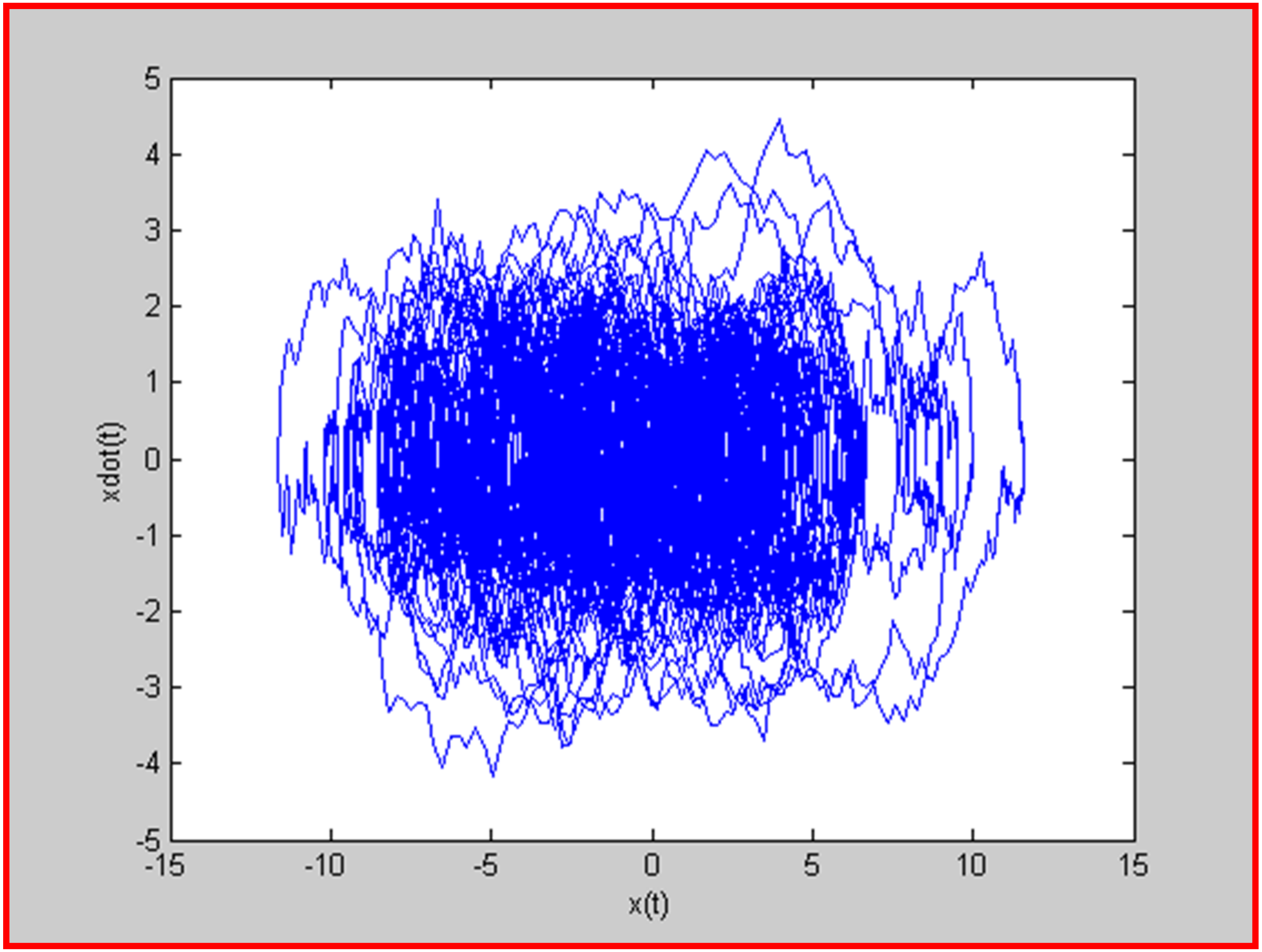


xdot(t)
time s
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
200
400
600
800
1000
1200

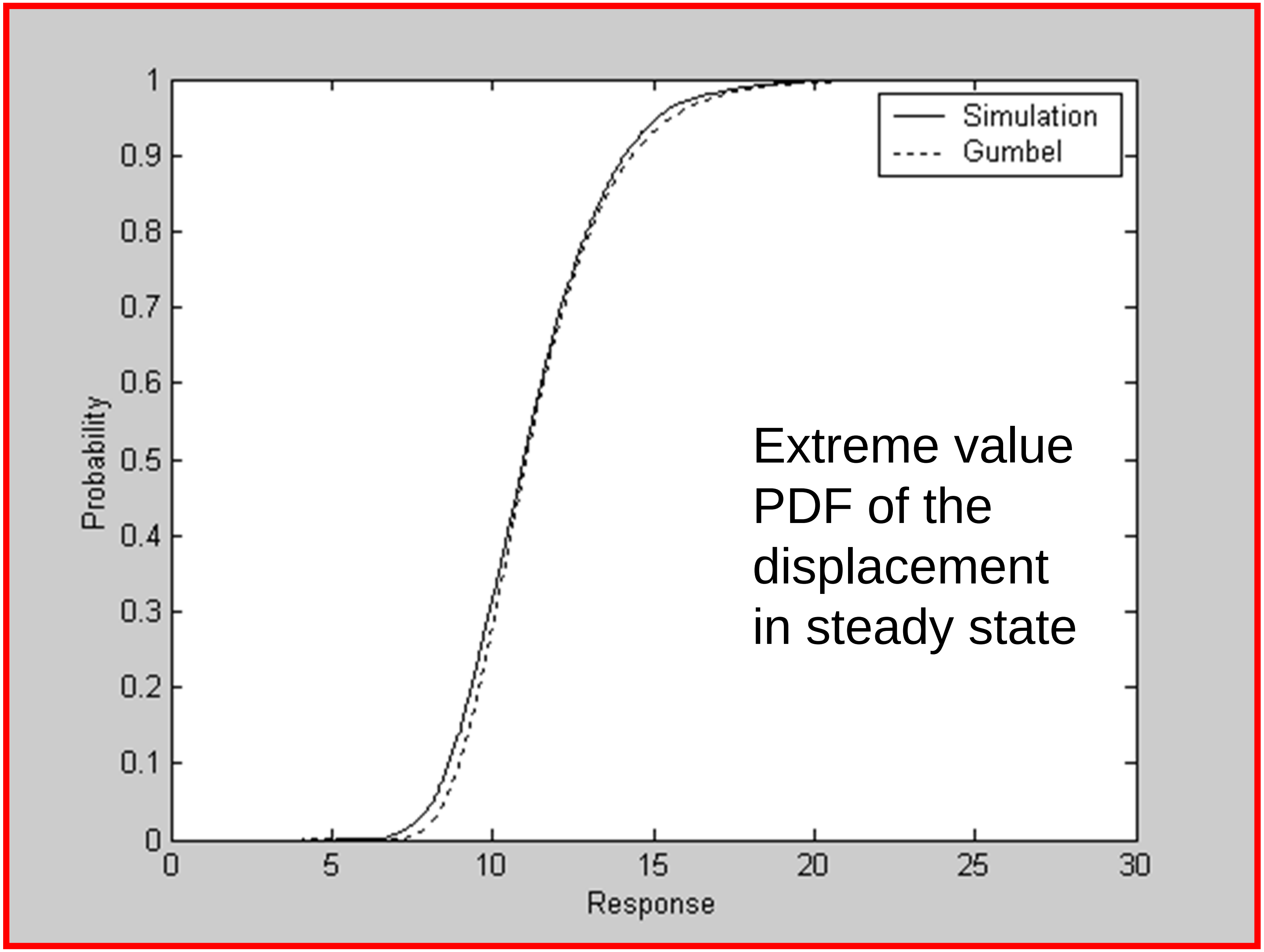


1.5
1
0.5
0
-0.5
-1
-1.5
z(t)
0
200
400
600
800
1000
1200
time s


Displacement
std dev
time s
0
0.5
1
1.5
2
2.5
3
3.5
4
0
200
400
600
800
1000
1200



5
4
3
2
1
0
-1
-2
-3
-4
-5
-15
-10
-5
0
5
10
15
xdot(t)
x(t)

