Stochastic Structural Dynamics

Lecture-28

Monte Carlo simulation approach-4

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Recall

Pseudorandom number generator

Simulation of random variables

- Scalar/vector
- Gaussian/non-Gaussian
- Completely specified/partially specified

Methods

- Transformation method
- Accept-Reject method

Fourier representation of a Gaussian random process

Let $X(t)$ be a zero mean, stationary, Gaussian random process defined as

$$X(t) = \sum_{n=1}^{\infty} a_n \cos \omega_n t + b_n \sin\omega_n t; \;\; \omega_n = n\omega_0$$

Assumptions

Here $a_n \sim N(0, \sigma_n), b_n \sim N(0, \sigma_n)$,

$\langle a_n a_k \rangle = 0 \forall n \neq k, \langle b_n b_k \rangle = 0 \forall n \neq k,$

$\langle a_n b_k \rangle = 0 \forall n, k = 1, 2, \cdots, \infty$

$$\Rightarrow \langle X(t) \rangle = \sum_{n=1}^{\infty} \left\{ \langle a_n \rangle \cos \omega_n t + \langle b_n \rangle \sin \omega_n t \right\} = 0$$

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$$\left\langle X(t)\,X(t+\tau)\right\rangle =$$

$$\left\langle \sum_{n=1}^{\infty}\left\{a_n \cos\omega_n t + b_n \sin\omega_n t\right\}\sum_{n=1}^{\infty}\left\{a_n \cos\omega_n (t+\tau) + b_n \sin\omega_n (t+\tau)\right\}\right\rangle$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\langle\left(a_n \cos\omega_n t + b_n \sin\omega_n t\right)\left(a_n \cos\omega_n (t+\tau) + b_n \sin\omega_n (t+\tau)\right)\right\rangle$$

$$\Rightarrow R_{XX}(\tau) = \sum_{n=1}^{\infty}\sigma_n^2 \cos\omega_n \tau$$

Fourier representation of a Gaussian random process (continued)

Consider the psd function

$$S_{XX}(\omega) = \sum_{n=1}^{\infty} S(\omega_n)\Delta\omega_n \delta(\omega - \omega_n)$$

$$\Rightarrow \tilde{R}_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty} S(\omega_n)\Delta\omega_n \delta(\omega - \omega_n)\cos(\omega\tau) d\omega$$

$$\Rightarrow \tilde{R}_{XX}(\tau) = \frac{1}{2\pi}\sum_{n=1}^{\infty} S(\omega_n)\Delta\omega_n \cos(\omega_n\tau)$$

Compare this with

$$R_{XX}(\tau) = \sum_{n=1}^{\infty} \sigma_n^2 \cos \omega_n \tau$$

By choosing $\sigma_n^2 = \dfrac{S(\omega_n)\Delta\omega_n}{2\pi}$, we see that the two ACF-s coincide.

