Stochastic Structural Dynamics

Lecture-29

Monte Carlo simulation approach-5

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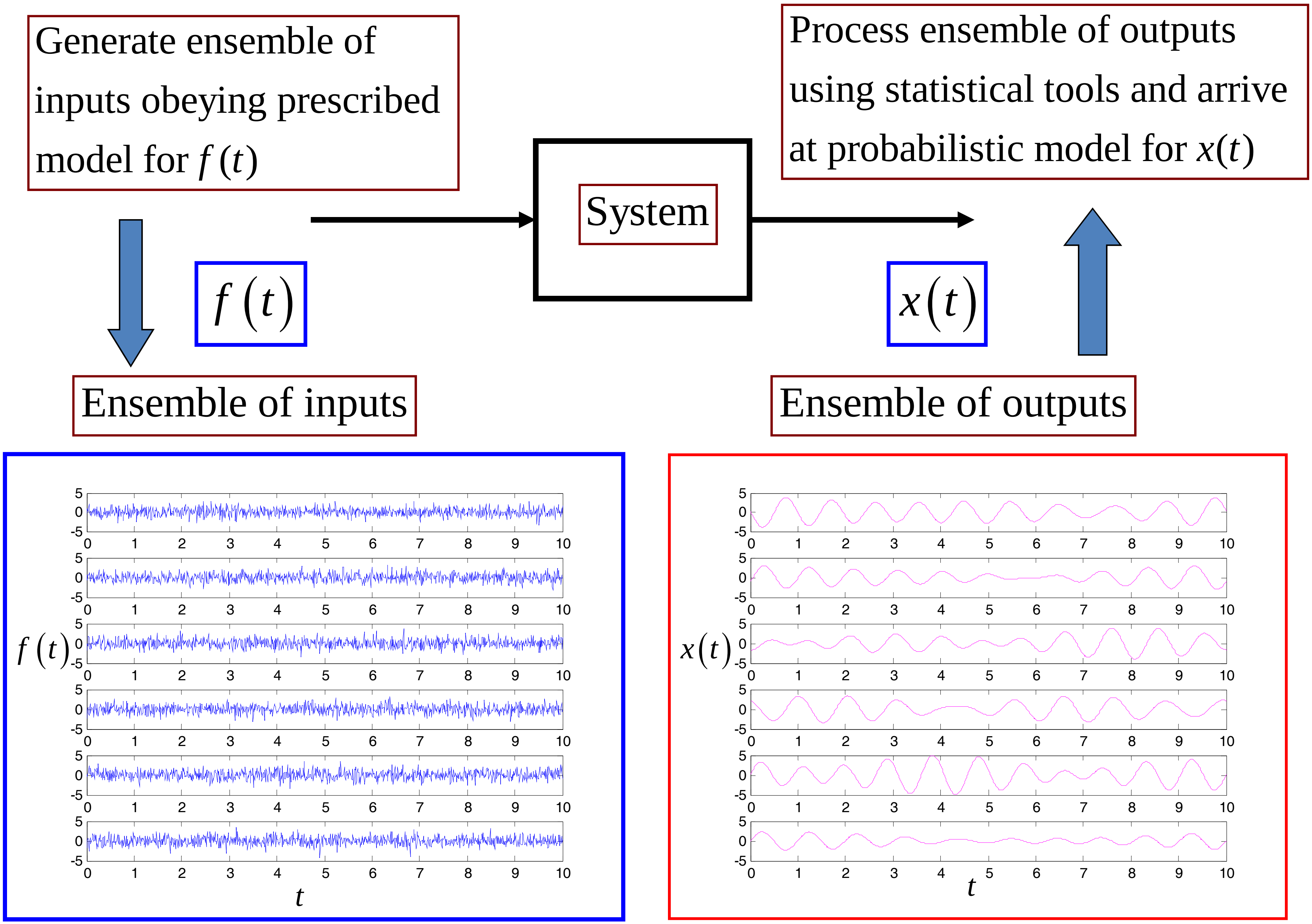

Generate ensemble of inputs obeying prescribed model for $f(t)$
System
Process ensemble of outputs using statistical tools and arrive at probabilistic model for $x(t)$
$f(t)$
$x(t)$
Ensemble of inputs
Ensemble of outputs
$f(t)$
$x(t)$
t
t

When does the Monte Carlo simulation procedure becomes useful?

Linear systems under Gaussian excitations

$$M\ddot{X} + C\dot{X} + KX = F(t); X(0) = X_0; \dot{X}\left(0\right) = \dot{X}_0$$

Exact solutions are available for

- Moments of response (mean, covariance matrix,...)
- PSD matrix of response in the steady state (when it exists)
- jpdf of response at different time instants

Monte Carlo simulation procedure does not offer any advantage if our interest is limited to the above quantities.

On the other hand, if we are interested in

- level crossing statistics
- first passage times
- distribution of peaks
- extreme value distribution

Monte Carlo simulation procedure offers useful means to tackle the problem even for LTI systems.

More complicated situations

- Nonlinear systems
- Parametric excitations
- Non-Gaussian excitations
- Randomly parametered systems

- Characterization of reliability measures of response
  - First passage times
  - Extreme value distributions

$\cdots$ MCS procedure wins.

Example

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]+P(t)\frac{\partial^2 y}{\partial x^2}+m(x)\frac{\partial^2 y}{\partial t^2}+c(x)\frac{\partial y}{\partial t}=f(x,t)+\xi(x,t)$$

$$y(x,0)=y_0(x)$$

$$\dot{y}(x,0)=\dot{y}_0(x)$$

$$\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]_{x=0}=k_2\left[\frac{\partial y}{\partial x}\right]_{x=0};\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]_{x=l}=-k_4\left[\frac{\partial y}{\partial x}\right]_{x=l}$$

$$\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]_{x=0}=k_1[y]_{x=0};\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]_{x=l}=-k_2[y]_{x=0}$$

Sources of randomness

- External excitations

$f(x,t)$: modeled as a space-time random field

$P(t)$ modeled as a random process evolving in time.

- Initial conditions: $y_0(x)$ & $\dot{y}_0(x)$: modeled as random fields evolving in space.

- Boundary conditions $\{k_i\}_{i=1}^{4}$: modeled as a vector of random variables

- System parameters: $EI(x), m(x)$ & $c(x)$: modeled as a vector of random fields

- Modeling error: $\xi(x,t)$: modeled as a space-time random field.

Procedures

- For simulating samples of random variables and random processes
- For solving sample problems in dynamic response simulation
- For processing samples of response time histories.

Two difficulties

- Treatment of spatially varying randomness:
  - **Stochastic FEM**
- Treatment of calculus associated with systems driven by white noise or filtered white noise excitations.
  - **Elements of calculus of Brownian motion processes**

Simulation of dynamical systems driven by white noise excitations

$$dX(t) = f\left[t, X(t)\right]dt + G\left[t, X(t)\right]dB(t); t \geq 0; X(0) = X_0$$

- Nonlinear sytems
- Parametric and (or) external excitations
- White noise or filtered white noise excitations
- Any desired measure of response (moments, reliability,...)

Major issue

How to take into account peculiarities of calculus associated with Brownian motion processes?

Reference

- P E Kloeden and E Platen, 1992, Numerical solution of stochastic differential equations, Springer-Verlag, Berlin.

Recall

Taylor's series for deterministic functions

Let $f(t)$ be a well behaved function (differentiable)

$$f(t) = f\left(t^* + t - t^*\right)$$

$$= f\left(t^*\right) + \left(t - t^*\right)\dot{f}\left(t^*\right) + \frac{\left(t - t^*\right)^2}{2!}\ddot{f}\left(t^*\right) + \frac{\left(t - t^*\right)^3}{3!}\dddot{f}\left(t^*\right) + \cdots$$

$$\bullet\ \Delta f(t) = f(t) - f\left(t^*\right); \Delta t = t - t^*$$

$$\Rightarrow \Delta f(t) = \Delta t \dot{f}\left(t^*\right) + \frac{(\Delta t)^2}{2!}\ddot{f}\left(t^*\right) + \frac{(\Delta t)^3}{3!}\dddot{f}\left(t^*\right) + \cdots$$

$$\bullet\ \Delta f(t) \to 0 \text{ as } \Delta t \to 0$$

$$\Rightarrow df(t) = \dot{f}(t)\,dt$$

RECALL

Simple random walk

Let $\left\{X_i\right\}_{i=1}^{\infty}$ be an iid sequence of random variables with

$$\mathrm{P}\left(X=\Delta x\right)=p$$

$$\mathrm{P}\left(X=-\Delta x\right)=q$$

such that $p+q=1.$

$$\left\langle X\right\rangle=\mathrm{P}\left(X=\Delta x\right)\left(\Delta x\right)+\mathrm{P}\left(X=-\Delta x\right)\left(-\Delta x\right)$$

$$=\Delta x\left(p-q\right)$$

$$\left\langle X^2\right\rangle=\mathrm{P}\left(X=\Delta x\right)\left(\Delta x\right)^2+\mathrm{P}\left(X=-\Delta x\right)\left(-\Delta x\right)^2$$

$$=\Delta x^2\left(p+q\right)$$

$$\mathrm{Var}\left(X\right)=\left\langle X^2\right\rangle-\left\langle X\right\rangle^2$$

$$=\Delta x^2\left(p+q\right)-\Delta x^2\left(p-q\right)^2$$

$$=\Delta x^2\left(p+q\right)^2-\Delta x^2\left(p-q\right)^2\quad\left(\because p+q=1\right)$$

$$=\Delta x^2\left[\left(p+q\right)^2-\left(p-q\right)^2\right]=4pq\Delta x^2$$

Let t be the time axis and let us divide the interval $(0, t)$ into n subintervals each of width Δt such that $n\Delta t = t$.

Define $S(t) = \sum_{i=1}^{n} X_i$

$$\Rightarrow \langle S(t) \rangle = \sum_{i=1}^{n} \langle X_i \rangle = \sum_{i=1}^{n} (p-q)\Delta x$$

$$= n(p-q)\Delta x$$

$$= t(p-q)\frac{\Delta x}{\Delta t}$$

$$\mathrm{Var}\left[S(t)\right] = 4tpq\Delta x^2$$

$$= 4tpq\frac{\Delta x^2}{\Delta t}$$

Remarks

•$S(t)$ is known as a simple random walk.

•$S(t)$ is a discrete state, disrete paramter random process.

•Consider the limit of $\Delta x \to 0$ as $\Delta t \to 0$

$\Rightarrow$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \langle S \rangle = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} t(p-q)\frac{\Delta x}{\Delta t}$$

and

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \mathrm{Var}\left[S(t)\right] = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} t4pq\frac{\Delta x^2}{\Delta t} \to 0$$

$\Rightarrow$

In the limit of $\Delta x \to 0$ as $\Delta t \to 0$, $S(t)$ becomes a deterministic function.

This is not an interesting limit from probabilistic point of view.

Wiener and Brownian motion Processes

Consider the following limit of the simple random walk

$\Delta x^2 \to 0$ as $\Delta t \to 0$

with

$$\Delta x = \sigma \Delta t; \quad p = \frac{1}{2}\left[1 + \frac{\mu\sqrt{\Delta t}}{\sigma}\right]; \quad q = \frac{1}{2}\left[1 - \frac{\mu\sqrt{\Delta t}}{\sigma}\right]$$

$\Rightarrow$

$$\langle S(t) \rangle \to \mu t$$

$$\mathrm{Var}\left[S(t)\right] \to \sigma^2 t$$

This is an interesting limit!

Remarks

- The resulting process is known as the Wiener process.
- This is a process with continuous state and continuous parameter.
- The process is a Gaussian process (central limit theorem).
- The process is nonstationary
- If $\mu = 0$, the process is known as a Brownian motion process.
- Without loss of generality we take $B(0) = 0$.

Ito's formula

Let $B(t)$: BMP

$$B(t) = B\left(t^* + t - t^*\right)$$

$$= B\left(t^*\right) + \left(t - t^*\right)\dot{B}\left(t^*\right) + \frac{\left(t - t^*\right)^2}{2!}\ddot{B}\left(t^*\right) + \frac{\left(t - t^*\right)^3}{3!}\dddot{B}\left(t^*\right) + \cdots$$

- $\Delta B(t) = B(t) - B\left(t^*\right); \Delta t = t - t^*$

$$\Rightarrow \Delta B(t) = \Delta t \dot{B}\left(t^*\right) + \frac{(\Delta t)^2}{2!}\ddot{B}\left(t^*\right) + \frac{(\Delta t)^3}{3!}\dddot{B}\left(t^*\right) + \cdots$$

- $\Delta B(t) \to 0$ as $\sqrt{\Delta t} \to 0$

$$\Rightarrow dB(t) = \dot{B}(t)dt + \frac{(\Delta t)^2}{2!}\ddot{B}(t)$$

Key rules

Let $B(t)$: BMP

$$\left[dB(t)\right]^2 = dt$$

$$dB(t)dt = 0$$

$$(dt)^2 = 0$$

Example : Scalar SDE

$$dx(t) = a(t)dt + b(t)dB(t); x(0) = x_0$$

$$\begin{aligned}
\left[dx(t)\right]^2 &= \left[a(t)dt + b(t)dB(t)\right]^2 \\
&= a^2(t)\left(dt\right)^2 + b^2(t)\left[dB(t)\right]^2 + 2a(t)b(t)dB(t) \\
&= b^2(t)\left[dB(t)\right]^2 \\
&= b^2(t)dt
\end{aligned}$$

dt

Consider $u\left[x(t)\right]$

$$du = u'\left[a(t)dt + b(t)dB(t)\right] + \frac{1}{2}u''\left[a(t)dt + b(t)dB(t)\right]^2$$

$$= u'\left[a(t)dt + b(t)dB(t)\right] + \frac{1}{2}u''b^2(t)dt$$

$$= \left[u'a(t) + \frac{1}{2}u''b^2(t)\right]dt + u'b(t)dB(t)$$

Note

$$\left[dx(t)\right]^3 = dx(t)\left[dx(t)\right]^2$$

$$= dx(t)b^2(t)dt$$

$$= 0$$

Consider $u(t) = B^2(t)$

$$du = u' dB(t) + \frac{1}{2} u'' \left[dB(t) \right]^2$$

$$= 2B(t) dB(t) + \frac{1}{2} 2dt$$

$$= 2B(t) dB(t) + \underbrace{dt}_{\text{New term}}$$

Consider $u(t) = \exp\left[B(t)\right]$

$$du = u' dB(t) + \frac{1}{2} u'' \left[dB(t)\right]^2$$

$$= \exp\left[B(t)\right] dB(t) + \underbrace{\frac{1}{2} \exp\left[B(t)\right] dt}_{\text{New term}}$$

$$u(x) = \ln x$$

$$du = \frac{dx}{x} - \frac{1}{2}\left(-\frac{1}{x^2}\right)(dx)^2$$

$$= \frac{dx}{x} + \underbrace{\frac{1}{2}\left(\frac{dx}{x}\right)^2}_{\text{New term}}$$

$$d(xy) = \frac{\partial}{\partial x}(xy)\,dx + \frac{\partial}{\partial y}(xy)\,dy +$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}(xy)(dx)^2 + \frac{1}{2}\frac{\partial^2}{\partial y^2}(xy)(dy)^2 + \frac{\partial^2}{\partial x \partial y}(xy)(dxdy)$$

$$= \underline{ydx + xdy} + \underbrace{dxdy}_{\text{New term}}$$

Proof that $\left[dB(t)\right]^2 = dt$

Consider the time interval 0 to t and divide into n intervals of width Δt such that $n\Delta t = t$.