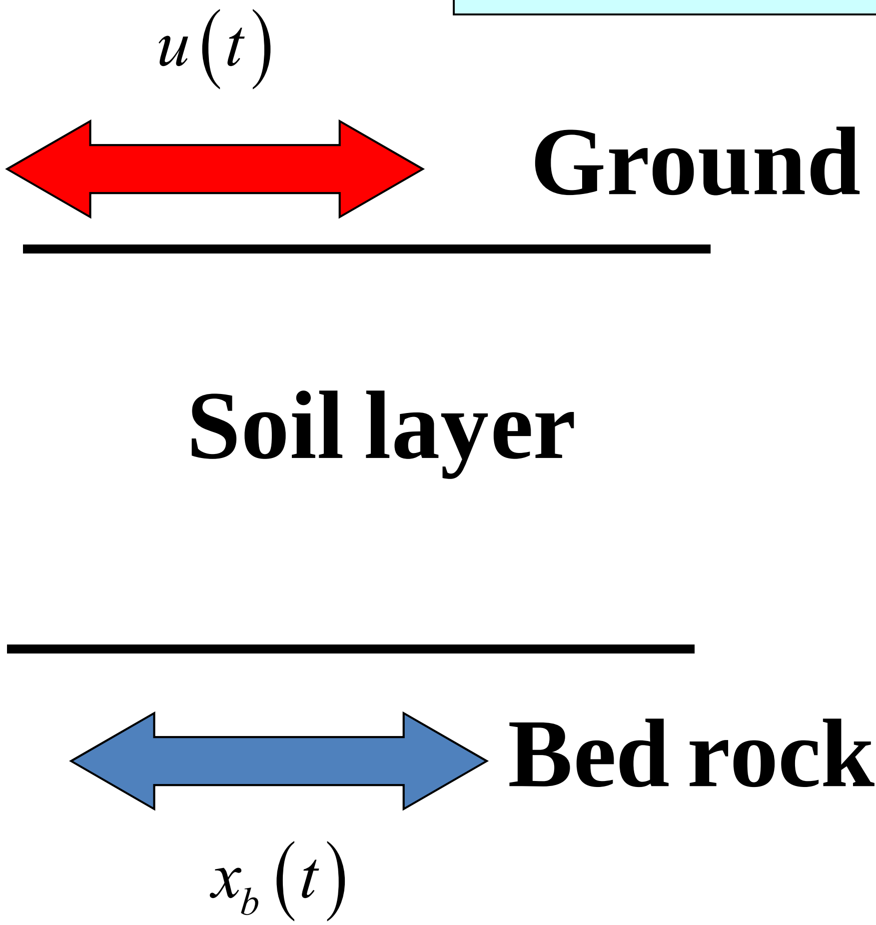


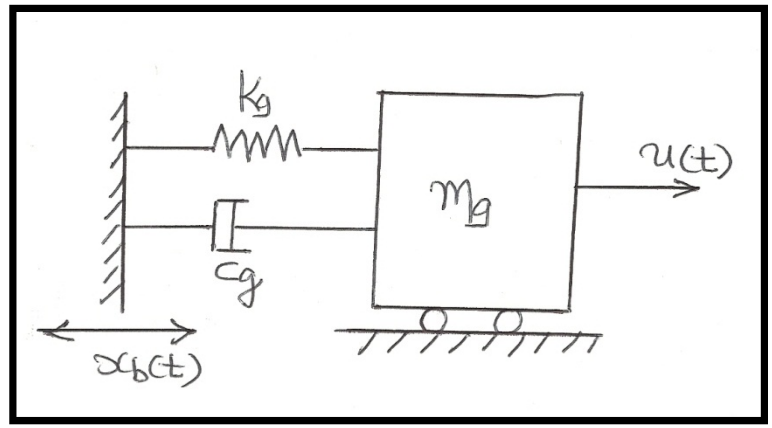
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Probability
0
5
10
15
20
25
30
Response
Simulation
Gumbel
Extreme value PDF of the displacement in steady state

Kanai – Tajimi & Clough and Penzien

Power spectral density function models

for free field earthquake ground acceleration





$$m_g\ddot{u}+c_g\left(\dot{u}-\dot{x}_b\right)+k_g\left(u-x_b\right)=0$$

$$\ddot{u}=-2\eta_g\omega_g\left(\dot{u}-\dot{x}_b\right)-\omega_g^2\left(u-x_b\right)$$

Let $v=u-x_b$

$$\Rightarrow \ddot{v}+2\eta_g\omega_g\dot{v}+\omega_g^2 v=-\ddot{x}_b$$

$$\ddot{u}=-2\eta_g\omega_g\dot{v}-\omega_g^2 v$$

$$\ddot{U}_T\left(\omega\right)=-\left(i2\eta_g\omega_g\omega+\omega_g^2\right)V_T\left(\omega\right)$$

$$=\left(i2\eta_g\omega_g+\omega_g^2\right)\frac{\ddot{X}_{bT}\left(\omega\right)}{\left(\omega_g^2-\omega^2\right)+i\left(2\eta_g\omega_g\omega\right)}$$

$$S\left(\omega\right)=\lim_{T\to\infty}\frac{1}{T}\left\langle\left|\ddot{U}_T\left(\omega\right)\right|^2\right\rangle$$

$$S\left(\omega\right)=I\frac{\left(\omega_g^4+4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2-\omega_g^2\right)^2+4\eta_g^2\omega_g^2\omega^2}$$

$$S(\omega) = I \frac{\left(\omega_g^4 + 4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\eta_g^2\omega_g^2\omega^2}$$

Clough and Penzien model

$$S(\omega) = I \frac{\left(\omega_g^4 + 4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\eta_g^2\omega_g^2\omega^2} \underbrace{|H_f(\omega)|^2}_{\text{High pass filter}}$$

$$= I \frac{\left(\omega_g^4 + 4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\eta_g^2\omega_g^2\omega^2} \underbrace{\frac{\left(\omega / \omega_f\right)^4}{\left[1 - \left(\omega / \omega_f\right)^2\right]^2 + 4\varsigma_f^2\left(\omega / \omega_f\right)^2}}_{\text{High pass filter}}$$

What is the role played by $\left|H_f(\omega)\right|^2$?

$$\left|H_f(\omega)\right|^2 = \frac{\left(\omega/\omega_f\right)^4}{\left[1-\left(\omega/\omega_f\right)^2\right]^2 + 4\varsigma_f^2\left(\omega/\omega_f\right)^2}$$

An artefact to remove singularity at ω=0 in the support displacement.

Introduction of non-stationarity and a time domain analysis

Digital simulation of earthquake ground motion using SDE approach

Filter from bed rock to ground level

$$m_1\ddot{z}_1 + c_1\left(\dot{z}_1 - \dot{x}_b\right) + k_1\left(z_1 - x_b\right) = 0$$

$$y_1 = z_1 - x_b$$

$$m_1\ddot{y}_1 + c_1\dot{y}_1 + k_1 y_1 = -m_1\ddot{x}_b$$

$$\ddot{y}_1 + 2\eta_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = -\ddot{x}_b = e\left(t\right)s\left(t\right)$$

$$\left\langle s\left(t\right)\right\rangle = 0; \left\langle s\left(t\right)s\left(t+\tau\right)\right\rangle = I\delta\left(\tau\right)$$

$e\left(t\right) =$ deterministic modulating function

$$\ddot{z}_1 = -2\eta_1\omega_1\dot{y}_1 - \omega_1^2 y_1$$

High pass filter

$$\ddot{y}_2 + 2\eta_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = 2\eta_1\omega_1\dot{y}_1 + \omega_1^2 y_1$$

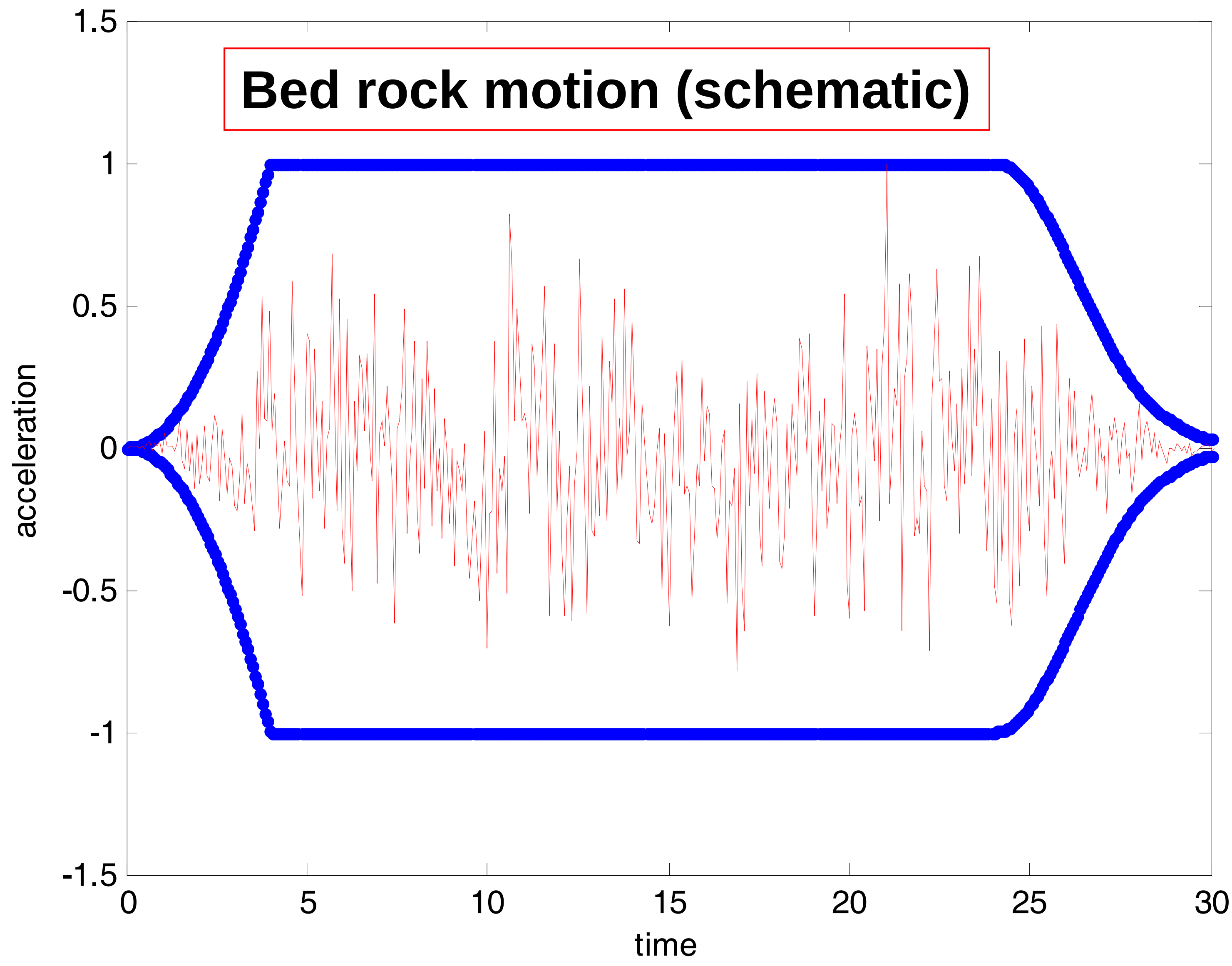
Examples for envelope function

$$e(t) = \left(\frac{t}{4}\right)^2 \text{ for } 0 < t < 4\text{s}$$

$$= 1 \text{ for } 4 < t < 24\text{s}$$

$$= \exp\left[-\frac{1}{2}(t-24)^2\right] \text{ for } t > 24 \text{ s}$$

$$e(t) = a\left[\exp(-\alpha t) - \exp(-\beta t)\right]$$


Bed rock motion (schematic)
acceleration
time
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30