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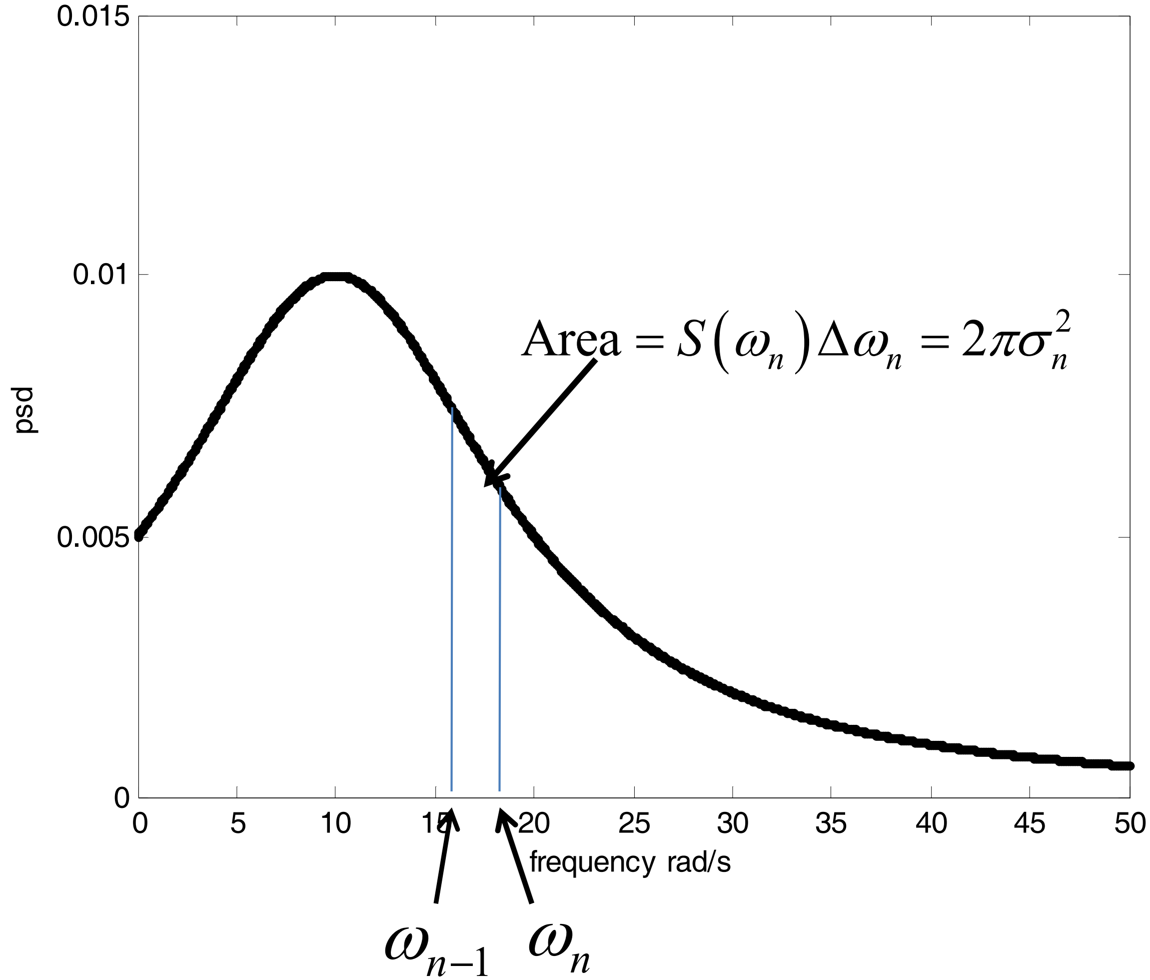
By discretizing the psd function as shown we can simulate samples of $X(t)$ using the Fourier representation

$$X(t)=\sum_{n=1}^{\infty}\{a_n\cos\omega_n t+b_n\sin\omega_n t\};\ \omega_n=n\omega_0$$



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Example

Simulate samples of a zero mean stationary Gaussian random process with following properties:

$$S(\omega) = \frac{I}{\sqrt{2\pi\alpha}} \exp\left[-\frac{\omega^2}{2\alpha^2}\right]; -\infty < \omega < \infty$$