Fix t. Define

$$\int_0^t \left[dB(t)\right]^2 = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \left[B(t_i) - B(t_{i-1})\right]^2$$

$$= \lim_{n\to\infty} \left(e_1^2 + e_2^2 + \cdots + e_n^2\right) \text{with } e_i = \left[B(t_i) - B(t_{i-1})\right]$$

$$\left(e_1^2 + e_2^2 + \cdots + e_n^2\right) = t\left\{\frac{1}{n}\left[\left(\frac{e_1^2}{\Delta t}\right) + \left(\frac{e_2^2}{\Delta t}\right) + \cdots + \left(\frac{e_n^2}{\Delta t}\right)\right]\right\}$$

$$\left(e_1^2+e_2^2+\cdots+e_n^2\right)=t\left\{\frac{1}{n}\left[\left(\frac{e_1^2}{\Delta t}\right)+\left(\frac{e_2^2}{\Delta t}\right)+\cdots+\left(\frac{e_n^2}{\Delta t}\right)\right]\right\}$$

Fix $t=n\Delta t$ and allow $n\to\infty$ & $\Delta t\to 0.$

$\Rightarrow\left\{\frac{1}{n}\left[\left(\frac{e_1^2}{\Delta t}\right)+\left(\frac{e_2^2}{\Delta t}\right)+\cdots+\left(\frac{e_n^2}{\Delta t}\right)\right]\right\}$ is the sample mean of $\chi^2(1)$ random variables. By law of large numbers

$$\lim_{n\to\infty}\left(e_1^2+e_2^2+\cdots+e_n^2\right)\to t.$$

$\Rightarrow\int_0^t\left[dB(t)\right]^2=t.$ We have $\int_0^t ds=t\Rightarrow\left[dB(t)\right]^2=dt$

Proof that $\left(dt\right)^2 = 0$

$$\int_0^t \left(dt\right)^2 = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \left(\Delta t\right)^2 = \lim_{\Delta t \to 0} \left[n\left(\Delta t\right)^2 \right]$$

$$= \lim_{\Delta t \to 0} \left(n\Delta t\right) \lim_{\Delta t \to 0} \Delta t$$

$$= t \times 0 = 0$$

Similarly it can be proved that

$$dB(t)dt = 0$$

[**Exercise**]

Recall: Integral version of Taylor's series

Let $x(t)$ be a scalar deterministic function.

Consider

$$\frac{dx}{dt} = a\left[x(t)\right]; x(t_0) = x_0; 0 \le t_0 \le T$$

This can also be written as

$$x(t) = x_0 + \int_{t_0}^{t} a\left[x(s)\right] ds.$$

Let $a\left[x(t)\right]$ be well behaved (sufficiently smooth).

Consider the function $f\left[x(t)\right]$.

$$\frac{d}{dt} f\left[x(t)\right] = \frac{\partial}{\partial x}\left\{ f\left[x(t)\right]\right\} \frac{dx}{dt}$$

$$= \frac{\partial}{\partial x}\left\{ f\left[x(t)\right]\right\} a\left[x(t)\right] = Lf\left[x(t)\right]$$

with $L = a\left[x(t)\right]\frac{\partial}{\partial x}$

$\Rightarrow$

$$f\left[x(t)\right] = f\left[x_0\right] + \int_{t_0}^{t} Lf\left[x(s)\right] ds$$

$$f\left[x(t)\right] = f\left[x_0\right] + \int_{t_0}^{t} Lf\left[x(s)\right]ds$$

$$\text{For } f\left[x(t)\right] = x(t),\ Lf\left[x(t)\right] = a\left[x(t)\right]\frac{\partial}{\partial x}x = a$$

$$\Rightarrow x(t) = x_0 + \int_{t_0}^{t} a\left[x(s)\right]ds$$

$$\text{Now consider } f = a\left[x(t)\right] \Rightarrow a\left[x(s)\right] = a(x_0) + \int_{t_0}^{s} La\left[x(z)\right]dz$$

$$\Rightarrow x(t) = x_0 + \int_{t_0}^{t}\left\{a(x_0) + \int_{t_0}^{s} La\left[x(z)\right]dz\right\}ds$$

$$\Rightarrow x(t) = x_0 + a(x_0)\int_{t_0}^{t} ds + \int_{t_0}^{t}\int_{t_0}^{s} La\left[x(z)\right]dzds$$

Now consider $f = La\left[x(t)\right]$

$$\Rightarrow La\left[x(z)\right] = La(x_0) + \int_{t_0}^{z} L^2 a\left[x(u)\right] du$$

$$\Rightarrow x(t) = x_0 + a(x_0)\int_{t_0}^{t} ds + \int_{t_0}^{t}\int_{t_0}^{s}\left\{La(x_0) + \int_{t_0}^{z} L^2 a\left[x(u)\right] du\right\} dzds$$

$$\Rightarrow x(t) = x_0 + a(x_0)\int_{t_0}^{t} ds + La(x_0)\int_{t_0}^{t}\int_{t_0}^{s} dzds + R_3$$

$$R_3 = \int_{t_0}^{t}\int_{t_0}^{s}\left\{\int_{t_0}^{z} L^2 a\left[x(u)\right] du\right\} dzds$$

In general for function $f\left[x(t)\right]$ that is $r+1$ times differentiable we get

$$f\left[x(t)\right]= f\left[x(t_0)\right]+\sum_{l=1}^{r}\frac{\left(t-t_0\right)^l}{l!}L^l f\left[x(t_0)\right]+$$

$$+\int_{t_0}^{t}\cdots\int_{t_0}^{s_2} L^{r+1} f\left[x(s_1)\right]ds_1\cdots ds_{r+1} \text{ for } t\in\left[t_0,T\right]\,\&\, r=1,2,3,\cdots$$

Question

How do we generalize this when $x(t)$ is a filtered white noise process?

Or, when

$$dx(t) = a\left[x\left(t\right)\right]dt + b\left[x\left(t\right)\right]dB\left(t\right); x\left(t_0\right) = x_0 \,?$$

$$\Rightarrow$$

Ito - Taylor expansion

Ito - Taylor's expansion and multiple stochastic integrals

$$dX(t) = a\left[X\left(t\right)\right]dt + b\left[X\left(t\right)\right]dB\left(t\right); X\left(t_0\right) = X_0$$

$$f\left[X(t)\right] = f\left[X(t_0 + t - t_0)\right]$$

$$= f\left[X(t_0)\right] + \int_{t_0}^{t}\left\{a\left[X\left(s\right)\right]\frac{\partial}{\partial x} f\left[X(s)\right] + \frac{1}{2}b^2\left[X\left(s\right)\right]\frac{\partial^2}{\partial x^2} f\left[X(s)\right]\right\}ds$$

$$+\int_{t_0}^{t} b\left[X\left(s\right)\right]\frac{\partial}{\partial x} f\left[X(s)\right]dB(s)$$

$$= f\left[X(t_0)\right] + \int_{t_0}^{t} L^0 f\left[X(s)\right]ds + \int_{t_0}^{t} L^1 f\left[X(s)\right]dB(s)$$

$$L^0 = a\left[X\left(s\right)\right]\frac{\partial}{\partial x} + \frac{1}{2}b^2\left[X\left(s\right)\right]\frac{\partial^2}{\partial x^2}$$

$$L^1 = b\left[X\left(s\right)\right]\frac{\partial}{\partial x}$$

$$dX(t) = a\left[X(t)\right]dt + b\left[X(t)\right]dB(t); X(t_0) = X_0$$

$$f\left[X(t)\right] = X(t)$$

$$\Rightarrow X(t) = X(t_0) + \int_{t_0}^{t} L^0 X(s)ds + \int_{t_0}^{t} L^1 X(s)dB(s)$$

$$L^0 = a\left[X(s)\right]\frac{\partial}{\partial x} + \frac{1}{2}b^2\left[X(s)\right]\frac{\partial^2}{\partial x^2}$$

$$L^1 = b\left[X(s)\right]\frac{\partial}{\partial x}$$

$$X(t) = X(t_0) + \int_{t_0}^{t} a\left[X(s)\right]ds + \int_{t_0}^{t} b\left[X(s)\right]dB(s)$$

OK

$$X(t) = X(t_0) + \int_{t_0}^{t} a\left[X(s)\right]ds + \int_{t_0}^{t} b\left[X(s)\right]dB(s)$$

Apply Ito's formula on $a\left[X(s)\right]$ and $b\left[X(s)\right]$

$\Rightarrow$

$$X(t) = X(t_0) + \int_{t_0}^{t} \left\{ a\left[X(t_0)\right] + \int_{t_0}^{s} L^0 a\left[X(z)\right]dz + \int_{t_0}^{s} L^1 a\left[X(z)\right]dB(z) \right\} ds$$

$$+ \int_{t_0}^{t} \left\{ b\left[X(t_0)\right] + \int_{t_0}^{s} L^0 b\left[X(z)\right]dz + \int_{t_0}^{s} L^1 b\left[X(z)\right]dB(z) \right\} dB(s)$$

$$X(t) = X(t_0) + a\left[X(t_0)\right]\int_{t_0}^{t} ds + b\left[X(t_0)\right]\int_{t_0}^{t} dB(s) + R$$

$$R = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a\left[X(z)\right]dzds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a\left[X(z)\right]dB(z)ds +$$

$$\int_{t_0}^{t}\int_{t_0}^{s} L^0 b\left[X(z)\right]dzdB(s) + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b\left[X(z)\right]dB(z)dB(s)$$

We can continue for instance by applying the Ito formula on $f = L^1 b\left[X(z)\right]$ to get

$$X(t) = X(t_0) + a\left[X(t_0)\right]\int_{t_0}^{t} ds + b\left[X(t_0)\right]\int_{t_0}^{t} dB(s)$$

$$+L^1 b\left[X(t_0)\right]\int_{t_0}^{t}\int_{t_0}^{s} dB(z)\,dB(u) + \overline{R}$$

$$\overline{R} = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a\left[X\left(z\right)\right] dz ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a\left[X\left(z\right)\right] dB(z) ds +$$

$$\int_{t_0}^{t}\int_{t_0}^{s} L^0 b\left[X\left(z\right)\right] dz dB(s) + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^0 L^1 b\left[X\left(u\right)\right] du dB(z) dB(s)$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^1 L^1 b\left[X\left(u\right)\right] dB(u) dB(z) dB(s)$$

Multiple Stochastic Integrals (MSI - s)

$$X(t) = X(t_0) + a\left[X(t_0)\right]\int_{t_0}^{t} ds + b\left[X(t_0)\right]\int_{t_0}^{t} dB(s)$$

$$+L^1 b\left[X(t_0)\right]\int_{t_0}^{t}\int_{t_0}^{s} dB(z)\,dB(u) + \overline{R}$$

Notice : the RHS has terms of the form

$$\int_{t_0}^{t} ds, \int_{t_0}^{t} dB(s), \text{ and } \int_{t_0}^{t}\int_{t_0}^{s} dB(z)\,dB(u).$$

These are called the multiple stochastic integrals.

The inclusion of higher order terms leads to more general forms of MSI-s.

Remarks

- Taylor’s series plays the central role in developing numerical integration schemes for ODE-s.
- Schemes with different “orders” can be derived by truncating the series at different levels.
- The numerical simulation of solutions of SDE-s is based on the application of truncated Ito-Taylor expansion

Consider the problem of numerical simulation of system governed by

$$dx(t) = a[x(t), t]dt + b[x(t), t]dB(t); x(t_0) = x_0$$

Sizes:

$$x\left(t\right) \sim d \times 1; dB(t) \sim m \times 1; a \sim d \times 1; b \sim d \times m$$

Time discretization:

$$0 = t_0 < t_1 < \cdots < t_N = T \text{ with } \Delta = T / N.$$

Notation: $Y_k\left(n\right) = x_k\left(t_n\right)$

1.5 order Strong Taylor scheme

$$Y_k(n+1) = Y_k(n) + a_k(n)\Delta + b_k(n)\Delta W + \frac{1}{2} L^1 b_k(n)\left\{\left(\Delta W\right)^2 - \Delta\right\}$$

$$+ L^1 a_k(n)\Delta Z + L^0 b_k(n)\left\{\Delta W \Delta - \Delta Z\right\} + \frac{1}{2} L^0 a_k(n)\Delta^2 + \frac{1}{2} L^1 L^1 b_k(n)\left\{\frac{1}{3}\left(\Delta W\right)^2 - \Delta\right\}\Delta W$$

$$L^0 = \frac{\partial}{\partial t} + \sum_{k=1}^{d} a_k \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k=1}^{d}\sum_{l=1}^{d} b_k b_l \frac{\partial^2}{\partial x_k \partial x_l}; L^1 = \sum_{k=1}^{d} b_k \frac{\partial}{\partial x_k}$$

$$\begin{Bmatrix} \Delta W \\ \Delta Z \end{Bmatrix} = \begin{bmatrix} \sqrt{\Delta} & 0 \\ 0.5\Delta^{1.5} & \frac{0.5\Delta^{1.5}}{\sqrt{3}} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}; \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \equiv N\left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Example

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = f(t)$$

$$x(0) = x_0 \;\; \& \quad \dot{x}(0) = \dot{x}_0 \qquad \langle f(t) \rangle = 0$$

$$< f(t_1) f(t_2) >= \sigma^2 \delta(t_1 - t_2)$$

$$\begin{Bmatrix} dx_1 \\ dx_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\eta\omega \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} dt + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} dB(t)$$

Discretization

$$Y_1(k+1) = Y_1(k) + a_1(k)\Delta + L^1 a_1(k)\Delta Z + \frac{1}{2} L^0 a_1(k)\Delta^2$$

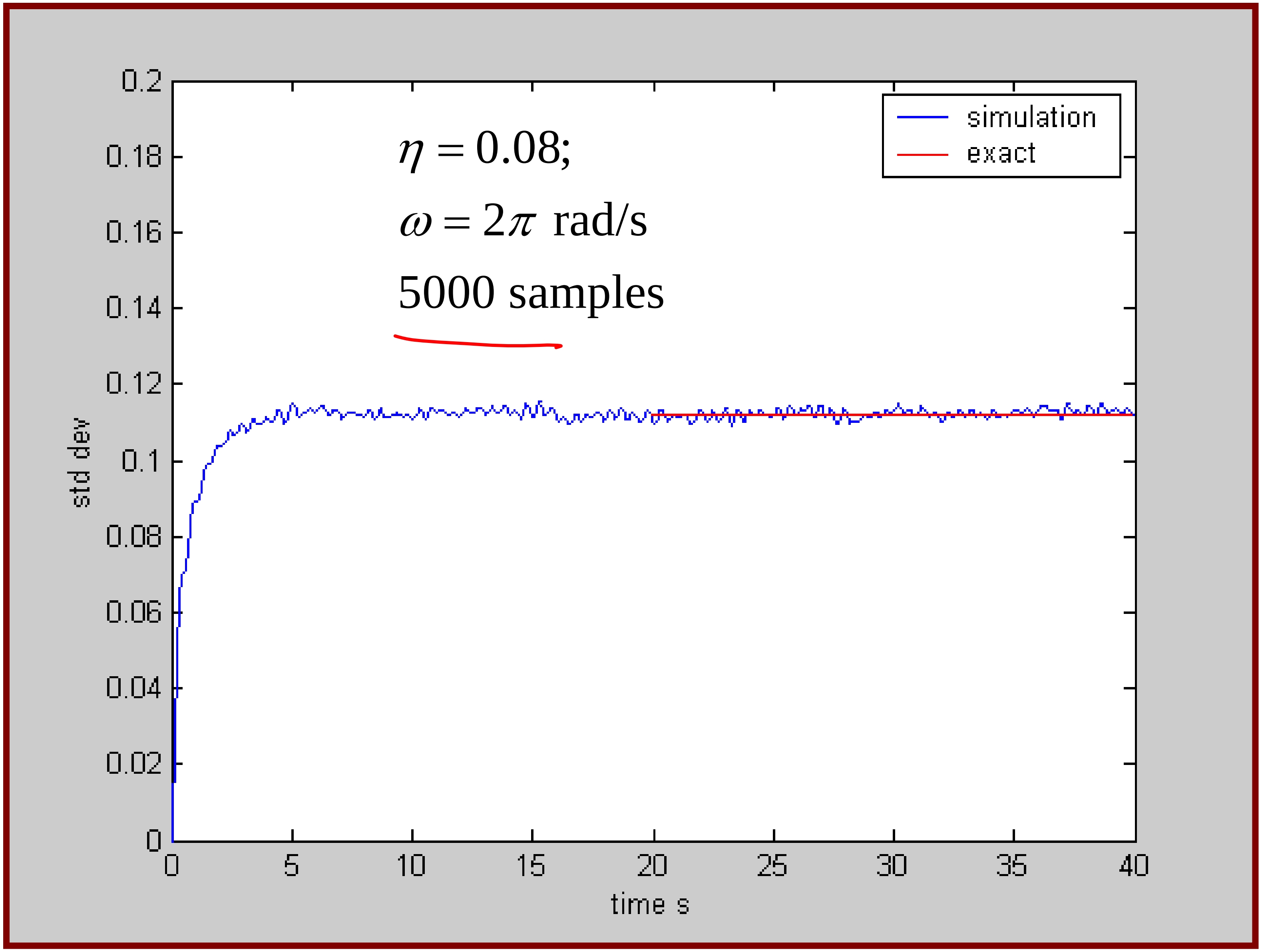
$$Y_2(k+1) = Y_2(k) + a_2(k)\Delta + b_2(k)\Delta W + L^1 a_2(k)\Delta Z + \frac{1}{2} L^0 a_2(k)\Delta^2$$

$$a_1(k) = Y_2(k);\ a_2(k) = -\left[2\eta\omega Y_2(k) - \omega^2 Y_1(k)\right];$$