$$\ddot{y}_1 + 2\eta_1\omega_1\dot{y}_1 + \omega_1^2 y_1 = e(t)s(t)$$

$$\ddot{y}_2 + 2\eta_2\omega_2\dot{y}_2 + \omega_2^2 y_2 = 2\eta_1\omega_1\dot{y}_1 + \omega_1^2 y_1$$

$$\begin{Bmatrix} \text{Ground displacement} \\ \text{Ground velocity} \\ \text{Ground acceleration} \end{Bmatrix} = \begin{Bmatrix} y_2(t) \\ \dot{y}_2(t) \\ \ddot{y}_2(t) \end{Bmatrix}$$

Introduce

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\eta_1\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_1^2 & 2\eta_1\omega_1 & -\omega_2^2 & -2\eta_2\omega_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} e(t)s(t)$$

$$\begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & -2\eta_1\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_1^2 & 2\eta_1\omega_1 & -\omega_2^2 & -2\eta_2\omega_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} dt + \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} e(t)\, dB(t)$$

$$\begin{Bmatrix} \text{Ground displacement} \\ \text{Ground velocity} \\ \text{Ground acceleration} \end{Bmatrix} = \begin{Bmatrix} x_3(t) \\ x_4(t) \\ a(t) \end{Bmatrix}$$

$$a(t) = -2\eta_2\omega_2 x_4 - \omega_1^2 x_3 + 2\eta_1\omega_1 x_2 + \omega_1^2 x_1$$

1.5 order Strong Taylor scheme

$$Y_k(n+1) = Y_k(n) + a_k(n)\Delta + b_k(n)\Delta W + \frac{1}{2}L^1 b_k(n)\left\{\left(\Delta W\right)^2 - \Delta\right\}$$

$$+L^1 a_k(n)\Delta Z + L^0 b_k(n)\left\{\Delta W \Delta - \Delta Z\right\} + \frac{1}{2}L^0 a_k(n)\Delta^2 + \frac{1}{2}L^1 L^1 b_k(n)\left\{\frac{1}{3}\left(\Delta W\right)^2 - \Delta\right\}\Delta W$$

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_k \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k=1}^{d}\sum_{l=1}^{d} b_k b_l \frac{\partial^2}{\partial x_k \partial x_l}; L^1 = \sum_{k=1}^{d} b_k \frac{\partial}{\partial x_k}$$

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$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_k \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k=1}^{d}\sum_{l=1}^{d} b_k b_l \frac{\partial^2}{\partial x_k \partial x_l}$$

$$= \frac{\partial}{\partial t} + x_2 \frac{\partial}{\partial x_1} + \left(-2\eta_1\omega_1 x_2 - \omega_1^2 x_1\right)\frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} +$$

$$+\left(-2\eta_2\omega_2 x_4 - \omega_2^2 x_3 + 2\eta_1\omega_1 x_2 + \omega_1^2 x_1\right)\frac{\partial}{\partial x_4}$$

$$L^0 a_1 = -2\eta_1\omega_1 x_2 - \omega_1^2 x_1$$

$$L^0 a_2 = x_2\left(-\omega_1^2\right) + \left(-2\eta_1\omega_1 x_2 - \omega_1^2 x_1\right)\left(-2\eta_1\omega_1\right)$$

$$L^0 a_3 = \left(-2\eta_2\omega_2 x_4 - \omega_2^2 x_3 + 2\eta_1\omega_1 x_2 + \omega_1^2 x_1\right)$$

$$L^0 a_4 = x_2\left(\omega_1^2\right) + \left(-2\eta_1\omega_1 x_2 - \omega_1^2 x_1\right)\left(2\eta_1\omega_1\right)$$

$$+x_4\left(-\omega_2^2\right) + \left(-2\eta_2\omega_2 x_4 - \omega_2^2 x_3 + 2\eta_1\omega_1 x_2 + \omega_1^2 x_1\right)\left(-2\eta_2\omega_2\right)$$

$$L^0 b_2 = \frac{\partial e}{\partial t} S_0; L^0 b_j = 0; j = 1, 3, 4$$

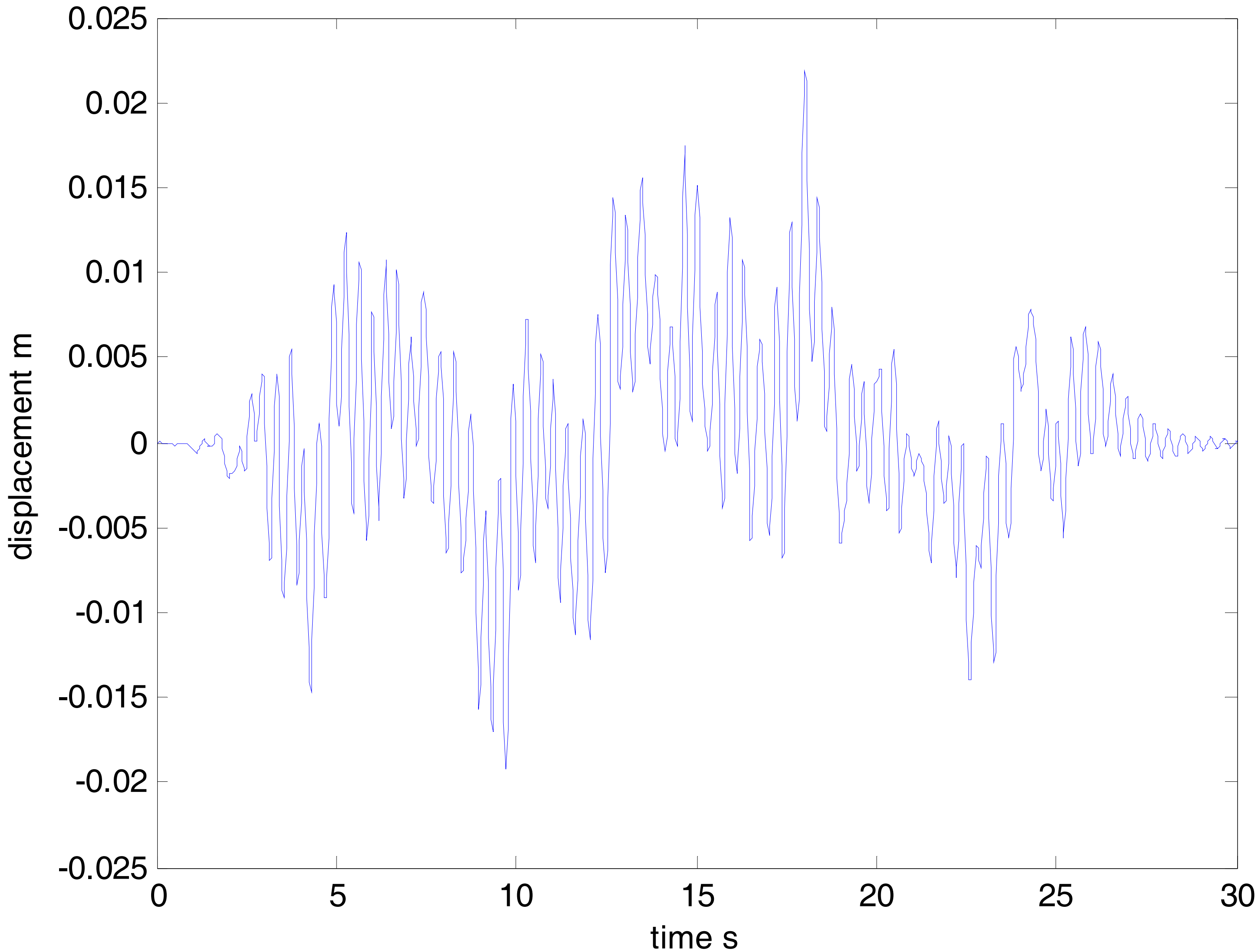
$$L^1 = \sum_{k=1}^{d} b_k \frac{\partial}{\partial x_k} = S_0 e(t) \frac{\partial}{\partial x_2}$$