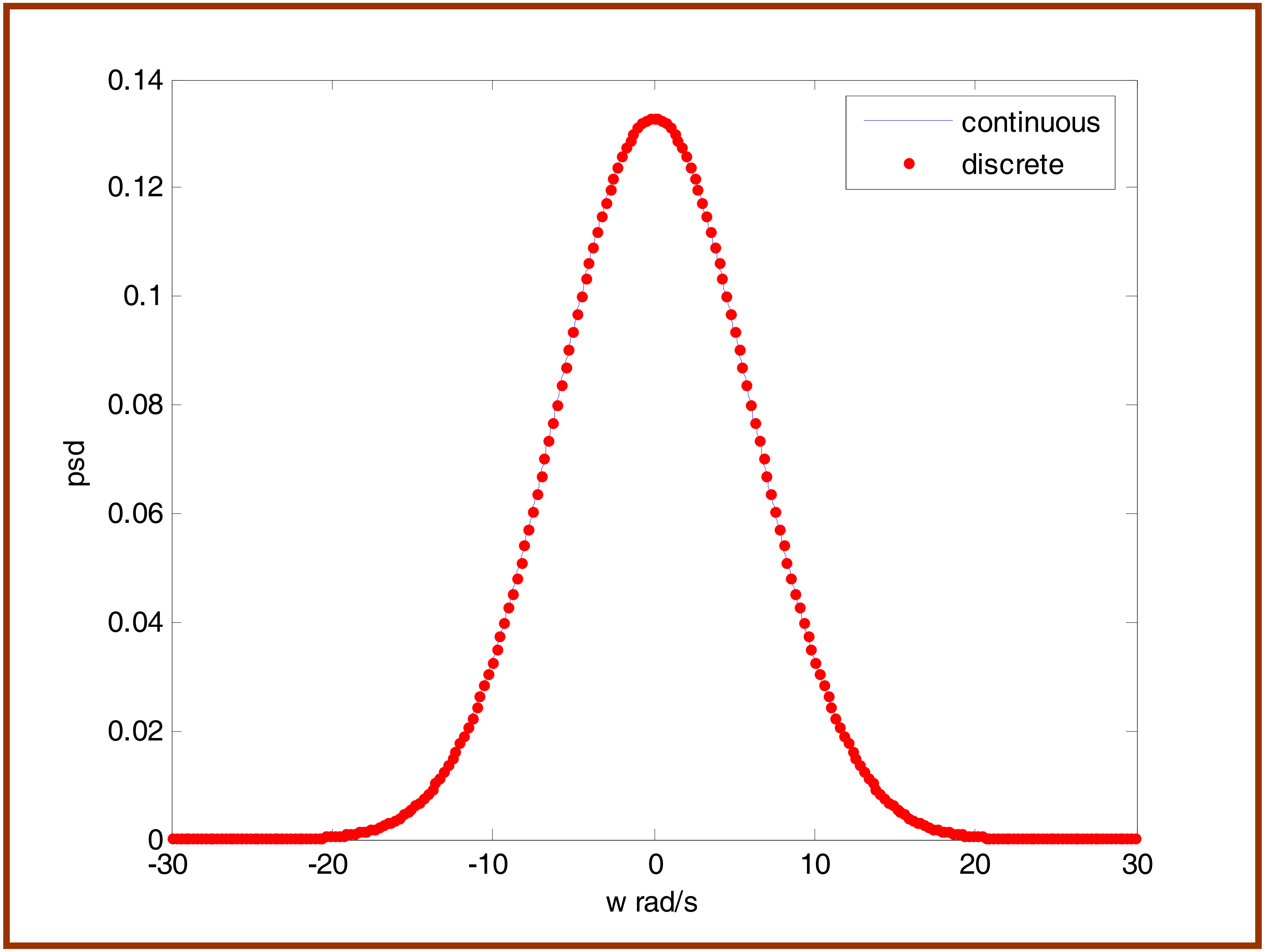
$$I = 2; \alpha = 6$$

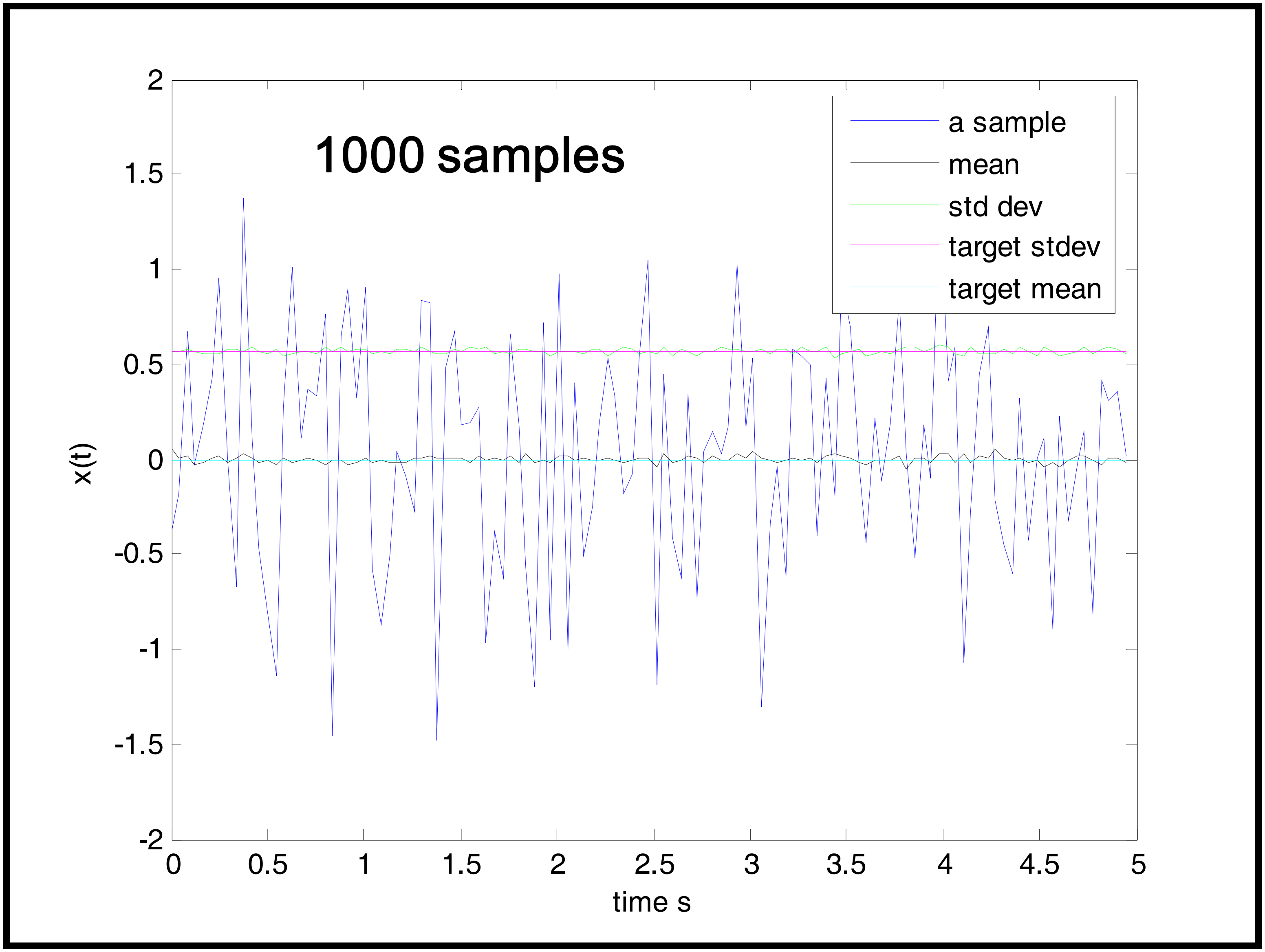
$$X(t) = \sum_{i=1}^{N} a_n \cos(\omega_n t) + b_n \sin(\omega_n t)$$