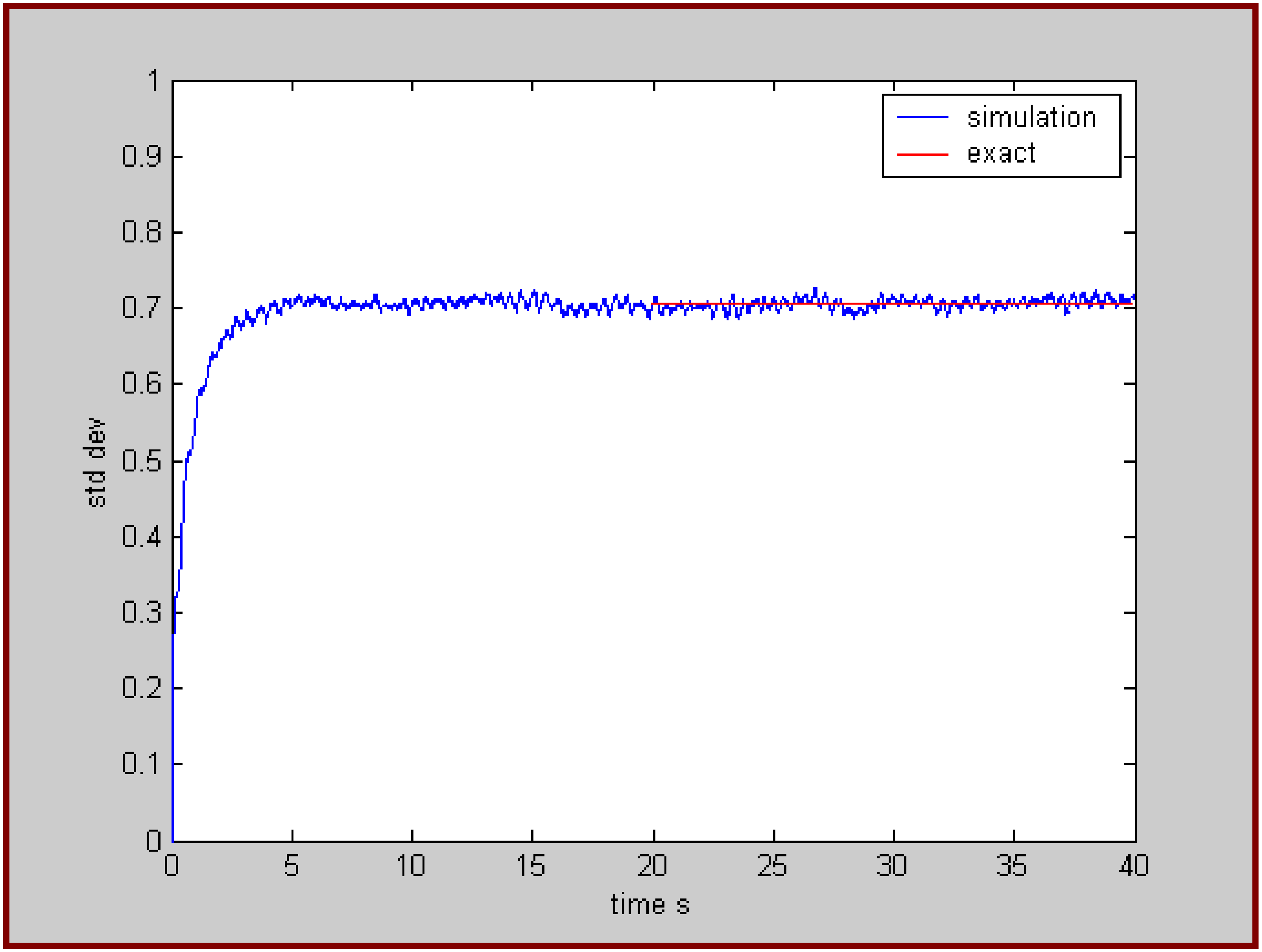
$$L^0 a_1(k) = a_1(k)\left(-\omega^2\right) + a_2(k)\left(-2\eta\omega\right);$$

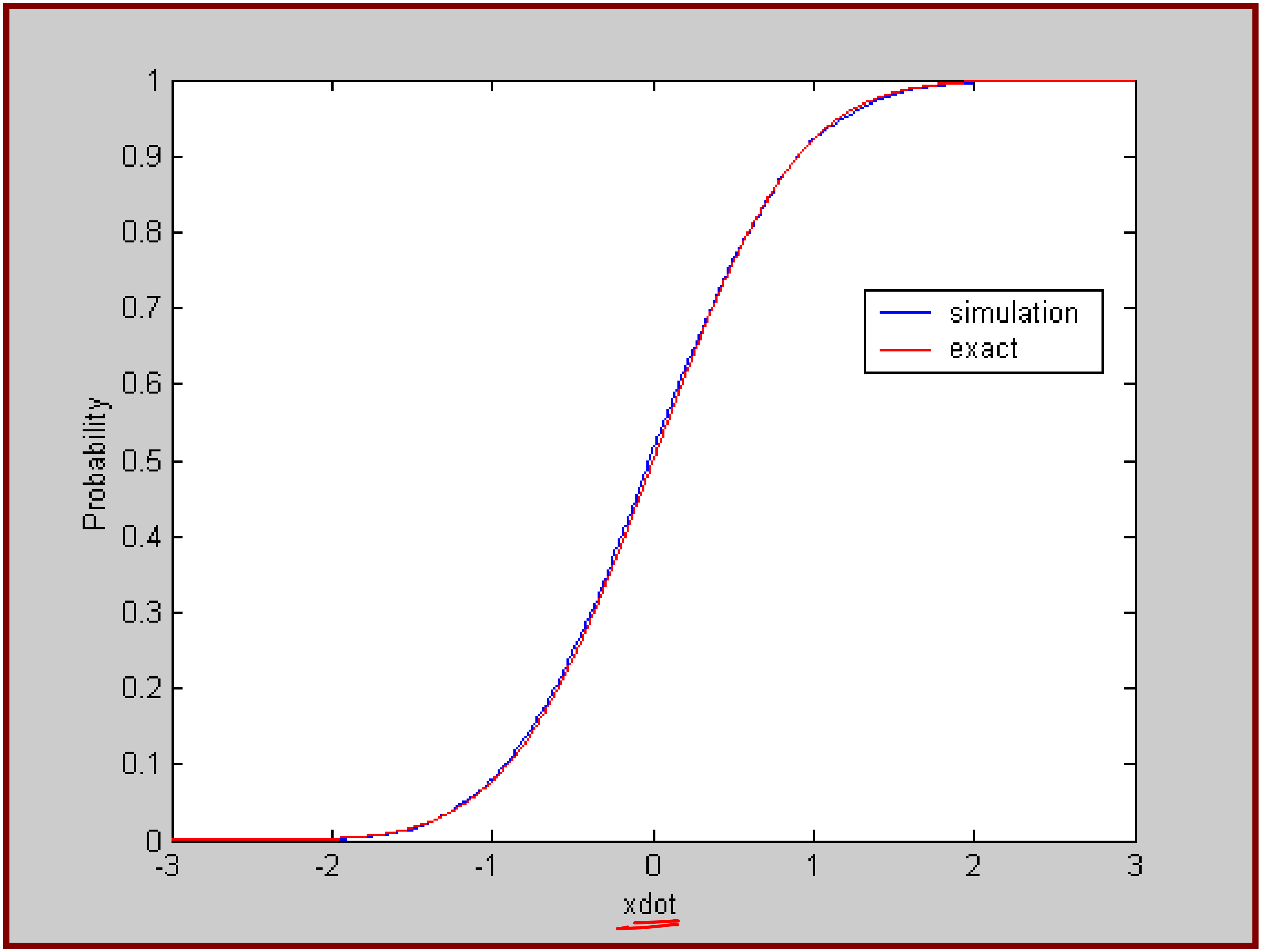
$$L^0 a_2(k) = a_1(k)\left(-\omega^2\right) + a_2(k)\left(-2\eta\omega\right);$$

$$L^1 a_1(k) = \sigma;\ L^2 a_2(k) = \sigma\left(-2\eta\omega\right)$$

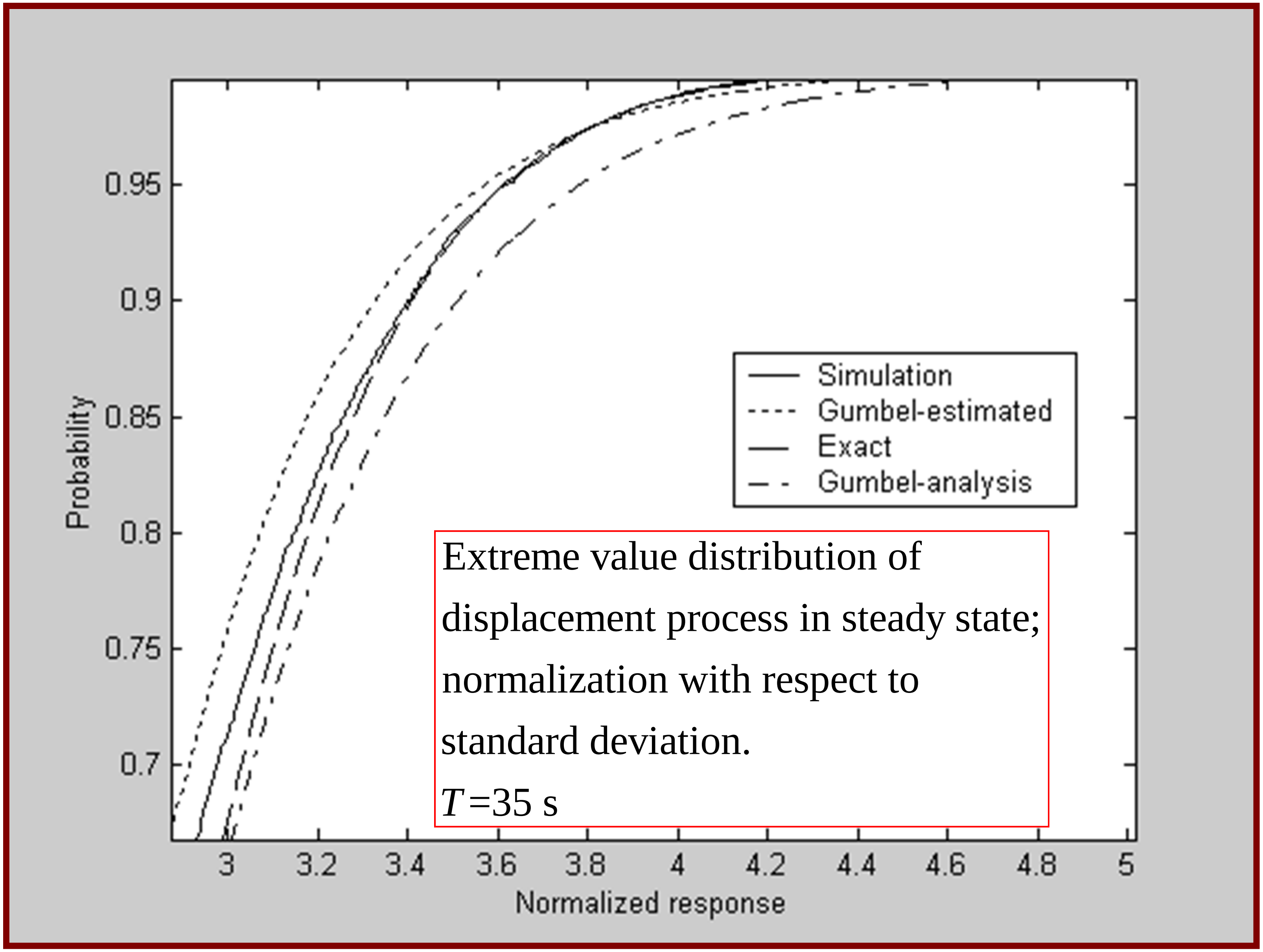


$\eta = 0.08;$
$\omega = 2\pi$ rad/s
5000 samples
simulation
exact
std dev
time s
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
25
30
35
40


simulation
exact
std dev
time s
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
5
10
15
20
25
30
35
40



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Probability
-3
-2
-1
0
1
2
3
xdot
simulation
exact


0.95
0.9
0.85
0.8
0.75
0.7
Probability
3
3.2
3.4
3.6
3.8
4
4.2
4.4
4.6
4.8
5
Normalized response
Simulation
Gumbel-estimated
Exact
Gumbel-analysis
Extreme value distribution of displacement process in steady state; normalization with respect to standard deviation.
T =35 s

Duffing Van Der Pol Oscillator under white noise

$$\ddot{x} + 2\eta\omega\dot{x} - \varepsilon\dot{x}(1-4\dot{x}^2) + \omega^2 x + \alpha x^3 = f(t)$$

$$x(0) = x_0 \ \& \quad \dot{x}(0) = \dot{x}_0$$

$$< f(t_1) f(t_2) >= \sigma^2 \delta(t_1 - t_2)$$

$$dx_1(t) = x_2 dt$$

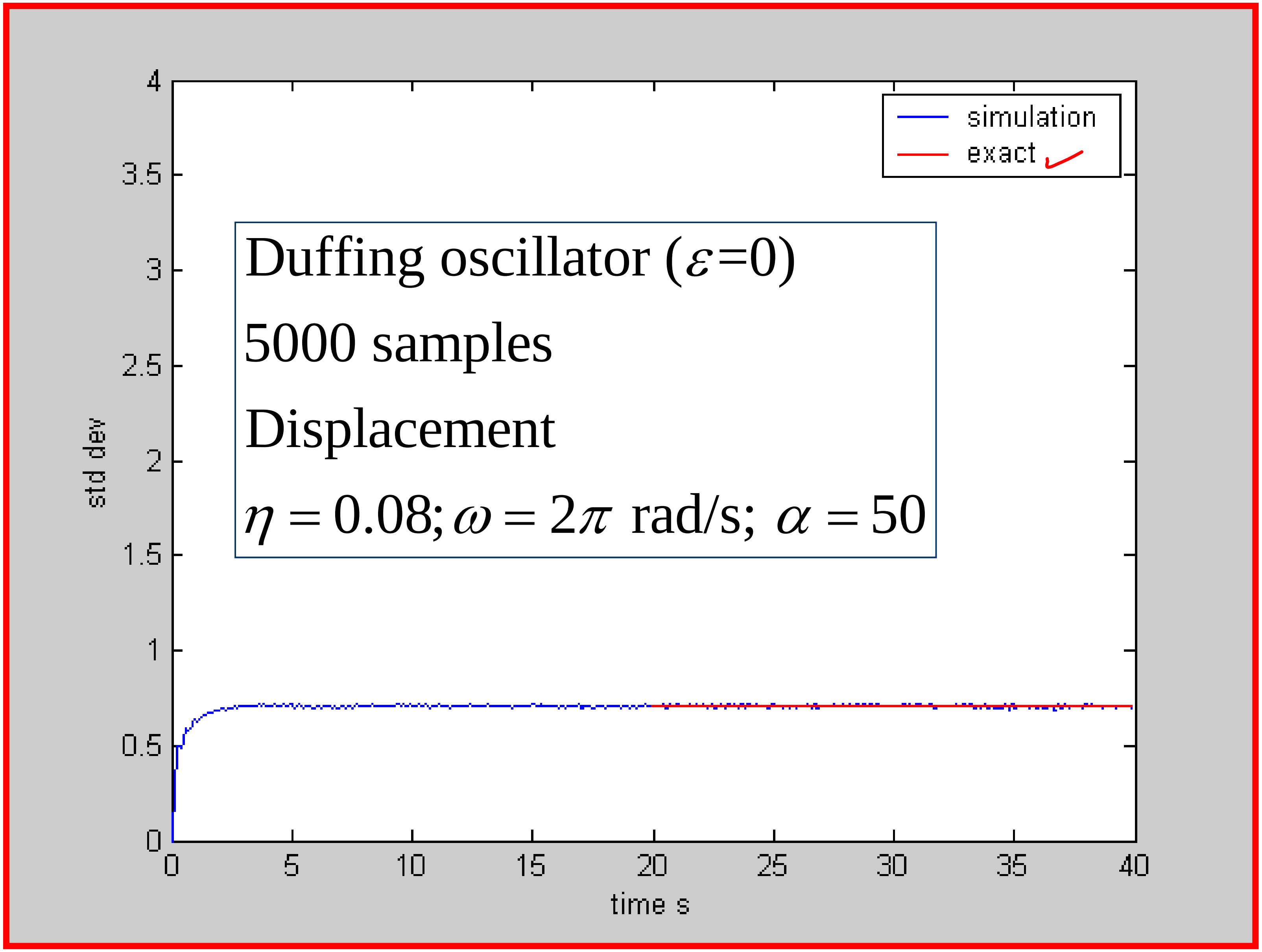
$$dx_2(t) = \left\{-2\eta\omega x_2 + \varepsilon x_2(1-4x_2^2) - \omega^2 x_1 - \alpha x_1^3\right\} dt + \sigma dw(t)$$