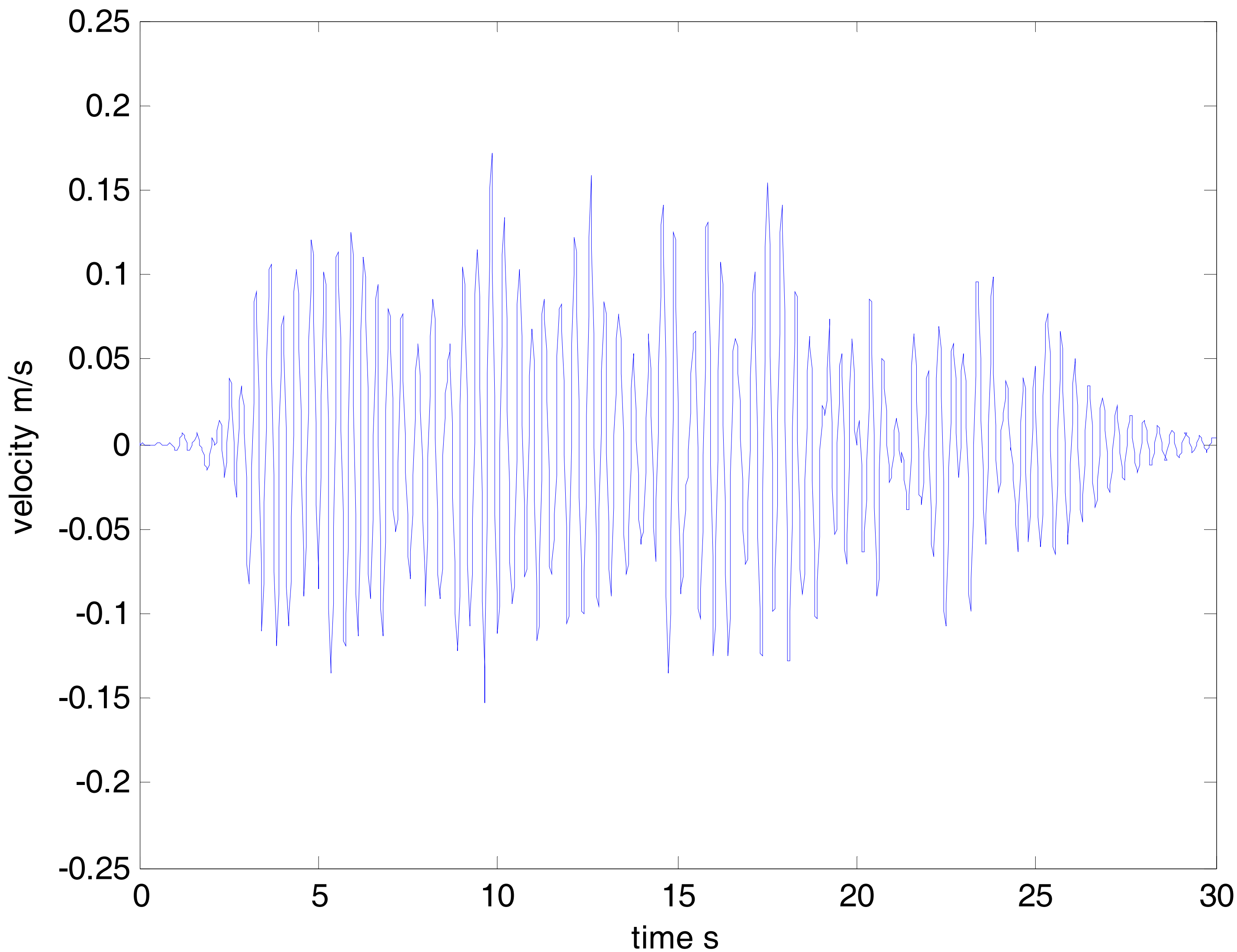
$$L^1 a_1 = S_0 e(t)$$

$$L^1 a_2 = (-2\eta_1 \omega_1) S_0 e(t)$$

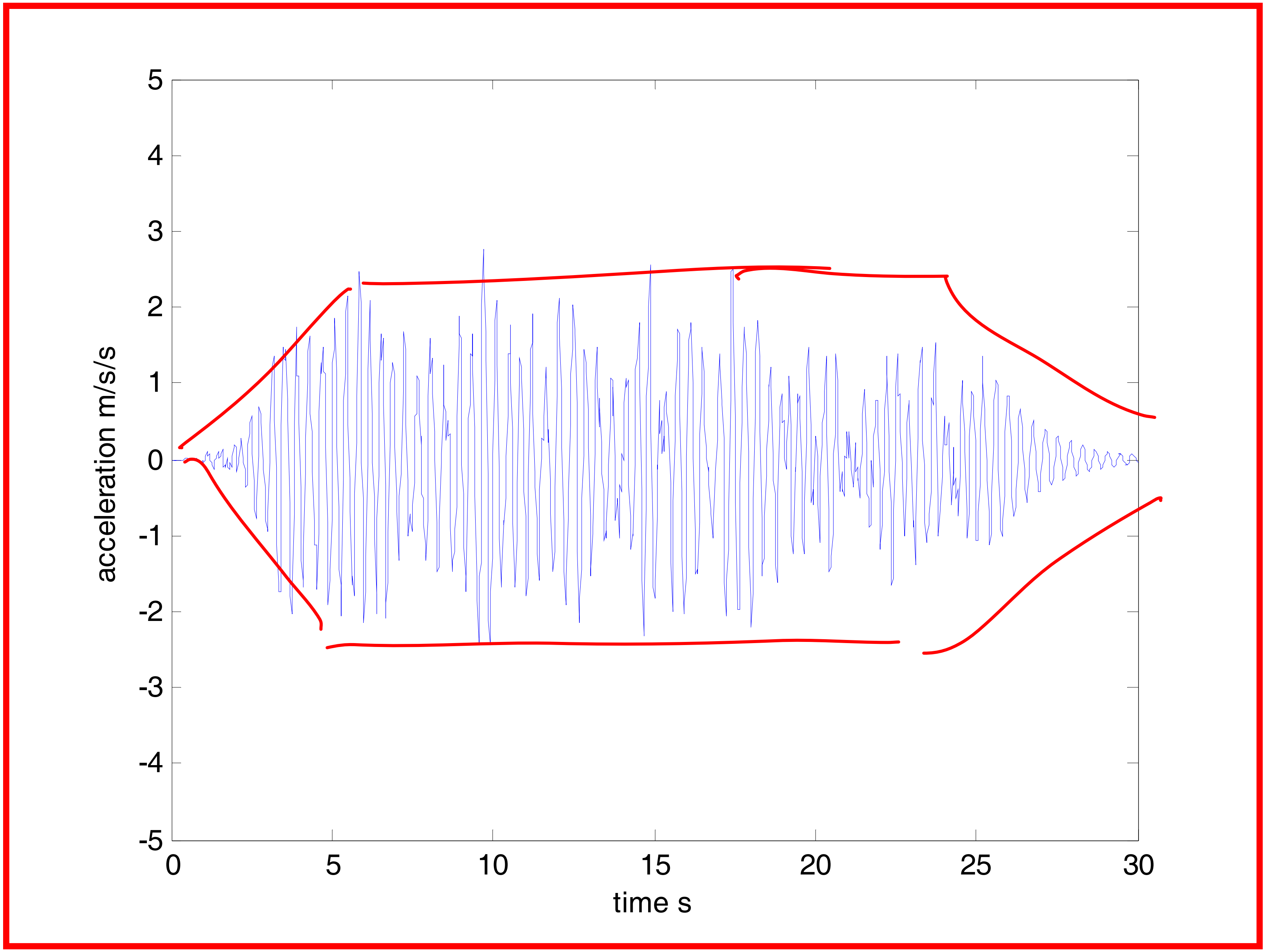
$$L^1 a_3 = 0$$

$$L^1 a_4 = (2\eta_1 \omega_1) S_0 e(t)$$

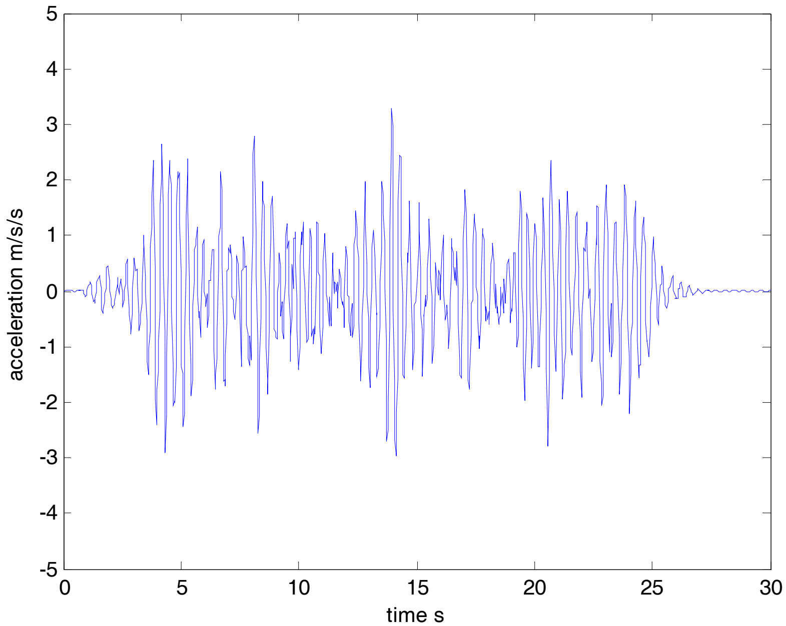

0.025
0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
-0.025
displacement m
0
5
10
15
20
25
30
time s