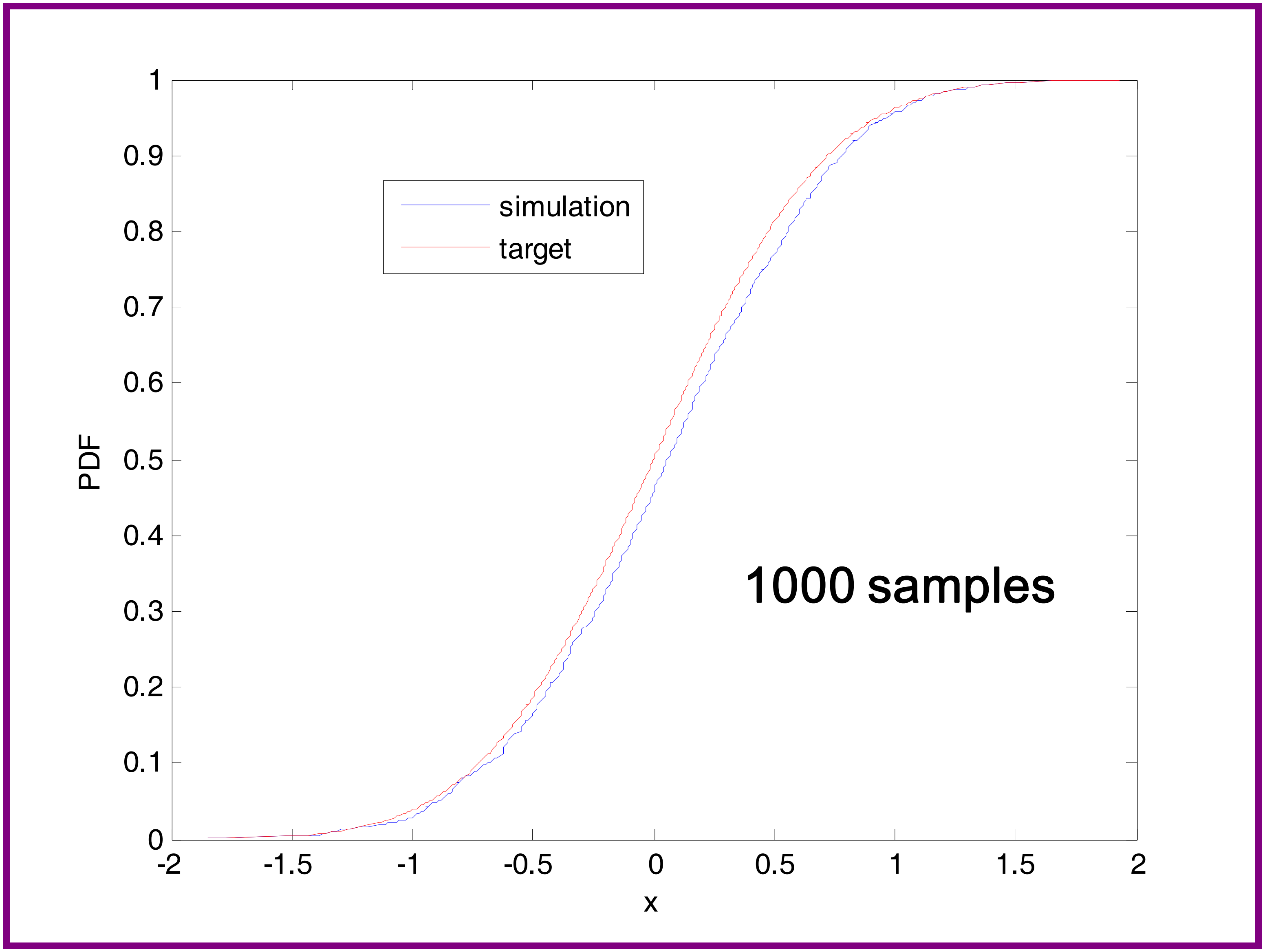
$$T_{\max} = 5s; N = 120; \omega_n = n\omega_0; \omega_0 = 0.2513 \text{ rad/s}$$