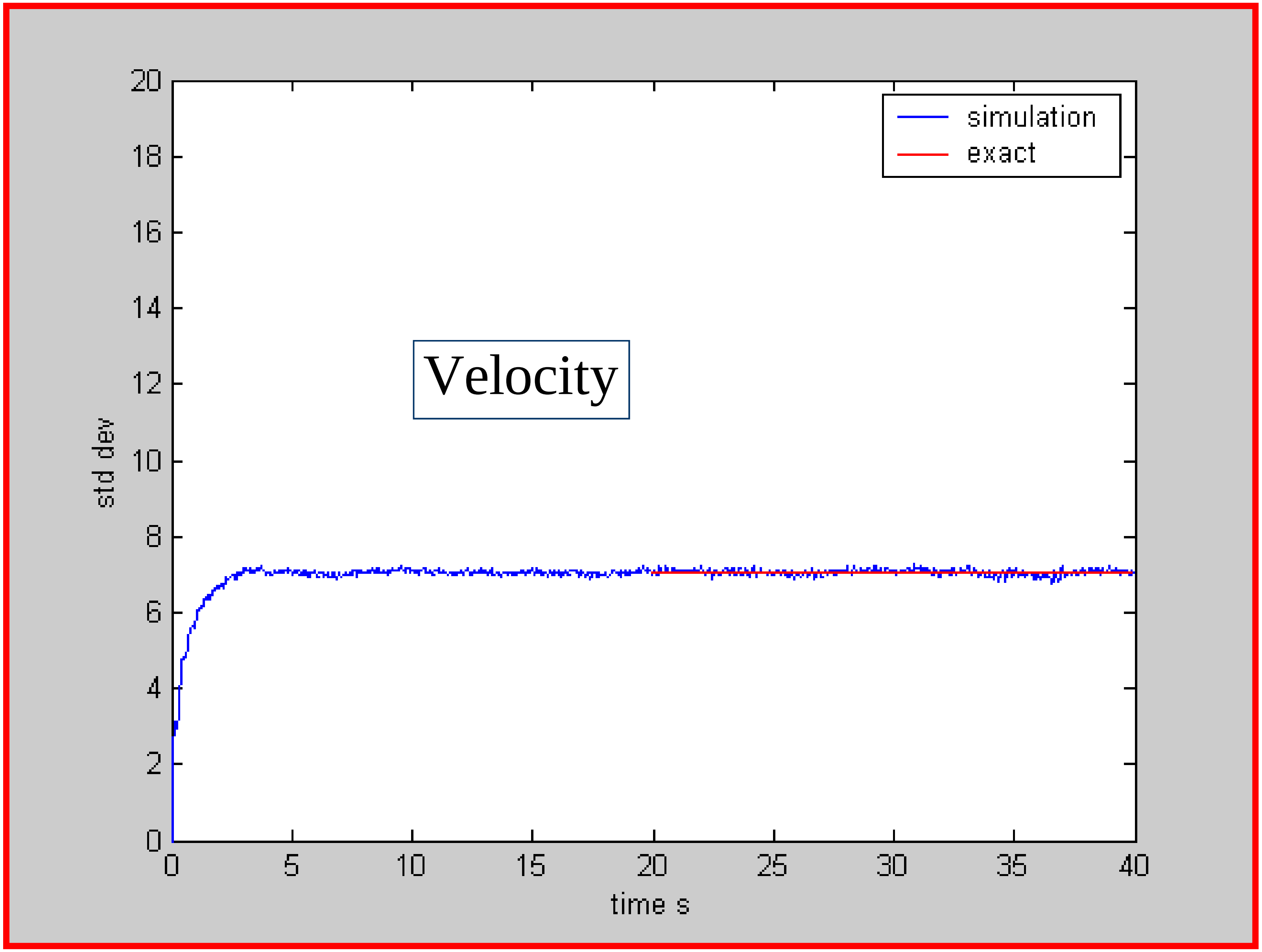
$$d = 2, m = 1,$$

$$a_1 = x_2$$

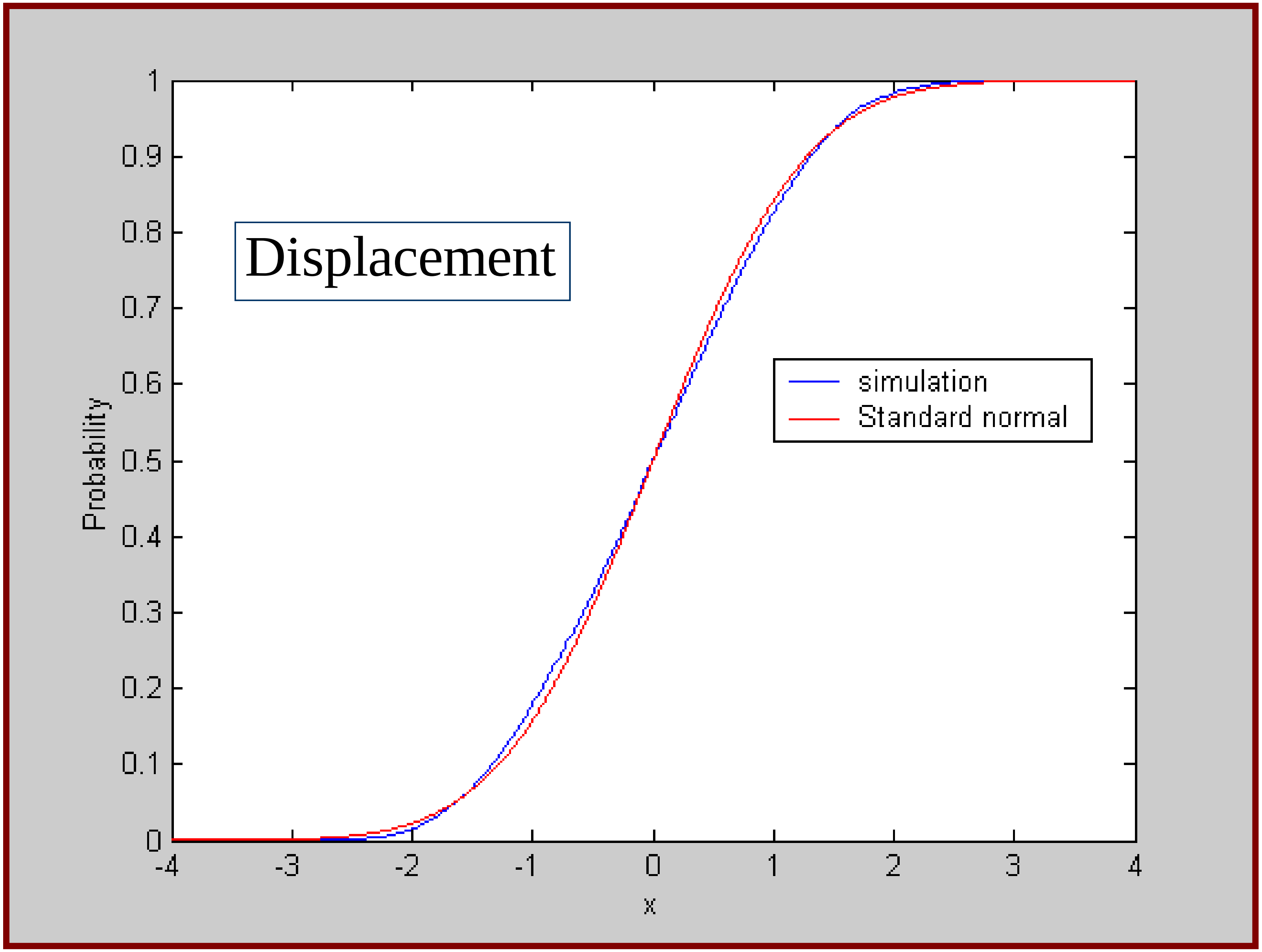
$$a_2 = -2\eta\omega x_2 + \varepsilon x_2(1-4x_2^2) - \omega^2 x_1 - \alpha x_1^3$$

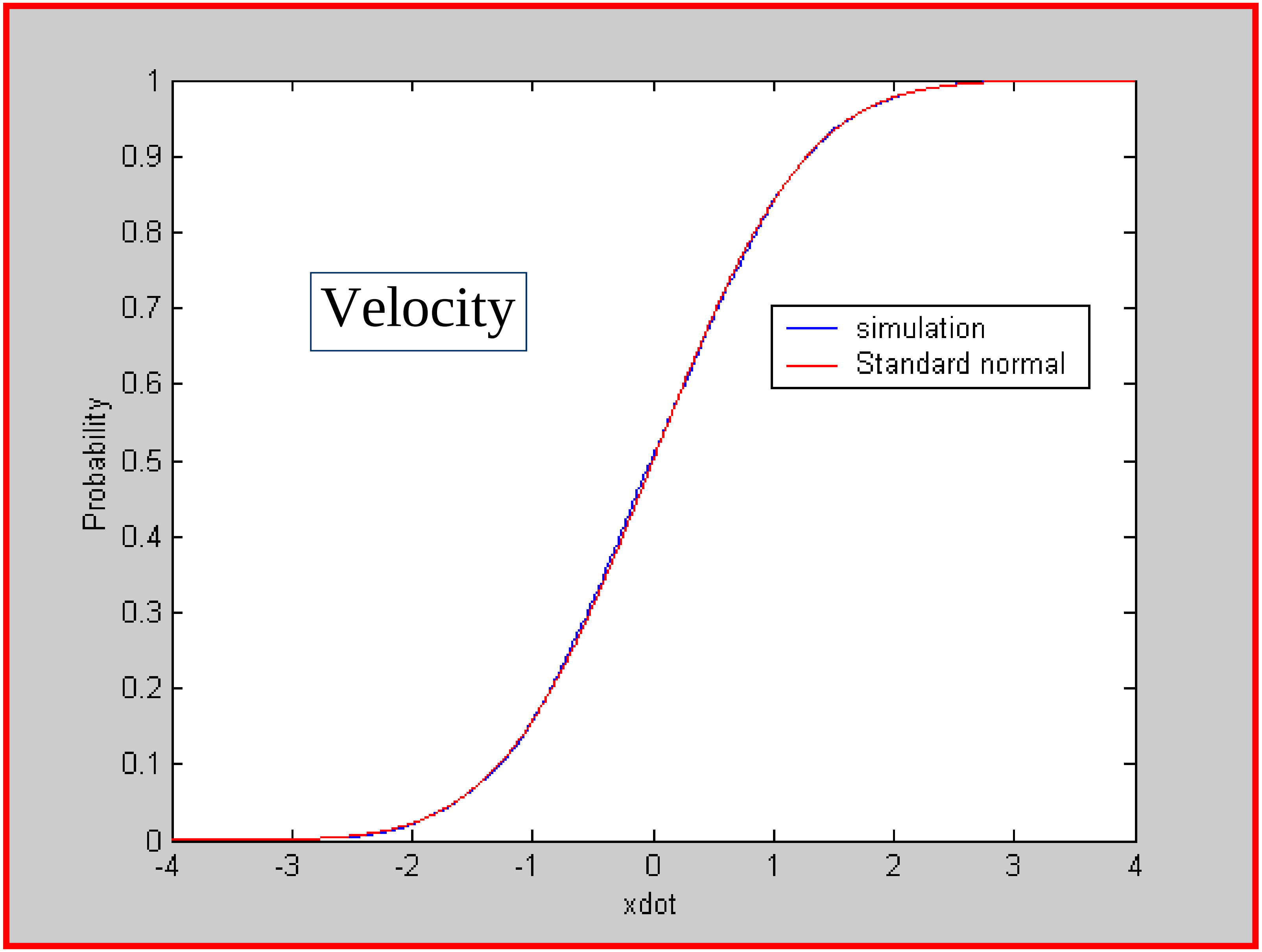
$$b_1 = 0; b_2 = \sigma$$


Duffing oscillator (ε=0)
5000 samples
Displacement
$\eta = 0.08; \omega = 2\pi$ rad/s; $\alpha = 50$
simulation
exact
std dev
time s
0
0.5
1
1.5
2
2.5
3
3.5
4
0
5
10
15
20
25
30
35
40