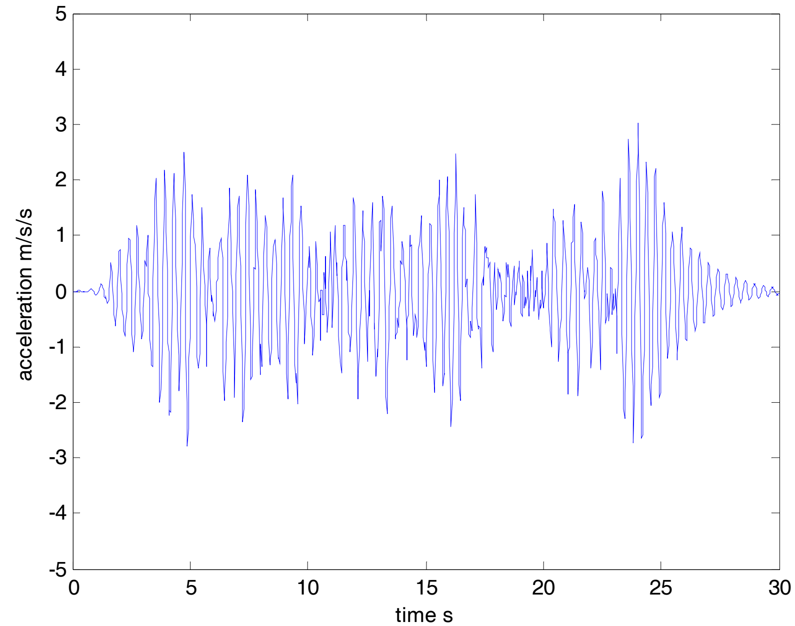


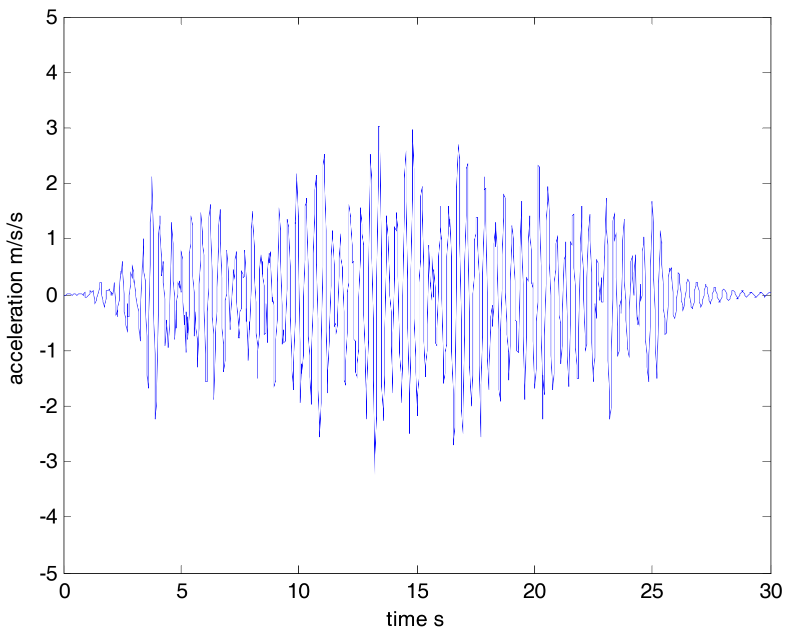
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
velocity m/s
0
5
10
15
20
25
30
time s

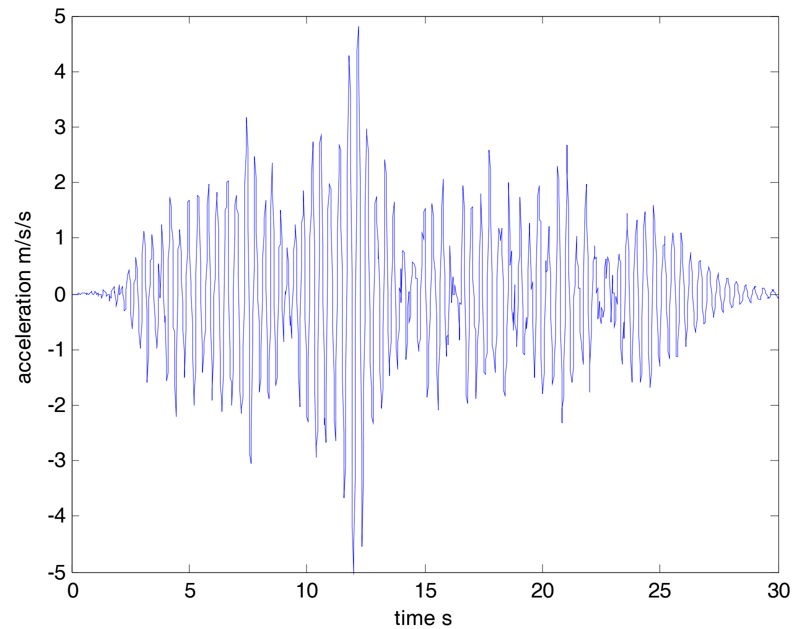


5
4
3
2
1
0
-1
-2
-3
-4
-5
acceleration m/s/s
0
5
10
15
20
25
30
time s


acceleration m/s/s
5 4 3 2 1 0 -1 -2 -3 -4 -5
0 5 10 15 20 25 30
time s


acceleration m/s/s
5 4 3 2 1 0 -1 -2 -3 -4 -5
0 5 10 15 20 25 30
time s


acceleration m/s/s
5 4 3 2 1 0 -1 -2 -3 -4 -5
0 5 10 15 20 25 30
time s


acceleration m/s/s
5 4 3 2 1 0 -1 -2 -3 -4 -5
0 5 10 15 20 25 30
time s

Remarks

Consider the nonstationary random process model

$$\ddot{x}_g(t) = e(t)s(t)$$

with

$e(t)$=deterministic envelope and

$s(t)$=zero mean, stationary, Gaussian random process with prescribed PSD function $S(\omega)$.

One could simulate samples of $s(t)$ by using

$$s(t) = \sum_{n=1}^{N} a_n \sin(\omega_n t) + b_n \cos(\omega_n t)$$

where $a_n, b_n \sim N\left(0, \sigma_n^2\right),\ a_n \perp a_k \forall n \neq k, b_n \perp b_k \forall n \neq k, \&$

$$a_n \perp b_k \forall n, k \in [1, N];\ \int_{\omega_n}^{\omega_{n+1}} S(\omega) d\omega = 2\pi\sigma_n^2$$

It is not obvious in this approach on how to simulate samples of $\left[x_g(t) \quad \dot{x}_g(t) \quad \ddot{x}_g(t)\right]^t$.

VARIANCE REDUCTION

$$P_f = \int_{g(x)<0} p_X(x)\,dx = \int_{-\infty}^{\infty} I[g(x)]\,p_X(x)\,dx = \left\langle I\left[g\left(X\right)\right]\right\rangle$$

$$\Theta = \sum_{i=1}^{n} \frac{1}{n} I\left[g\left(X_i\right)\right]$$

$$\left\langle \Theta \right\rangle = \sum_{i=1}^{n} a_i \left\langle I\left[g\left(X_i\right)\right]\right\rangle = P_F \sum_{i=1}^{n} a_i$$

Θ is an unbiased estimator with optimal sampling variance

$$\mathrm{Var}\left(\Theta\right) = \sum_{i=1}^{n} \frac{1}{n^2} P_F (1 - P_F) = \frac{P_F (1 - P_F)}{n}$$

Illustration

$$\sigma = \sqrt{\frac{\mathrm{P_F}\left(1-P_F\right)}{n}} \Rightarrow$$

Coefficient of variation $\zeta = \frac{\sigma}{m} = \frac{1}{P_F}\sqrt{\frac{P_F\left(1-P_F\right)}{n}}$

$$\Rightarrow \zeta = \sqrt{\frac{\left(1-P_F\right)}{P_F n}} \approx \frac{1}{\sqrt{P_F n}} \text{ (for small } P_F\text{)}$$

$\Rightarrow$ Suppose $\zeta = 0.10 \& P_F \approx 10^{-5}$

$\Rightarrow$ Number of samples needed $n \approx 10^7$.

Similarly, for $\zeta = 0.01,\ P_F \approx 10^{-5}$

$\Rightarrow$ Number of samples needed $n \approx 10^9$.

Remarks

(1) Variance of estimator $\left(=\dfrac{P_F(1-P_F)}{n}\right)$ is independent of size of basic random variable vector X.

(2) If this variance is large, the utility of estimator becomes questionable.

(3) It appears that, in order to reduce the variance of the estimator we need to increase sample size n.

(4) Question: Can we reduce the variance of the estimator without increasing n?

$\Rightarrow$ **Variance reduction techniques**.

Problem of variance reduction: how to reduce $\mathrm{Var}(\Theta)$ without increasing sample size?

$$P_F = \int_{-\infty}^{\infty} I\{g(x) \le 0\} p_X(x) dx$$

This is re-written as

$$P_F = \int_{-\infty}^{\infty} \frac{I\{g(x) \le 0\} p_X(x)}{h_V(x)} h_V(x) dx$$

where $h_V(x)$ is a valid pdf and satisfies the condition

$p_X(x) > 0 \Rightarrow h_V(x) > 0.$

$$\Rightarrow P_F = \int_{-\infty}^{\infty} F(x) h_V(x) dx \text{ where}$$

$$F(x) = \frac{I\{g(x) \le 0\} p_X(x)}{h_V(x)}.$$

$$\Rightarrow P_F = \langle F(X) \rangle_h$$

$\langle \bullet \rangle_h$ = Expectation defined with respect to the pdf $h_V(x)$.