$$\omega_{\max} = 0.2513 \times 120 = 30.1593 \text{ rad/s.}$$

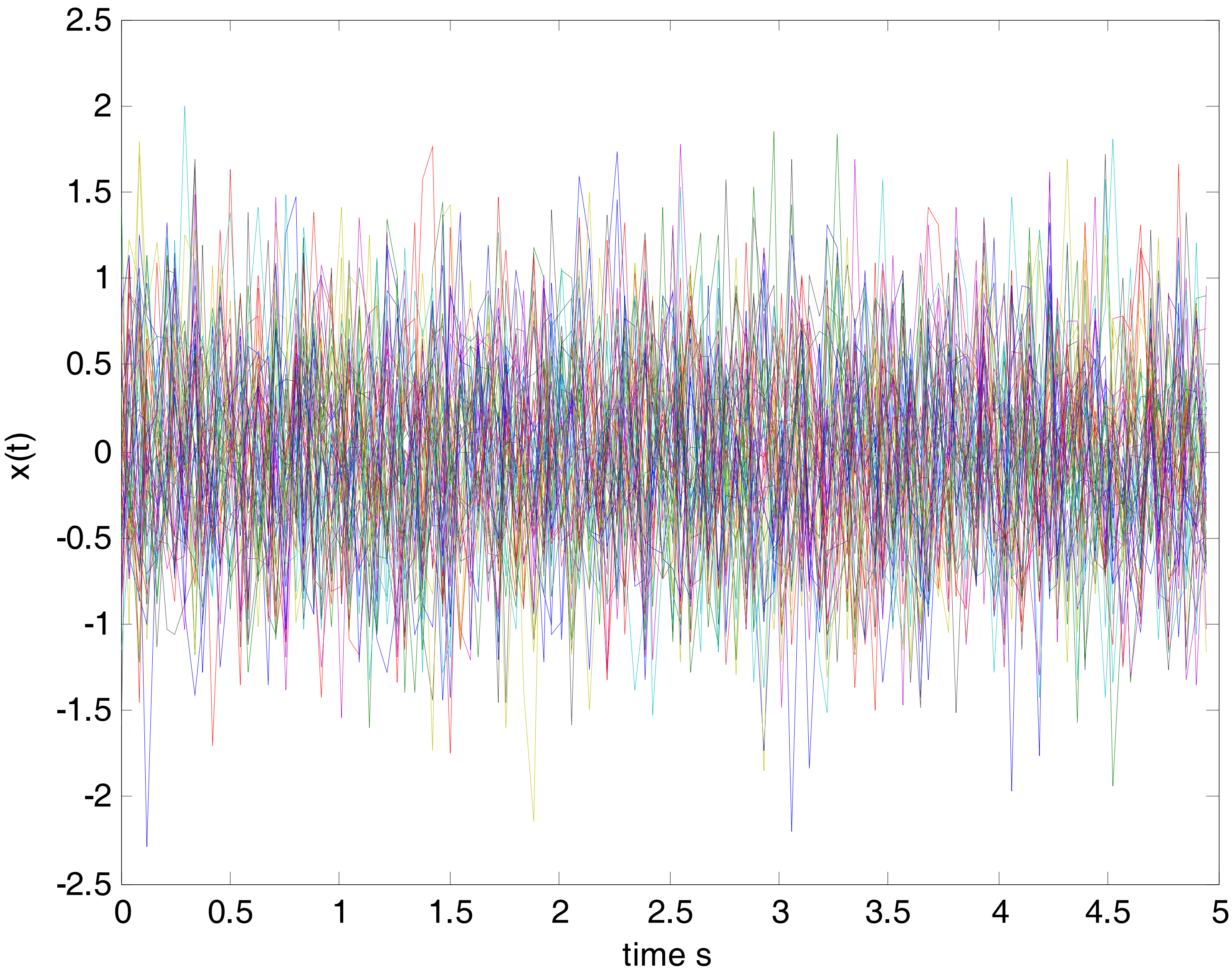
$$\Delta t = 0.0419 \text{ s}$$


0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
psd
-30
-20
-10
0
10
20
30
w rad/s
continuous
discrete


1000 samples
a sample
mean
std dev
target stdev
target mean
x(t)
time s