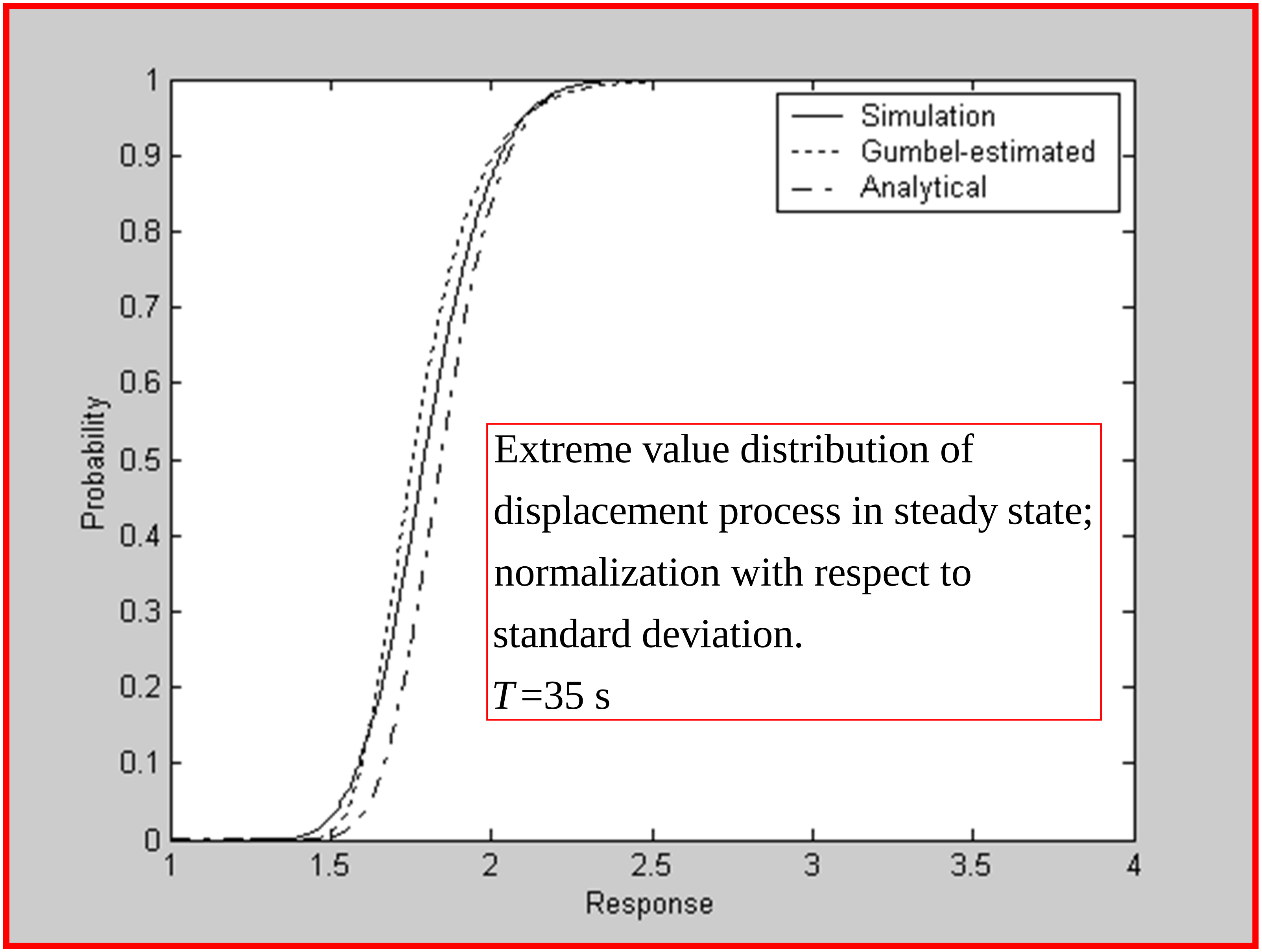


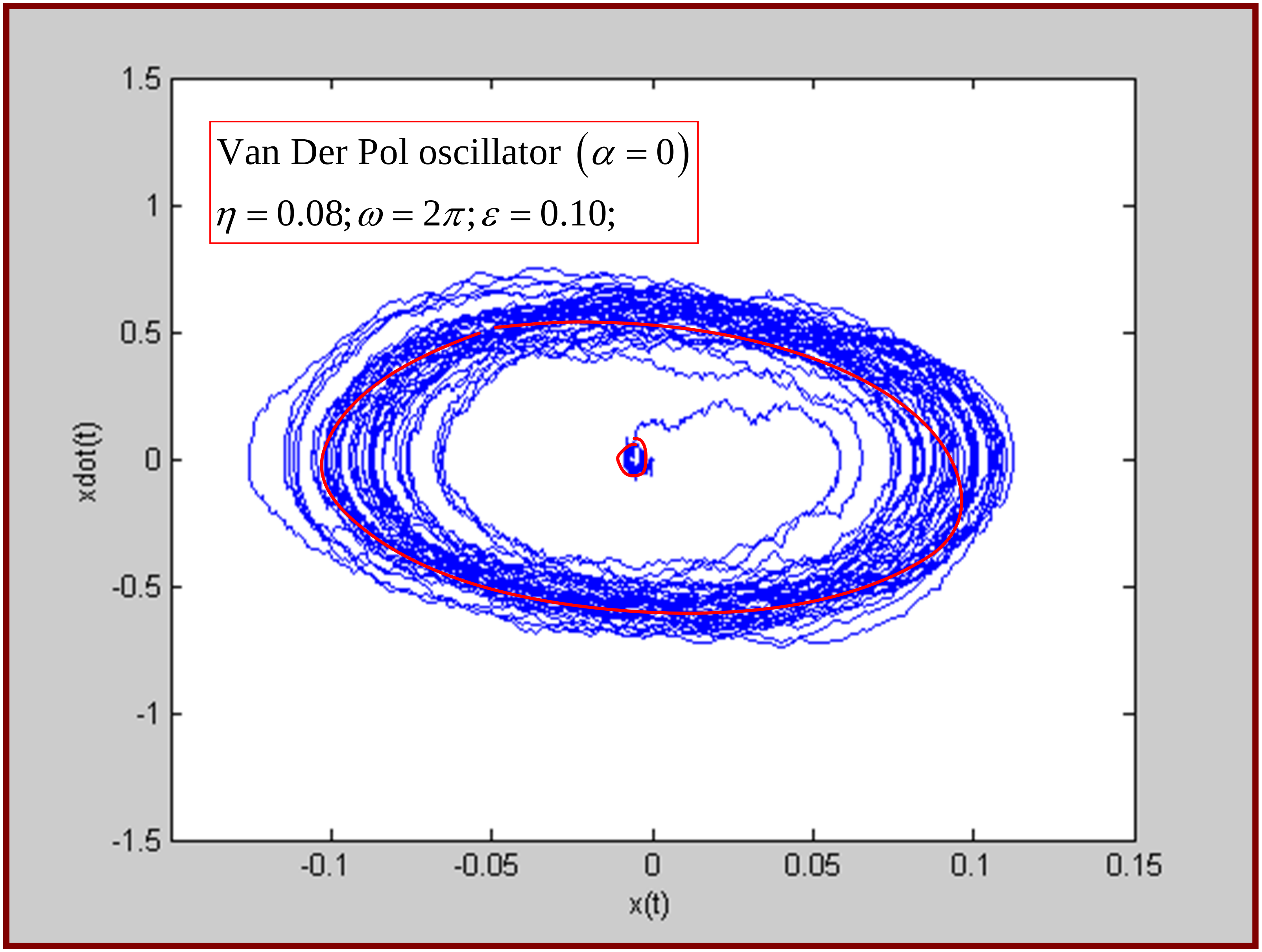
Velocity
simulation
exact
std dev
time s
0
2
4
6
8
10
12
14
16
18
20
0
5
10
15
20
25
30
35
40


Displacement
simulation
Standard normal
Probability
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4

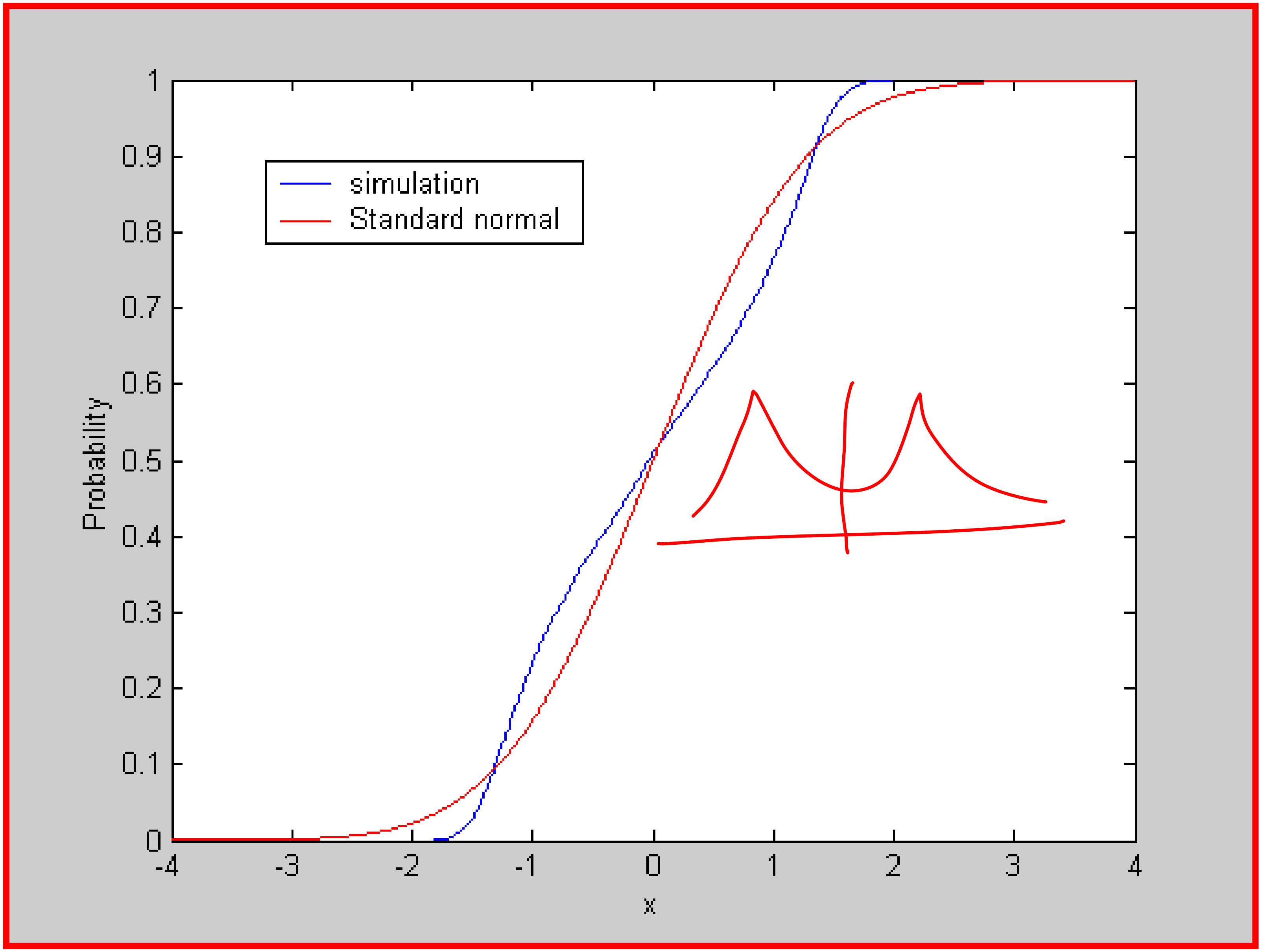


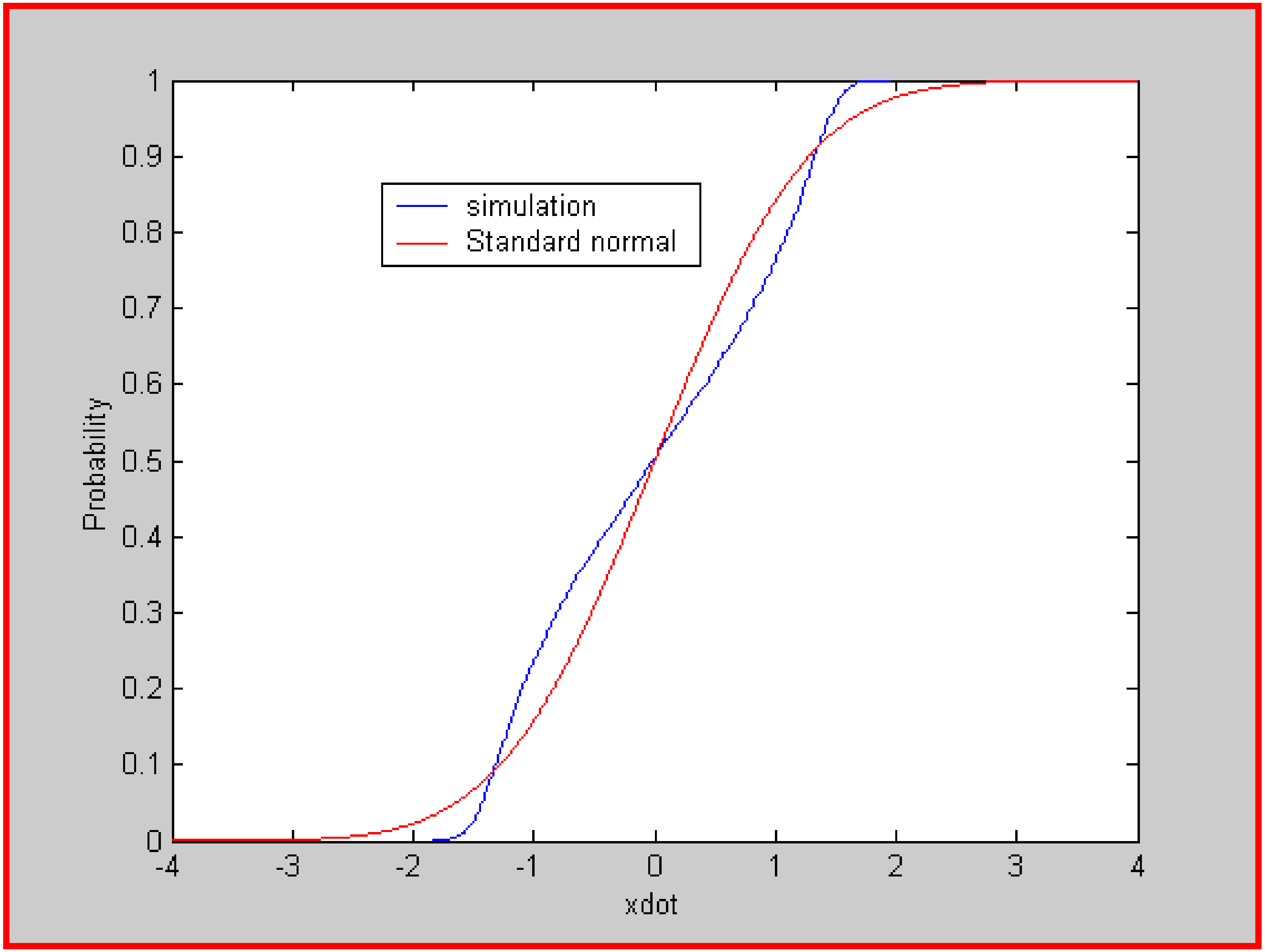
Velocity
simulation
Standard normal
Probability
xdot
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4


Simulation
Gumbel-estimated
Analytical
Probability
Response
Extreme value distribution of displacement process in steady state; normalization with respect to standard deviation.
T=35 s

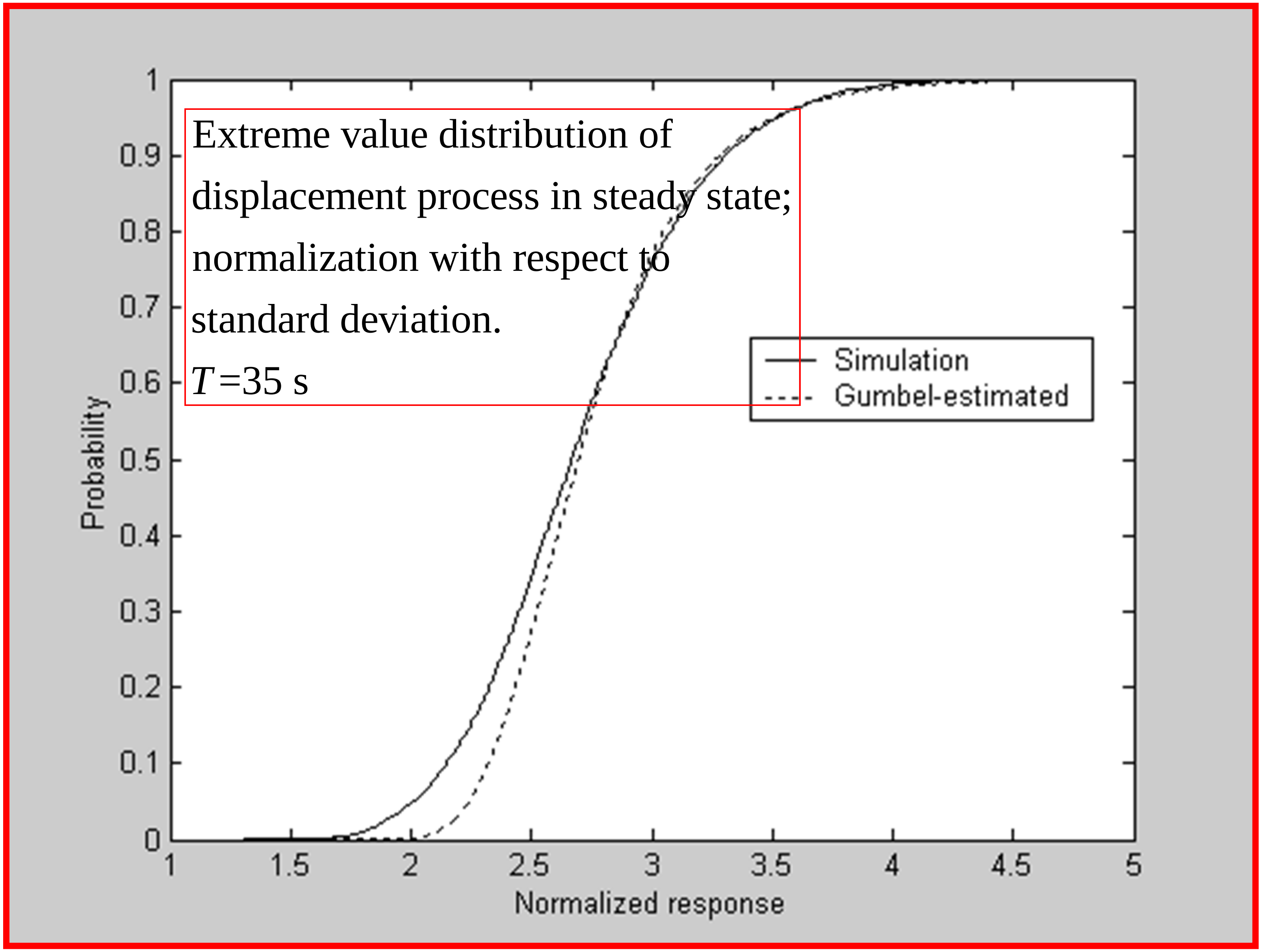


Van Der Pol oscillator $(\alpha = 0)$
$\eta = 0.08; \omega = 2\pi; \varepsilon = 0.10;$
1.5
1
0.5
0
-0.5
-1
-1.5
xdot(t)
-0.1
-0.05
0
0.05
0.1
0.15
x(t)