Note: at this stage the function $h_V(x)$ is yet undefined and needs to be suitably selected.

Let $J = \frac{1}{N}\sum_{i=1}^{N} F\left(V_i\right)$ where $\left\{\mathrm{V_i}\right\}_{i=1}^{N}$ are drawn from $h_V\left(x\right)$.

We have shown that J is an unbiased estimator for P_F which minimzes the sampling variance with the lowest sampling variance being

$$\mathrm{Var}\left(J\right) = \frac{\mathrm{Var}\left[F(V)\right]}{N}.$$

$$\mathrm{Var}\left[F(V)\right] = \left\langle \left\{ \frac{I\left[g\left(V\right) \le 0\right] p_X\left(V\right)}{h_V\left(V\right)} - P_F \right\}^2 \right\rangle$$

We now select $h_V(v)$ such that $\text{Var}[F(V)]$ is minimized. Clearly if we select

$$h_V(v) = \frac{I[g(v) \le 0]\, p_X(v)}{P_F}$$

it follows $\text{Var}[F(V)] = 0$.

This would mean that even with one sample we will get the exact estimate of P_F.

The pdf $h_V(v)$ is called the **ideal importance sampling density function (ispdf)**.

Remarks

- The construction of the ideal ispdf requires the knowledge of probability of failure – the very quantity being sought in the first place.

- The ideal ispdf cannot be realized in practice.

- However, the fact that it is guaranteed to exist itself is an assuring idea: one could look for suboptimal solutions. Here the sampling variance may not be reduced to zero but one could attempt to reduce it.

Evaluation of $I = \int_0^1 x^2 dx$ **revisited**.

$\langle x^2 \rangle_{U(0,1)}$

$$I = \int_0^1 x^2 dx = \int_0^1 \frac{x^2}{\pi(x)} \pi(x)\, dx = \left\langle \frac{X^2}{\pi(X)} \right\rangle_{\pi}$$

Here $\pi(x)$ = a valid pdf defined over 0 to 1.

$\hat{I} = \frac{1}{N} \sum_{i=1}^{N} \frac{X_i^2}{\pi(X_i)}$ where $\{X_i\}_{i=1}^{N}$ are samples drawn from $\pi(x)$.

.

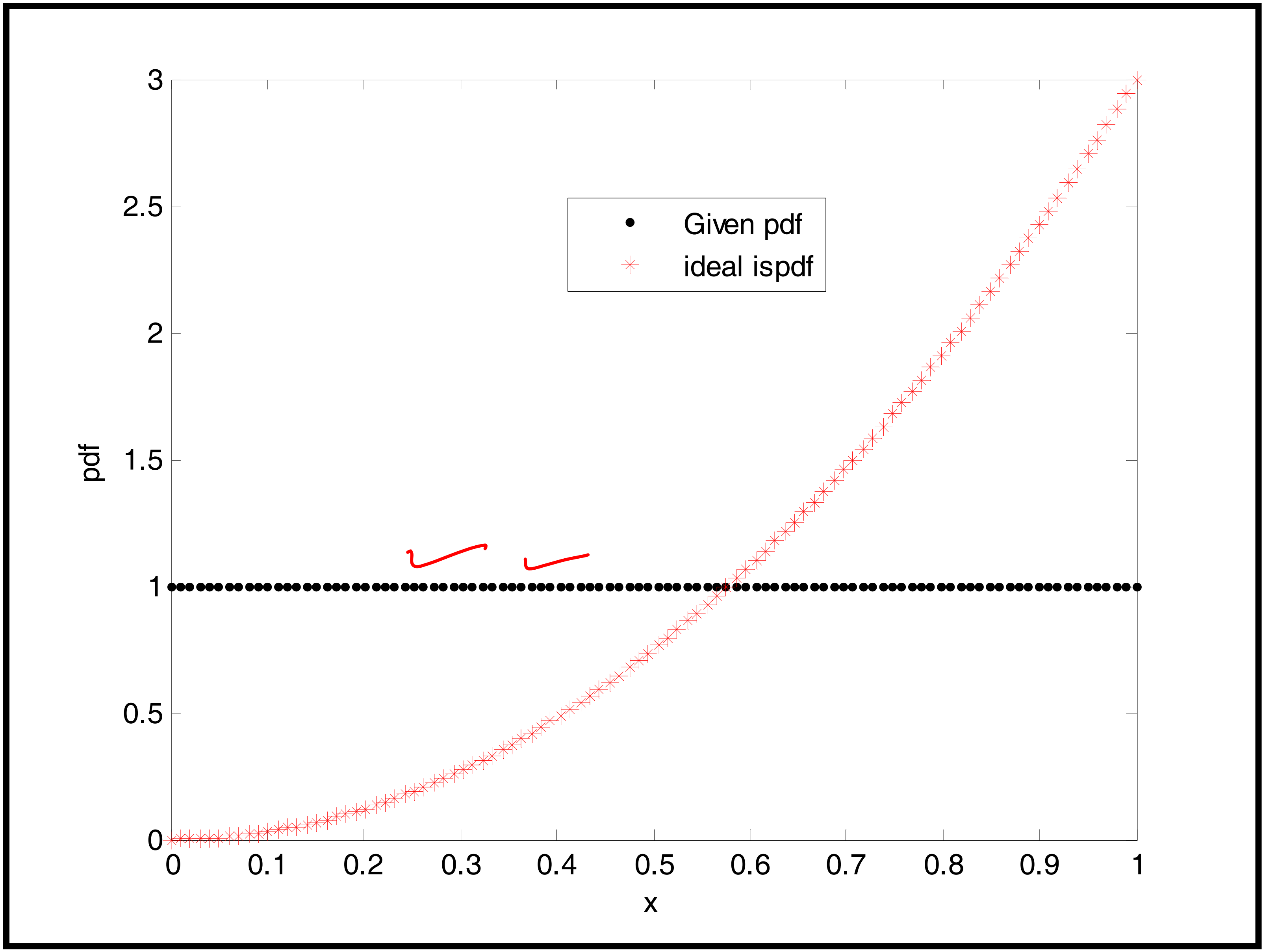
Let $\pi(x) = 3x^2; 0 < x \le 1.$

$$I = \int_0^1 \frac{x^2}{3x^2} \pi(x)\, dx$$

$\hat{I} = \frac{1}{N}\sum_{i=1}^{N} \frac{X_i^2}{3X_i^2} = \frac{1}{3}$ for any value of N and hence for N=1.

$\pi(x) = 3x^2; 0 < x \le 1$ is the ideal ispdf.

Catch: the definition of this ispdf requires the knowledge of I being evaluated.



Given pdf
ideal ispdf
pdf
x
0
0.5
1
1.5
2
2.5
3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

$$\pi(x) = \alpha x^2; 0 < x < 1$$

$$\int_0^1 \pi(x)dx = 1 \Rightarrow \int_0^1 \alpha x^2 dx = 1 \Rightarrow \alpha = \frac{1}{\int_0^1 x^2 dx} = 3.$$

Remarks :

(a) Variance reduction can be viewed as a means to use known information about the problem.

(b) If nothing is known about the problem, variance reduction is not achievable.

(c) At the other extreme, that is, when everything about the problem is known, variance reduces to zero but then simulation itself is not needed.

(d) How do we get information about the problem?

- Perform a few cycles of brute force simulations and learn something about the problem.

Let X~N(0,1)

Consider the evaluation of

$$I = P(X > \beta) = \int_{\beta}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

Let $\beta = 3$.

$I_{exact} = \mathbf{0.00134989803163.}$

$\hat{I}$ using 296318 samples (cov = 0.05) = **0.00134989803163**

Model -1

While running the simulation in the above step, we collected samples lying in the region $X > \beta$. The mean and standard deviation of this sample set was found, a normal pdf was fitted using these moments and this pdf was used as the ispdf.

With this, $\hat{I} = \mathbf{0.0133997065428 5}$ (1000 samples).