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



simulation
target
1000 samples
PDF
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
x(t)
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
time s

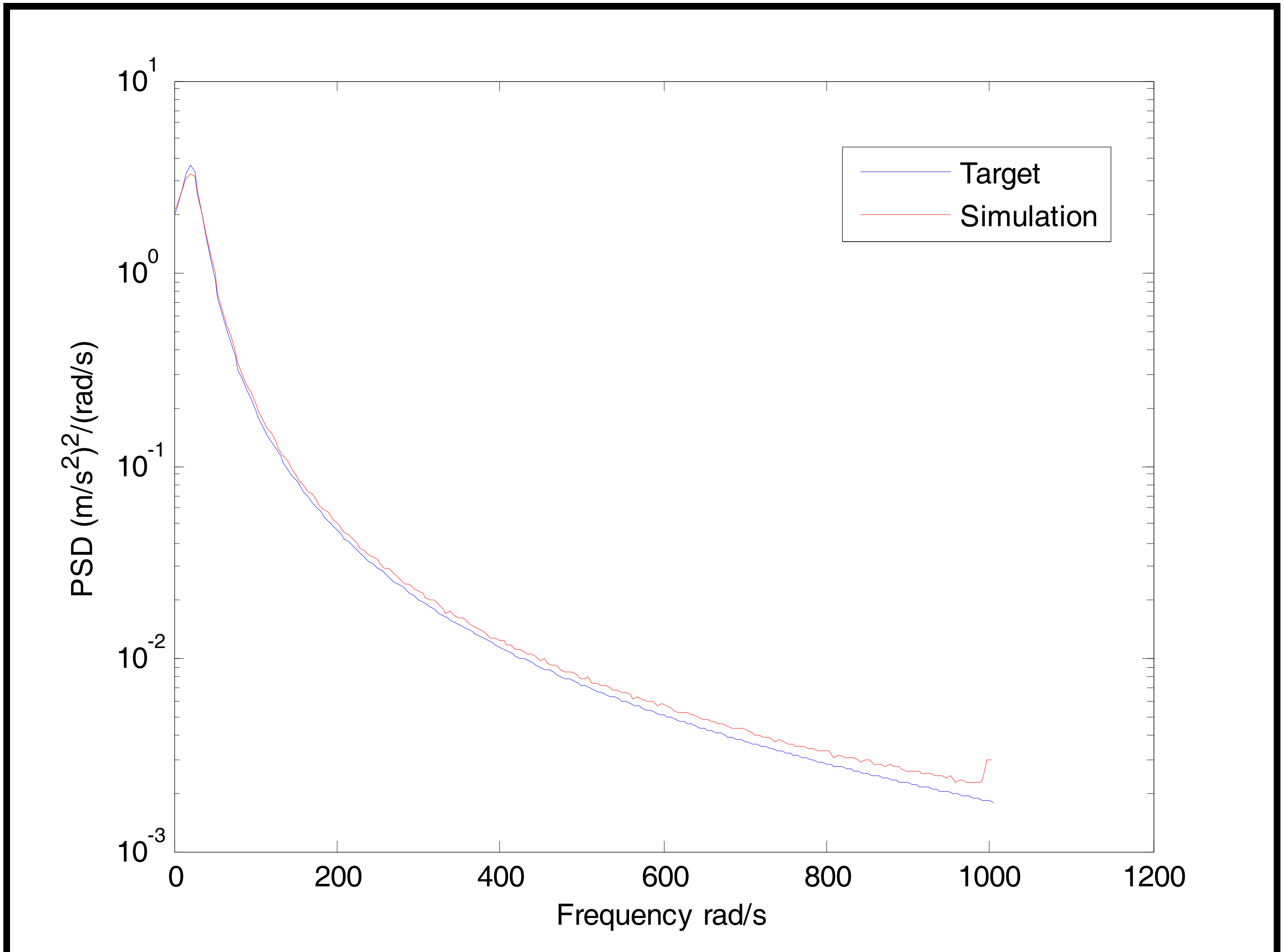
$$m\ddot{u} + c_g\left(u - \dot{x}_b\right) + k_g\left(u - x_b\right) = 0$$