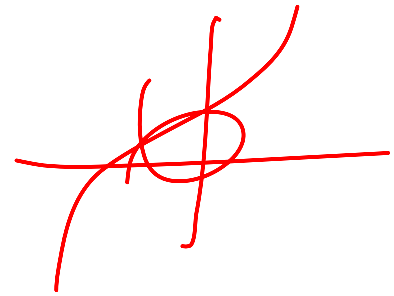

1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Probability
-4
-3
-2
-1
0
1
2
3
4
x
simulation
Standard normal



simulation
Standard normal
Probability
xdot
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4

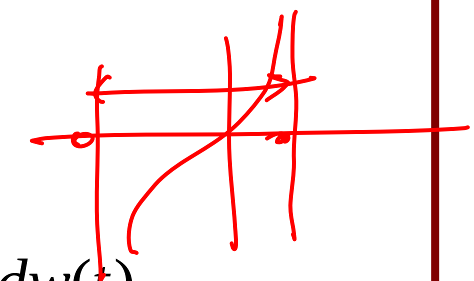

Extreme value distribution of
displacement process in steady state;
normalization with respect to
standard deviation.
T =35 s
Simulation
Gumbel-estimated
Probability
Normalized response
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1
1.5
2
2.5
3
3.5
4
4.5
5

System with tangent stiffness



$$\ddot{x} + 2\alpha\dot{x} + \frac{2d\omega^2}{\pi}\tan\left(\frac{\pi x}{2\alpha}\right) = f(t)$$

$$x(0) = x_0; \dot{x}(0) = \dot{x}_0 \; < f(t) >= 0; < f(t_1) f(t_2) >= \sigma^2\delta(t_1 - t_2)$$



$$dx_1(t) = x_2 dt$$

$$dx_2(t) = \left(-2\alpha x_2 - \frac{2d\omega^2}{\pi m}\tan\left\{\frac{\pi x_1}{2d}\right\}\right)dt + \sigma dw(t)$$

$$a_1 = x_2; a_2 = -2\alpha x_2 - \frac{2d\omega^2}{\pi m}\tan\left\{\frac{\pi x_1}{2d}\right\}$$

$$b_1 = 0; b_2 = \sigma$$

Notations

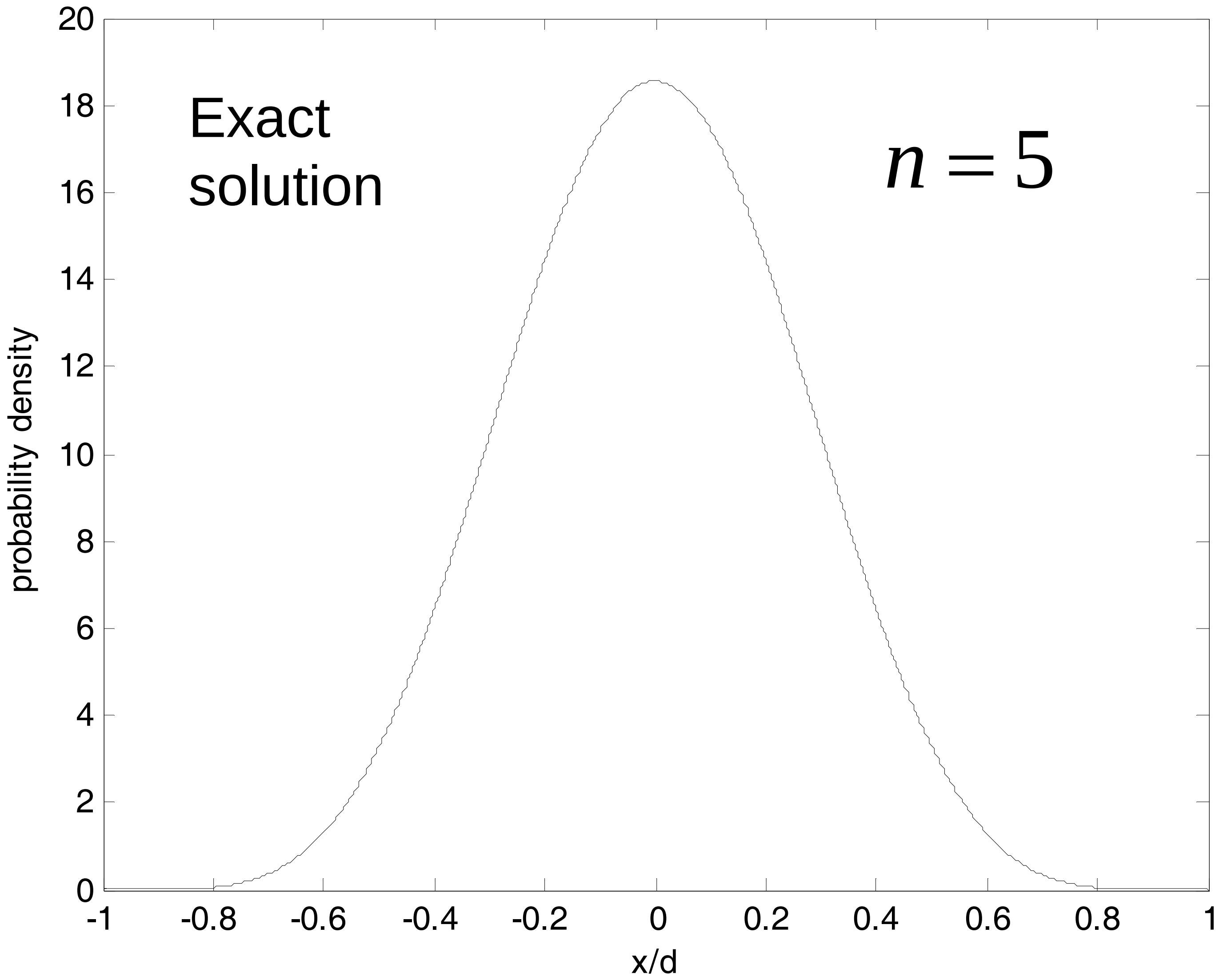
$$\sigma_0^2 = \frac{\pi S}{2\alpha\omega^2}; \quad \sigma^2 = 2\pi S; \quad n = \frac{4d^2}{\pi^2 \sigma_0^2}$$

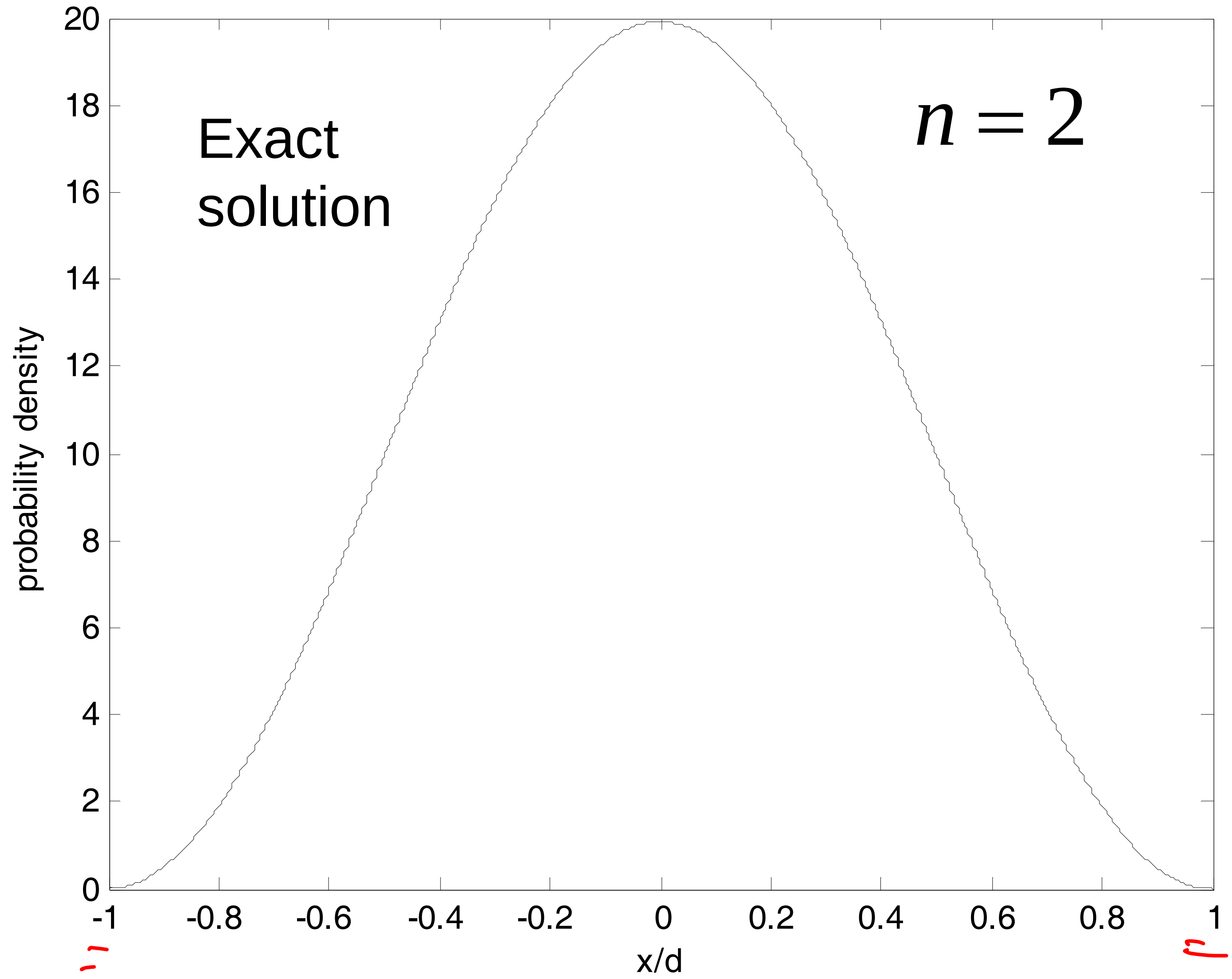
Numerical values

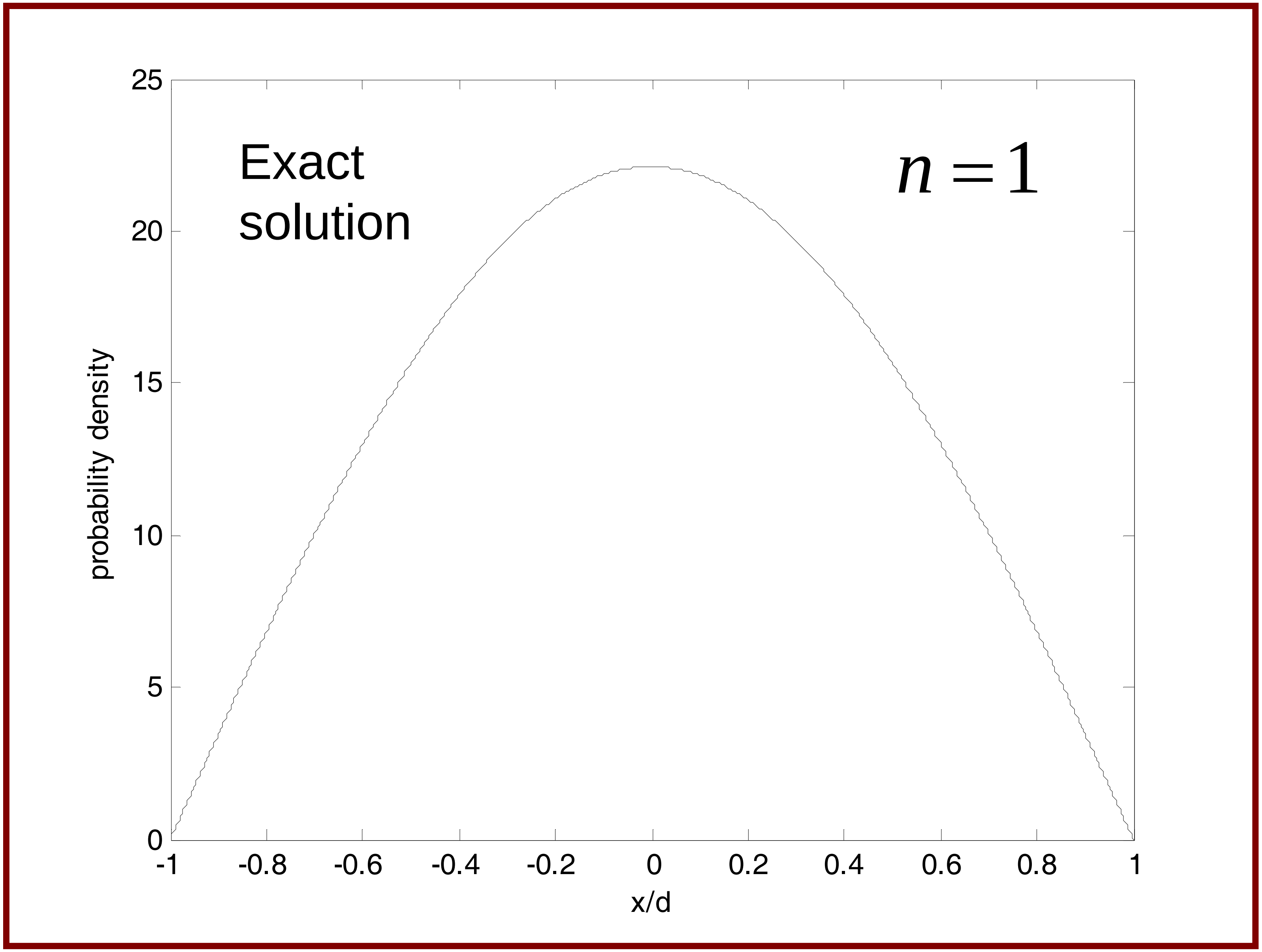
$S = 0.0040, T = 30 \text{ s}, t_0 = 5s, \omega = 10 \text{ rad/s},$