Model 2

The ispdf was taken here as N(m,s) with

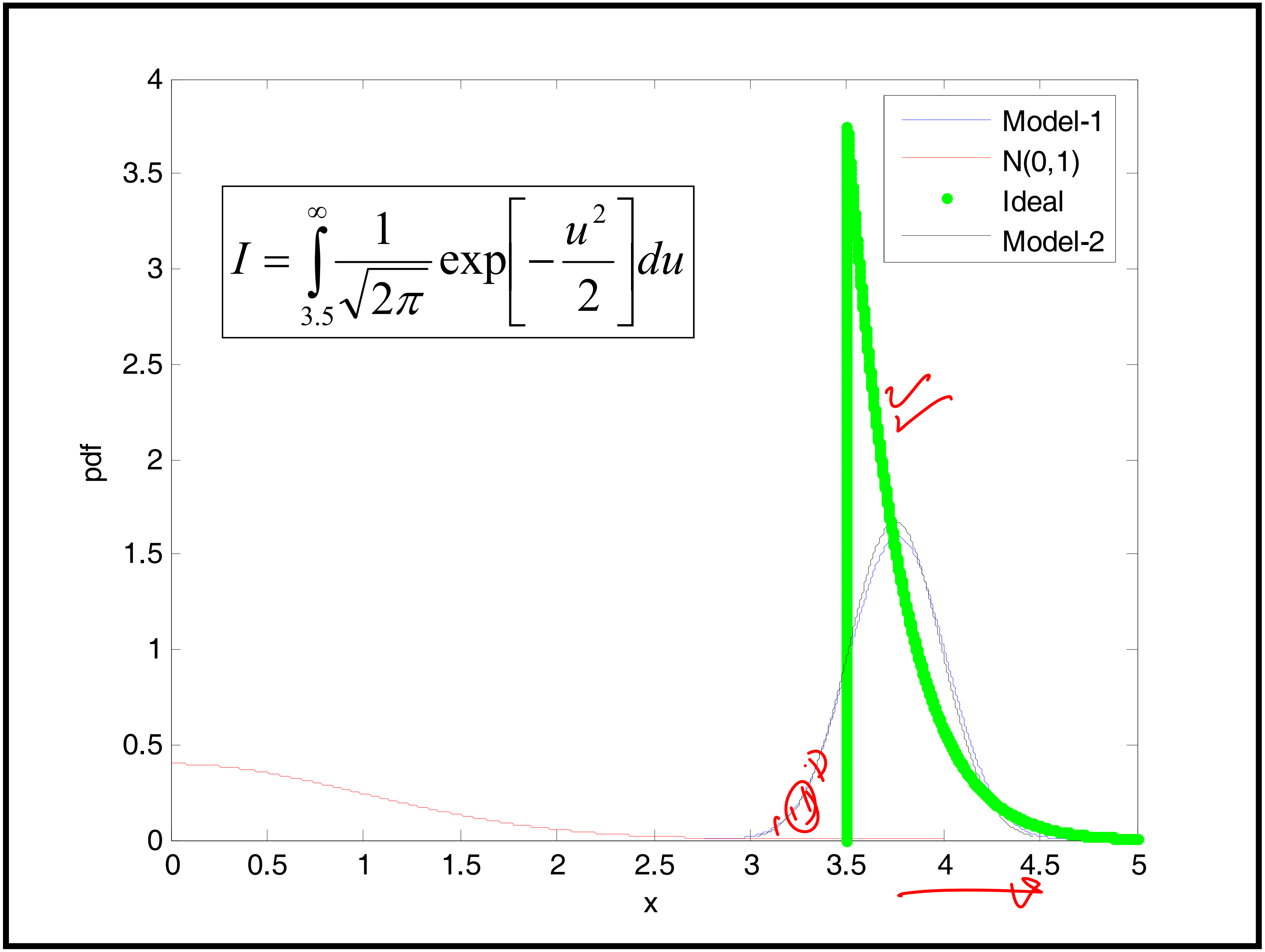
$$\mathrm{m} = \langle \mathrm{X} | \mathrm{X} > \beta \rangle = \frac{\exp\left[-\left(\frac{\beta^2}{2}\right)\right]}{\sqrt{2\pi}\Phi(-\beta)}$$

$$s = \left\langle (X-m)^2 \mid \mathrm{X} > \beta \right\rangle = 1 + \beta m - m^2.$$

With this, $\hat{\mathrm{I}} =$ **0.01436317152000** (1000 samples)

The ideal ispdf

$$h_{\mathrm{V}}^{\mathrm{ideal}}(x) = \frac{1}{\sqrt{2\pi}\Phi(-\beta)} \exp\left[-\frac{x^2}{2}\right] U(x-\beta); \beta < x < \infty$$



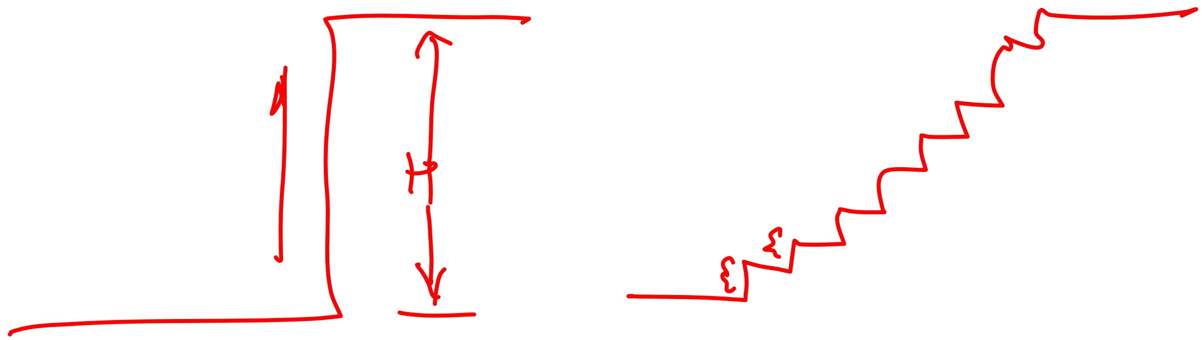
Model-1
N(0,1)
Ideal
Model-2
pdf
x

Sub-set simulations using Markov Chain Monte Carlo (MCMC)

- S K Au and J L Beck, 2001, Estimation of small failure probabilities in high dimension by subset simulation, Probabilistic Engineering Mechanics, 16, 263-277
- J S Liu, 2001, Monte Carlo strategies in scientific computing, Springer, NY.

Basic idea

- Small failure probability can be expressed as a product of larger conditional failure probabilities.
- These larger conditional failure probabilities can be estimated with lesser computational effort.
- The method is applicable to a wide class of problems



Overiview of MCMC simulation method

Let X be a $d \times 1$ vector of random variables with jpdf $p_X(x) = \pi(x)$.

This pdf could be specified as $\pi(x) = k\tilde{\pi}(x)$ where k could be unknown.

Objective

To simulate samples of X and to evaluate $\mathrm{E}\left[f(X)\right]$.

According to MCMC,

$$\mathrm{E}\left[f(X)\right] \approx \frac{1}{n-m}\sum_{i=1}^{n} f\left[X(t_i)\right]$$

where $t_0 < t_1 < t_2 < \cdots < t_n$ and $X(t_0), X(t_1), \cdots, X(t_n)$ form a Markov Chain with stationary pdf $\pi(x) = k\tilde{\pi}(x)$.

Question

How to form a Markov chain whose stationary pdf $\pi(x) = k\tilde{\pi}(x)$ is specified?

Recall

Markov Property

A scalar random process $X(t)$ is said to possess Markov property if

$$P\left[X\left(t_n\right) \le x_n \mid X\left(t_{n-1}\right) \le x_{n-1}, X\left(t_{n-2}\right) \le x_{n-2}, \cdots, X\left(t_1\right) \le x_1\right]$$
$$= P\left[X\left(t_n\right) \le x_n \mid X\left(t_{n-1}\right) \le x_{n-1}\right]$$

for any n and any choice of $0 < t_1 < t_2 < \cdots < t_n$.

$$\underbrace{p\left(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n\right)}_{\text{Multi-dimensional jpdf}} = \underbrace{p(x_1; t_1)}_{\text{Initial pdf}} \underbrace{\prod_{\nu=2}^{n} p\left(x_\nu; t_\nu \mid x_{\nu-1}; t_{\nu-1}\right)}_{\text{Product of transistional pdfs}}$$

Consistency condition for a vector Markov process (CKS equation)

$$p\left(x_2;t_2 \mid x_1;t_1\right)=\int p\left(x_2;t_2 \mid x;\tau\right)p\left(x;\tau \mid x_1;t_1\right)dx$$

$$t_1=t_0, t_2 \to \infty \Rightarrow p\left(x_2;t_2 \mid x_1;t_1\right) \to p\left(x_2;t_2\right)$$

$$\Rightarrow$$

$$p\left(x_2;t_2\right)=\int \underbrace{p\left(x_2;t_2 \mid x;\tau\right)}_{\text{KERNEL}}p\left(x;\tau\right)dx$$

This is can be written in the form

$$\pi\left(y\right)=\int \underbrace{A\left(x,y\right)}_{\text{KERNEL}}\pi\left(x\right)dx$$

Metropolis - Hastings algorithm

1. Initialize x_0; set $t = t_0$.

2. Define a d-dimensional pdf $q(\bullet | X_t = x_t)$ called the proposal pdf.

   Draw a sample y from $q(\bullet | X_t = x_t)$.

   $\left[\text{For example } q(\bullet | X_t = x_t) \sim N\{\bullet, x_t, \sigma^2 \Sigma\} \right]$

3. Let U~$\mathbb{U}[0,1]$. Simulate a sample u from U~$\mathbb{U}[0,1]$.

4. Define $\alpha(x, y) = \min\left[1, \frac{\pi(y) q(x | y)}{\pi(x) q(y | x)}\right]$.

5. If $u < \alpha(x, y)$, set $X_{t+1} = y$; else $X_{t+1} = x_t$.

6. Increment $t \to t+1$. If $t = T_{\max}$, exit; else go to 2.

Explanation

We need to show that the stationary pdf of X_t

[simulated as per the algorithm outlined in the previous slide]

is $\pi(x)$.

We have $X_{t+1} = Y$ if $U \leq \alpha(x, y)$

$= X_t$ otherwise

$$\Rightarrow p_{X_{t+1}}(x_{t+1} \mid X_t = x_t) = q(y \mid X_t = x_t)\alpha(x_t, y)$$

$$+\delta_{x_{t+1}}(x_t)\left[1 - \int q(y \mid X_t = x_t)\alpha(x_t, y)\,dy\right]$$

where

$\delta_{x_{t+1}}(x_t) = I[x_{t+1} = x_t]$ with $I[\bullet]$ being the indicator function.

Alternative notation

$$p(y \mid x) = q(y \mid x)\alpha(x, y) + \delta_y(x)\left[1 - \int q(y \mid x)\alpha(x, y)\,dy\right] = A(x, y)$$

We have

$$\int \pi(x) A(x,y)\, dx = \pi(y)$$

Condition of detailed balance

$$\pi(x) A(x,y) = \pi(y) A(y,x)$$

If this condition is satisfied we get

$$\int \pi(x) A(x,y)\, dx = \int \pi(y) A(y,x)\, dx = \pi(y) \int A(y,x)\, dx = \pi(y)$$

Question:

Does the function

$$p(y|x) = q(y|x)\alpha(x,y) + \delta_y(x)\left[1 - \int q(y|x)\alpha(x,y)\, dy\right] = A(x,y)$$

satisfy the condition of detailed balance?

$$p(y|x) = q(y|x)\alpha(x,y) + \delta_y(x)\left[1 - \int q(y|x)\alpha(x,y)\,dy\right]$$

Let us consider the two terms separately for checking the condition of detailed balance.

$$A(x,y) = q(y|x)\alpha(x,y) = q(y|x)\min\left[1, \frac{\pi(y)q(x|y)}{\pi(x)q(y|x)}\right]$$

$$\Rightarrow A(x,y)\pi(x) = \min\left[\pi(x)q(y|x), \pi(y)q(x|y)\right]$$

Similarly,

$$A(y,x) = q(x|y)\alpha(y,x) = q(x|y)\min\left[1, \frac{\pi(x)q(y|x)}{\pi(y)q(x|y)}\right]$$

$$\Rightarrow A(y,x)\pi(y) = \min\left[\pi(x)q(y|x), \pi(y)q(x|y)\right]$$

$$\Rightarrow A(x,y)\pi(x) = A(y,x)\pi(y)$$

$\Rightarrow$ The first term satisfies the condition of detailed balance.

How about the second term?

$$A(x,y)=\delta_y(x)\left[1-\int q(y\mid x)\alpha(x,y)\,dy\right]$$

$$\Rightarrow \pi(x)A(x,y)=\delta_y(x)\left[\pi(x)-\int q(y\mid x)\pi(x)\alpha(x,y)\,dy\right]$$

$$=\delta_y(x)\left[\pi(x)-\int q(y\mid x)\pi(x)\min\left[1,\frac{\pi(y)q(x\mid y)}{\pi(x)q(y\mid x)}\right]dy\right]$$

$$=\delta_y(x)\left[\pi(x)-\int \min\left[\pi(x)q(y\mid x),\pi(y)q(x\mid y)\right]dy\right]$$

Similarly, $A(y,x)=\delta_x(y)\left[1-\int q(x\mid y)\alpha(y,x)\,dx\right]$

$$\pi(y)A(y,x)=\delta_x(y)\left[\pi(y)-\int q(x\mid y)\pi(y)\alpha(y,x)\,dx\right]$$

$$=\delta_y(x)\left[\pi(y)-\int \min\left[\pi(y)q(x\mid y),\pi(x)q(y\mid x)\right]dx\right]$$

Notice: for the non-zero terms inside the bracket, $x = y$.

$\Rightarrow$ Detailed balance is satisfied by the second term also.

Subset simulation : motivation

$m\ddot{y}+c\dot{y}+ky+f\left[y,\dot{y},t\right]=q(t);y(0),\dot{y}(0)$ specified

$q(t)$: zero mean, stationary Gaussian random process.

$$q(t)=\sum_{n=1}^{N_0} a_n \cos(\omega_n t)+b_n \sin(\omega_n t)$$

where $a_n, b_n \sim N\left(0,\sigma_n^2\right)$, $a_n \perp a_k \forall n\neq k, b_n \perp b_k \forall n\neq k, \&$

$a_n \perp b_k \forall n,k \in [1,N]$; $\int_{\omega_n}^{\omega_{n+1}} S_{qq}(\omega)d\omega = 2\pi\sigma_n^2$

Let $z(t)=h\left[y(t),\dot{y}(t),t\right]$ a metric of system performance.

We are interested in estimating $\mathrm{P}\left[z(t)\leq z^* \forall t\in[0,T]\right]$.

Note : The system parameters could also be random (θ).

$$1 - P_F = P\left[z(t) \le z^* \forall t \in [0,T] \right]$$

$$= P\left[\max_{t \in [0,T]} z(t) \le z^* \right]$$

$$= P\left[Z_m(X) - z^* \le 0 \right]$$

$$= P\left[g(X) > 0 \right]$$

$$Z_m(X) = \max_{t \in [0,T]} z(t)$$

$$g(X) = z^* - Z_m(X)$$

$$X = \left\{ (a_n, b_n)_{n=1}^{N_0}, \theta, z^* \right\}$$

$$P_F = \int_{-\infty}^{\infty} I\left[g(x) \le 0 \right] p_X(x)\, dx$$

$$P_F = \int_{-\infty}^{\infty} I\left[g(x) \le 0\right] p_X(x)\,dx$$

$$\hat{P}_F = \frac{1}{N}\sum_{i=1}^{N} I\left[g\left(X^{(i)}\right) \le 0\right]$$

Remark

- $\hat{P}_F$ is an unbiased and consistent estimator of P_F with minimum variance. The optimal variance is given by

$$\sigma^2_{\hat{P}_F} = \frac{P_F(1-P_F)}{n}.$$

Illustration

$$\sigma_{\hat{P}_F} = \sqrt{\frac{\mathrm{P_F}\left(1-P_F\right)}{n}} \Rightarrow$$

Coefficient of variation $\zeta = \frac{\sigma}{m} = \frac{1}{P_F}\sqrt{\frac{P_F\left(1-P_F\right)}{n}}$

$\Rightarrow \zeta = \sqrt{\frac{\left(1-P_F\right)}{P_F n}} \approx \frac{1}{\sqrt{P_F n}}$ (for small P_F)

$\Rightarrow$ Suppose $\zeta = 0.10 \& P_F \approx 10^{-5}$

$\Rightarrow$ Number of samples needed $n \approx 10^7$.

Similarly, for $\zeta = 0.01,\ P_F \approx 10^{-5}$

$\Rightarrow$ Number of samples needed $n \approx 10^9$.

Subset simulations

$$F = \left[g(X) \le 0 \right] = \text{Failure event}$$

Define

$$F_1 \supset F_2 \supset \cdots \supset F_m = F \text{ such that}$$

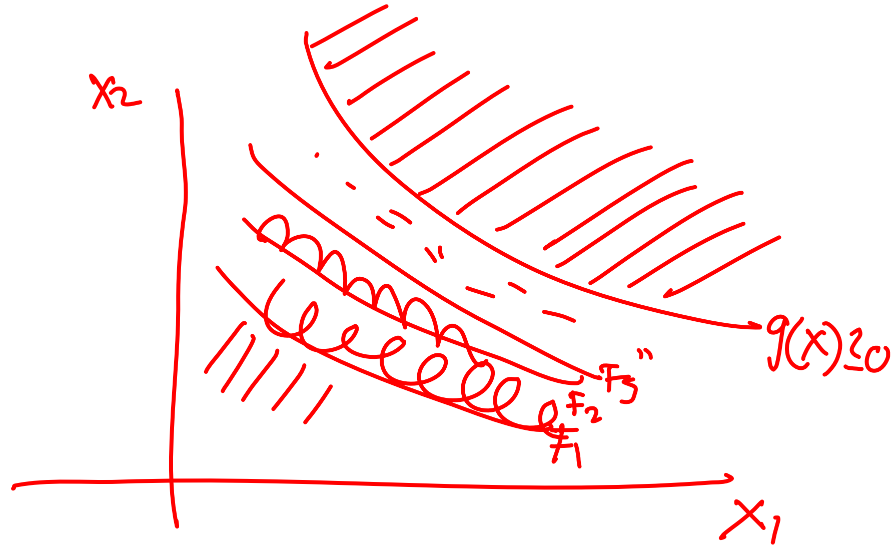
$$F_k = \bigcap_{i=1}^{k} F_i, k = 1, 2, \cdots, m$$

$$P_F = P(F_m) = P\left(\bigcap_{i=1}^{m} F_i \right)$$

$$= P\left(F_m \mid \bigcap_{i=1}^{m-1} F_i \right) P\left(\bigcap_{i=1}^{m-1} F_i \right)$$

$$= P(F_m \mid F_{m-1}) P\left(\bigcap_{i=1}^{m-1} F_i \right)$$

$$= P(F_1) \prod_{i=1}^{m-1} P(F_{i+1} \mid F_i)$$



$P_F = 10^{-6}$

$= P_{F_1} \, P_2 \, P_3 \cdots P_m$

10^{-1} $\quad$ 10^{-1}

Remarks

$$P_F = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1} \mid F_i)$$

If F_i-s are configured such that $P(F_{i+1} \mid F_i)$ and $P(F_1)$ are much larger than P_F, then we will be able to estimate P_F in terms of product of "large" probabilities.

Suppose, $P_F \sim 10^{-6}$, then we could obtain an estimate of P_F as $10^{-6} \sim (10^{-1})\times(10^{-1})\times(10^{-1})\times(10^{-1})\times(10^{-1})\times(10^{-1})$.

Estimation of probability of failure of the order of 0.1 can be easily done using MCS because the failure events here are more frequent.

Remarks (continued)

$$P_F = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1} \mid F_i)$$

$P(F_1)$ can be estimated using a "brute force" Monte Carlo.

$P(F_{i+1} \mid F_i), i = 1, 2, \cdots, m-1$ can be estimated using MCMC.