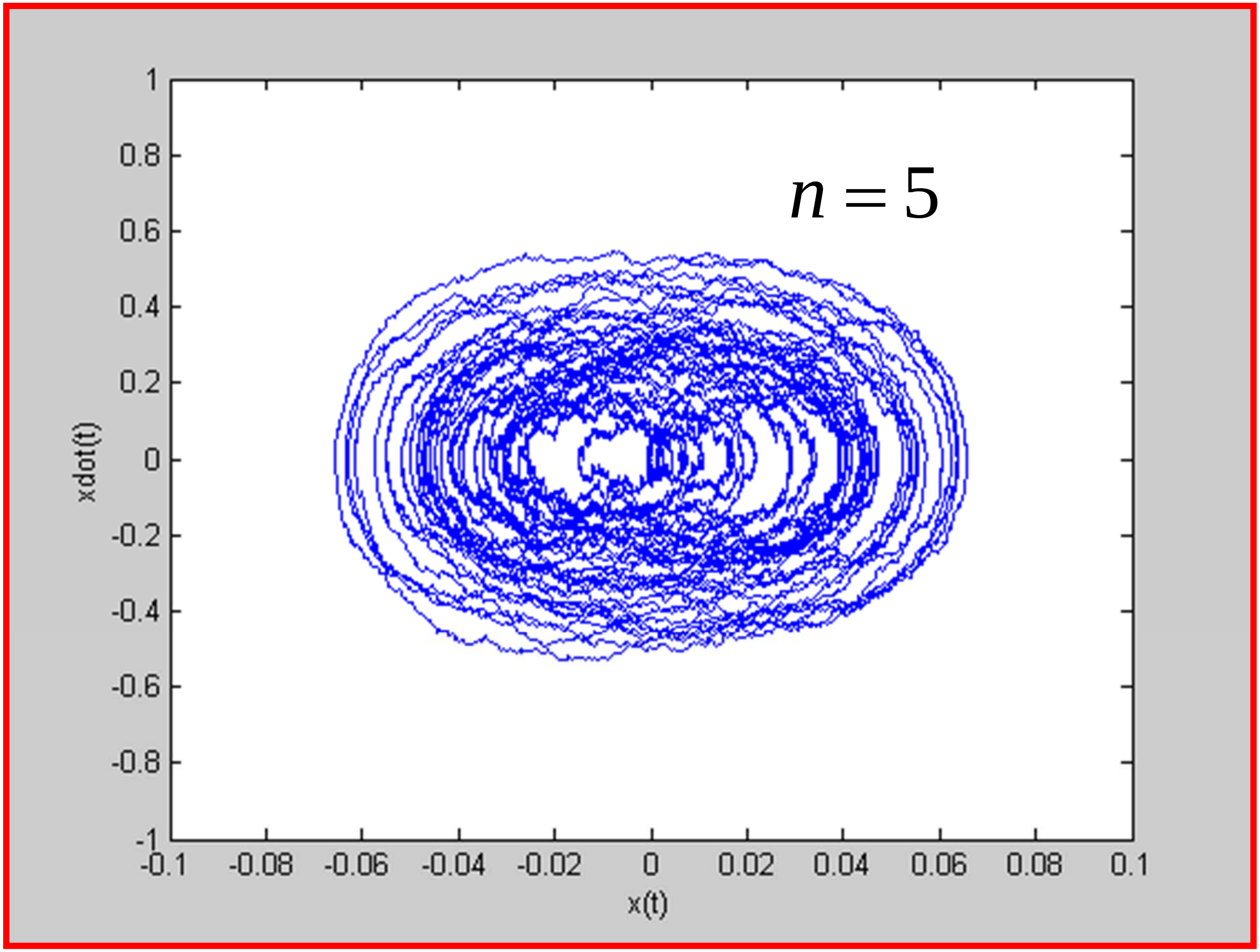
and $\alpha = 0.05\omega$

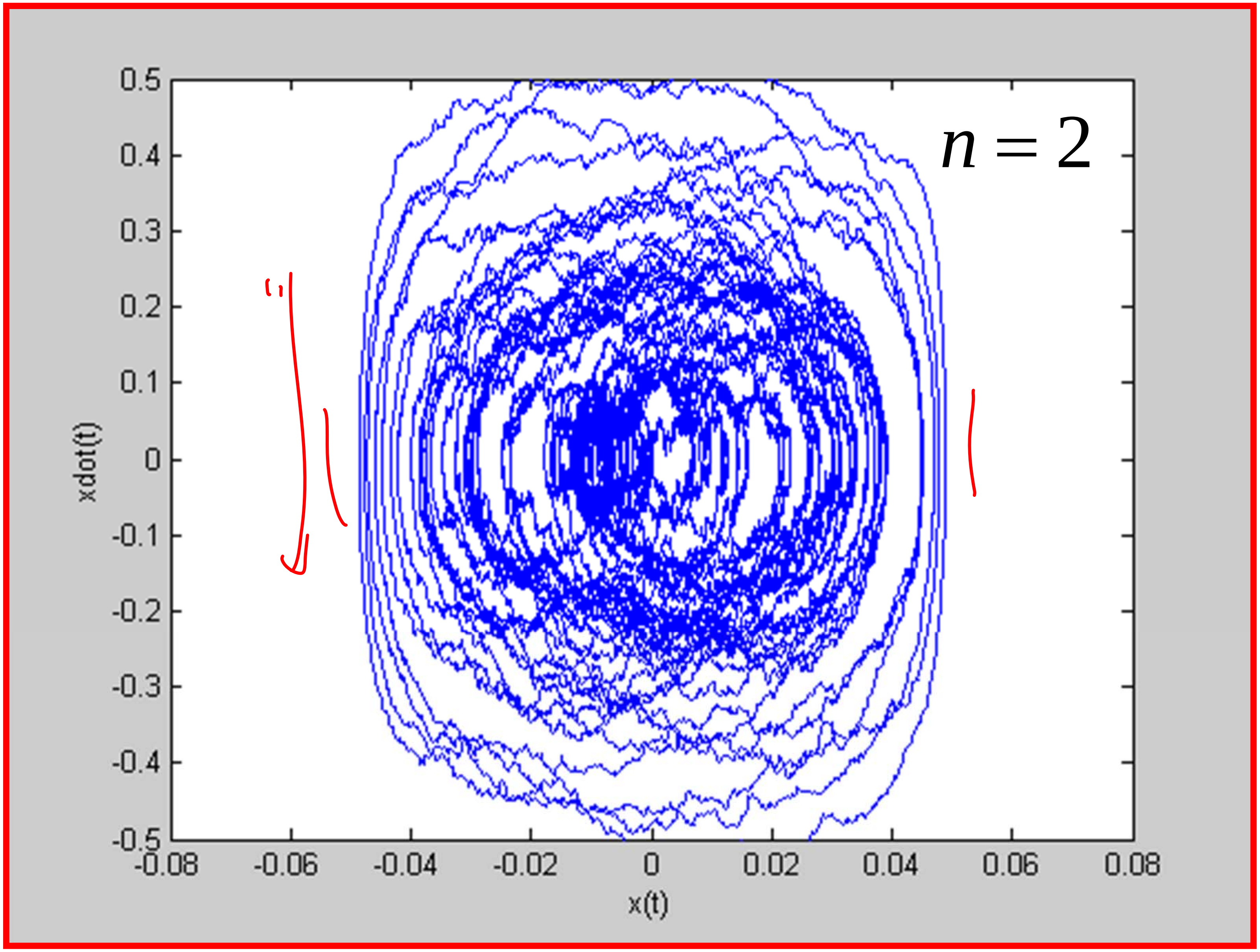
5000 samples.

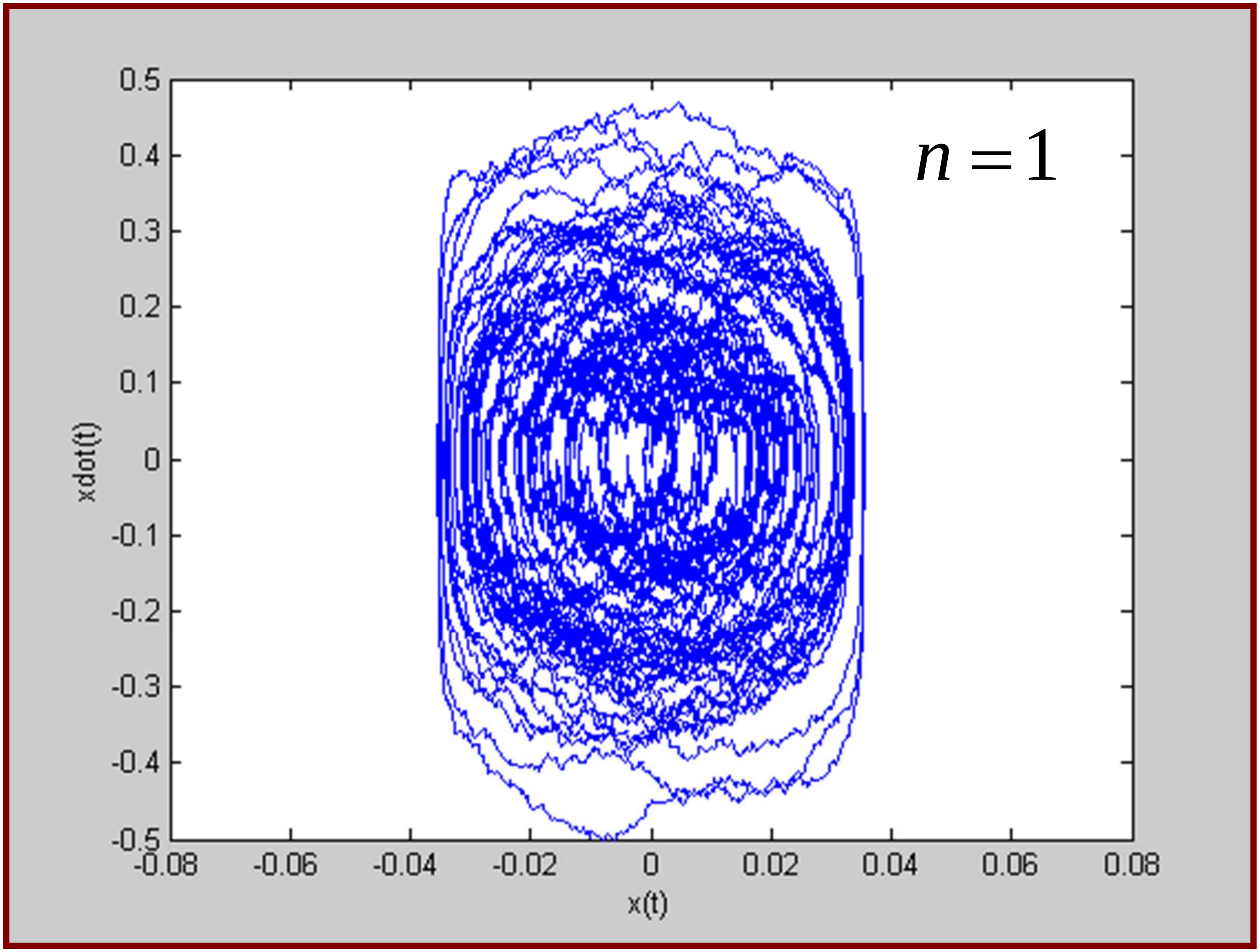

Exact
solution
$n = 5$
probability density
x/d
20
18
16
14
12
10
8
6
4
2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Exact
solution
$n = 2$
probability density
x/d
20
18
16
14
12
10
8
6
4
2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

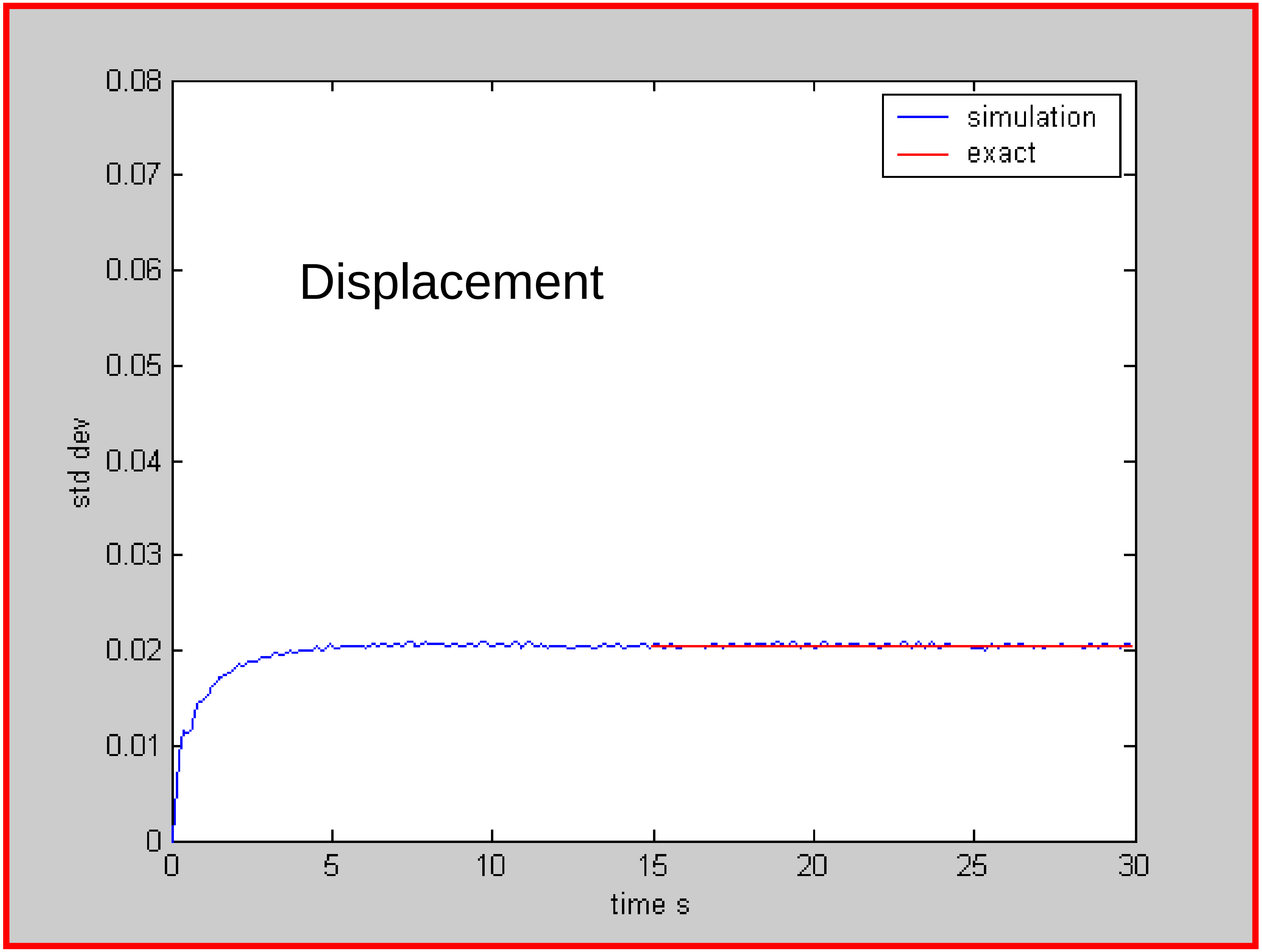

Exact
solution
$n = 1$
probability density
x/d
25
20
15
10
5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

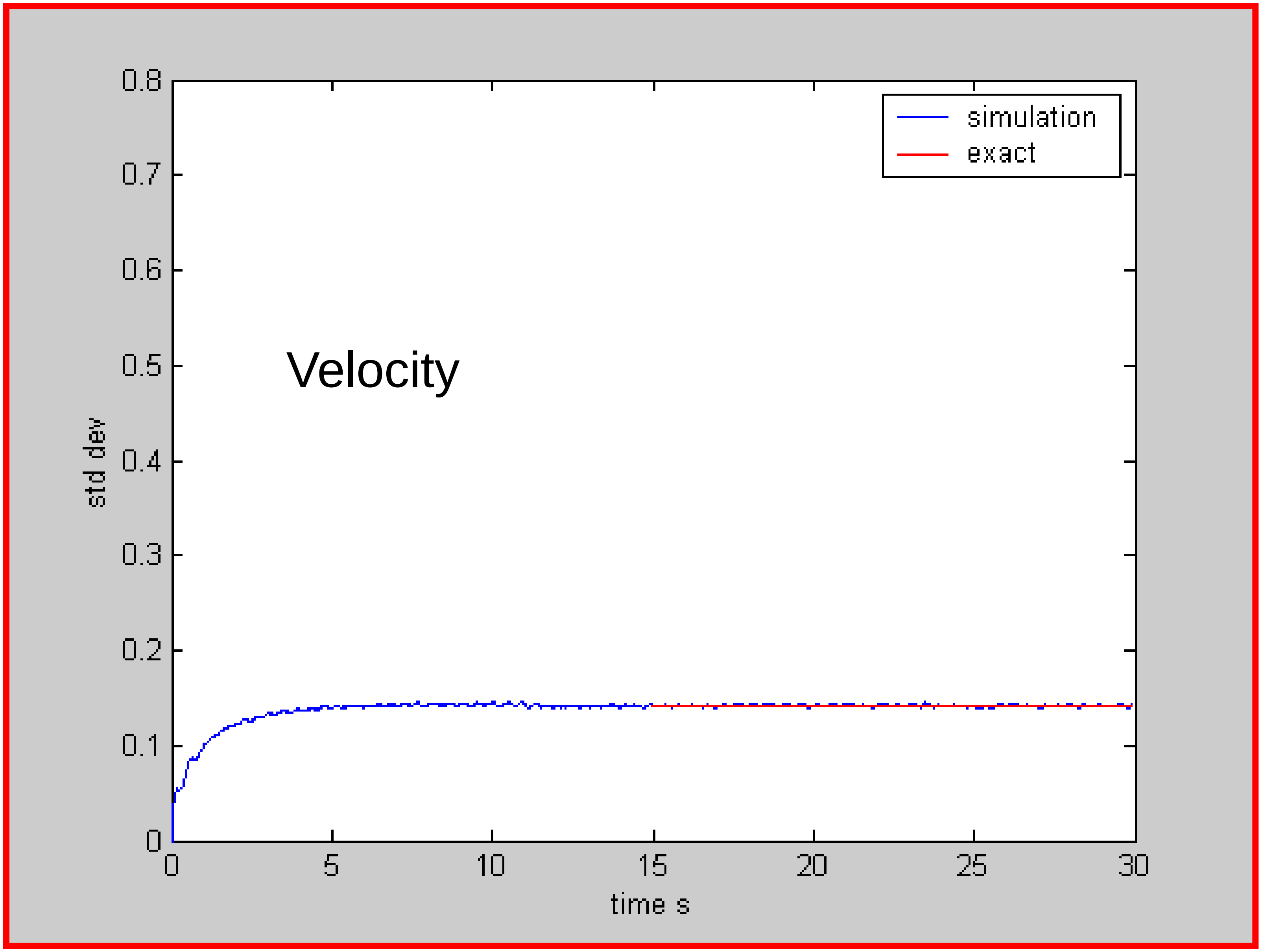

$n = 5$
xdot(t)
x(t)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.1
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.1


$n = 2$
xdot(t)
x(t)
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08



0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
xdot(t)
-0.08
-0.06
-0.04
-0.02
0
0.02
0.04
0.06
0.08
x(t)
$n=1$


Displacement
simulation
exact
std dev
time s
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
5
10
15
20
25
30



0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
std dev
Velocity
simulation
exact
0
5
10
15
20
25
30
time s

2-dof system with cubic nonlinearities

$$\ddot{x}_1 + 2\eta_1\omega_1\dot{x}_1 + \alpha_1 x_1^3 + \alpha_2 x_1^2 x_2 + \alpha_3 x_1 x_2^3 + \alpha_4 x_2^3 = f_1(t)$$

$$\ddot{x}_2 + 2\eta_2\omega_2\dot{x}_2 + \beta_1 x_1^3 + \beta_2 x_1^2 x_2 + \beta_3 x_1 x_2^3 + \beta_4 x_2^3 = f_2(t)$$

$$x_i(0) = x_{i0}; \dot{x}_i(0) = \dot{x}_{i0}; i = 1, 2$$

$$< f_i(t) >= 0; i = 1, 2$$

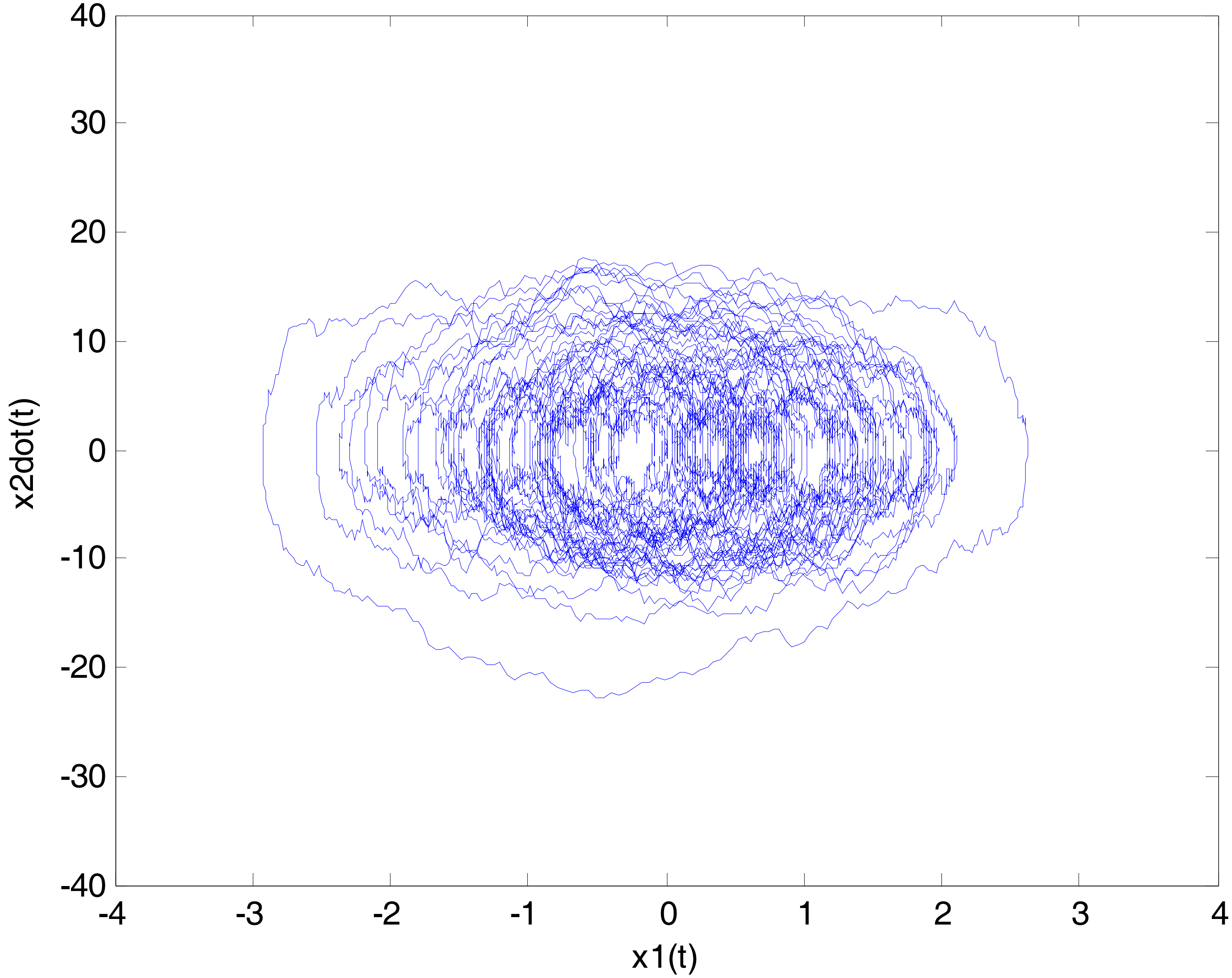
$$< f_1(t_1) f_2(t_2) >= 0$$

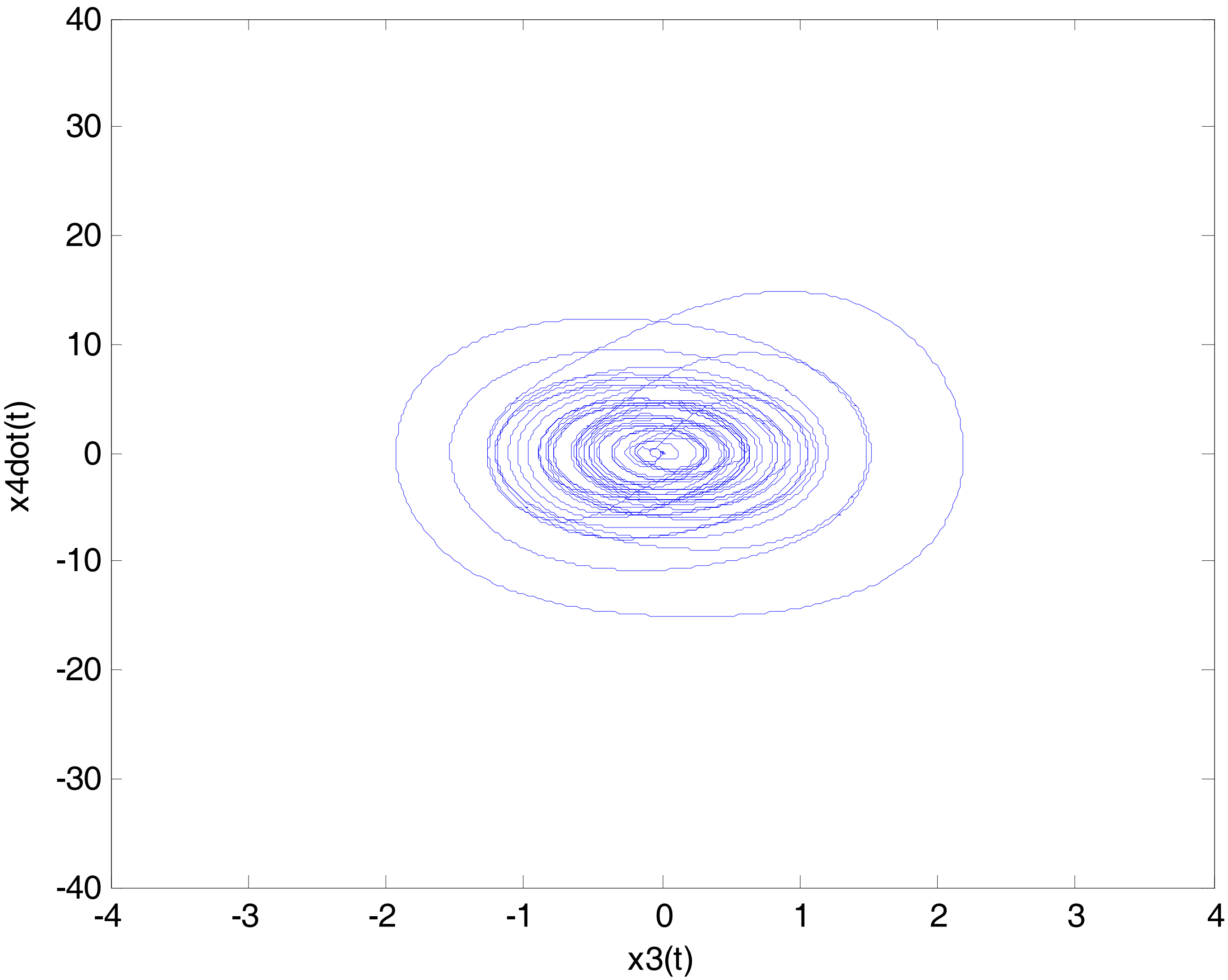
$$< f_i(t_1) f_i(t_2) >= \sigma_i^2 \delta(t_1 - t_2)$$

Recast as SDE - s

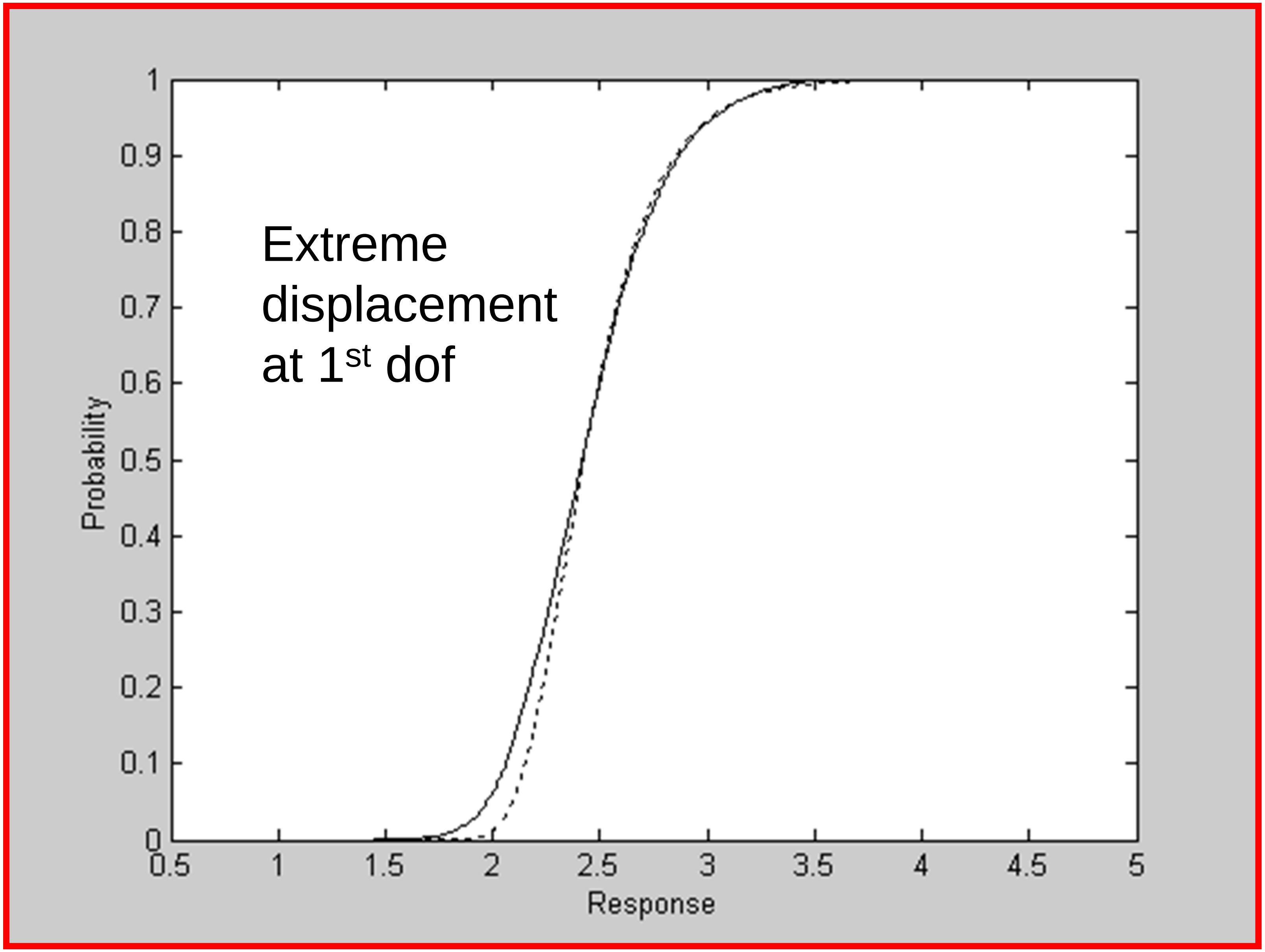
$$dy_1(t) = y_2 dt$$
$$dy_2(t) = \left(-2\eta_1\omega_1 y_2 - \omega_1^2 y_1 - \alpha_1 y_1^3 - \alpha_2 y_1^2 y_3 - \alpha_3 y_1 y_2^2 - \alpha_4 y_2^3\right) dt + \sigma_1 dB_1(t)$$
$$dy_3(t) = y_4 dt$$
$$dy_4(t) = \left(-2\eta_1\omega_1 y_3 - \omega_2^2 y_3 - \beta_1 y_1^3 - \beta_2 y_1^2 y_3 - \beta_3 y_1 y_2^2 - \beta_4 y_2^3\right) dt + \sigma_2 dB_2(t)$$

$\omega_1 = \omega_2 = 2\pi$ rad/s; $\eta_1 = 0.08, \eta_2 = 0.05, \alpha_1 = 2, \alpha_2 = 4, \alpha_3 = 5, \alpha_4 = 2.5,$
$\beta_1 = 4, \beta_2 = 5, \beta_3 = 1.5, \beta_4 = 4, \sigma_1 = 10, \sigma_2 = 0$
5000 samples


40
30
20
10
0
-10
-20
-30
-40
x2dot(t)
-4
-3
-2
-1
0
1
2
3
4
x1(t)



40
30
20
10
0
-10
-20
-30
-40
x4dot(t)
-4
-3
-2
-1
0
1
2
3
4
x3(t)


Extreme displacement at 1st dof
Probability
Response
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5