$$\Rightarrow \ddot{u} + 2\eta_g\omega_g\dot{u} + \omega_g^2 u = 2\eta_g\omega_g\dot{x}_b + \omega_g^2 x_b$$

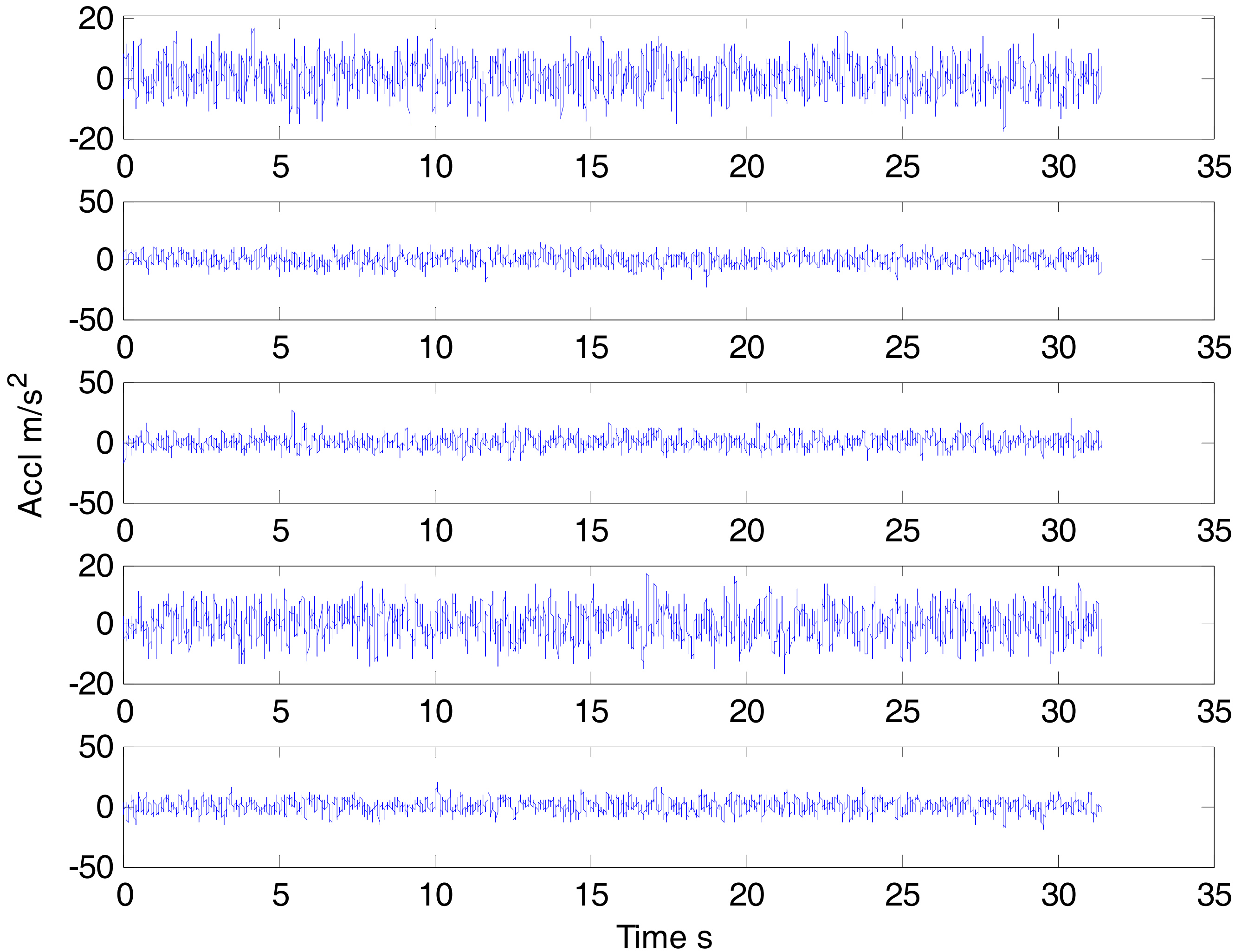
Let $y = \ddot{u}$

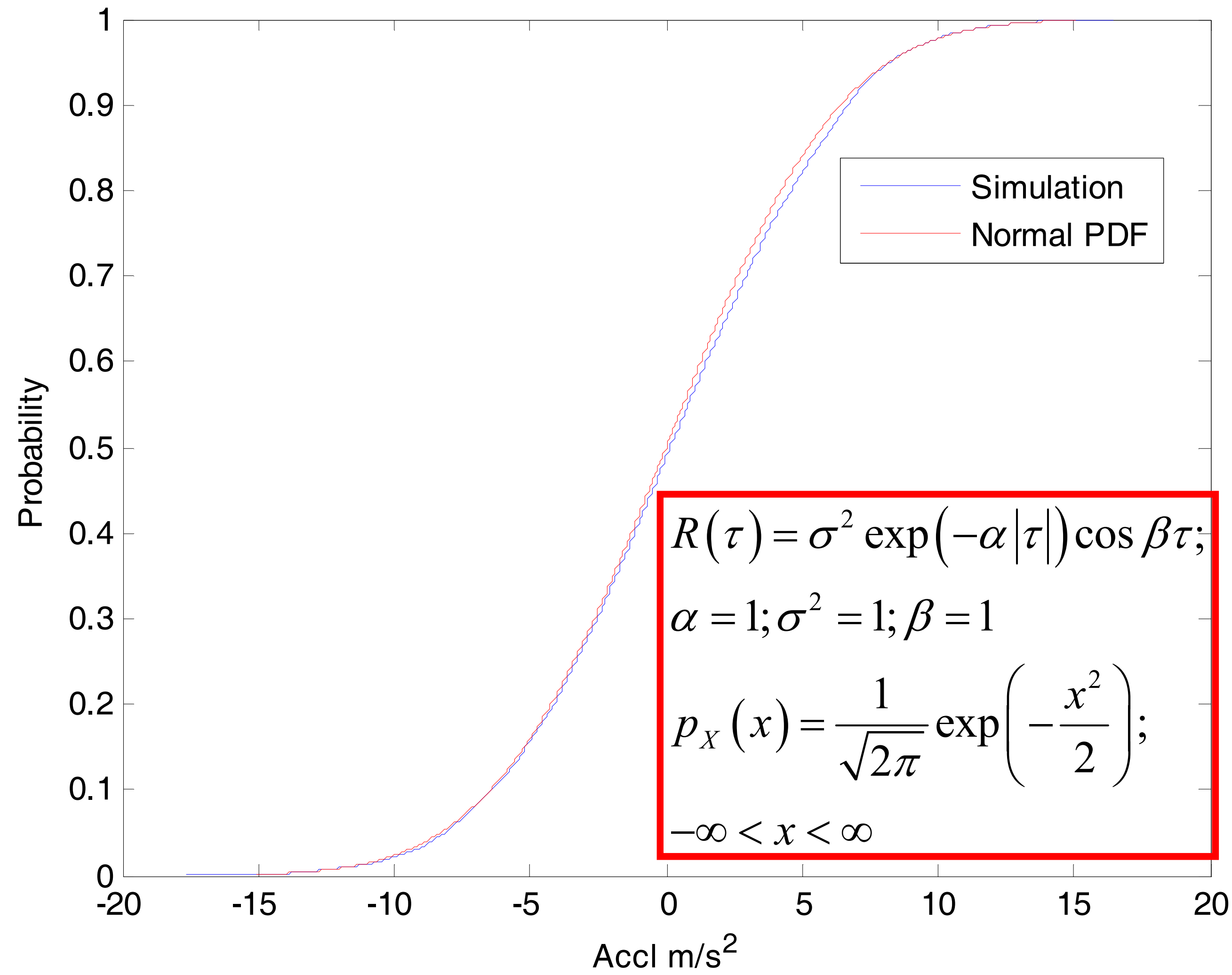
$$S_{yy}\left(\omega\right) = I\frac{\left(\omega_g^4 + 4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\eta_g^2\omega_g^2\omega^2}$$

$I = 1; \omega_g = 8\pi$ rad/s; $\eta_g = 0.6$



10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
PSD $(m/s^2)^2/(rad/s)$
0
200
400
600
800
1000
1200
Frequency rad/s
Target
Simulation


Accl m/s^2
Time s
20
0
-20
50
-50
0 5 10 15 20 25 30 35

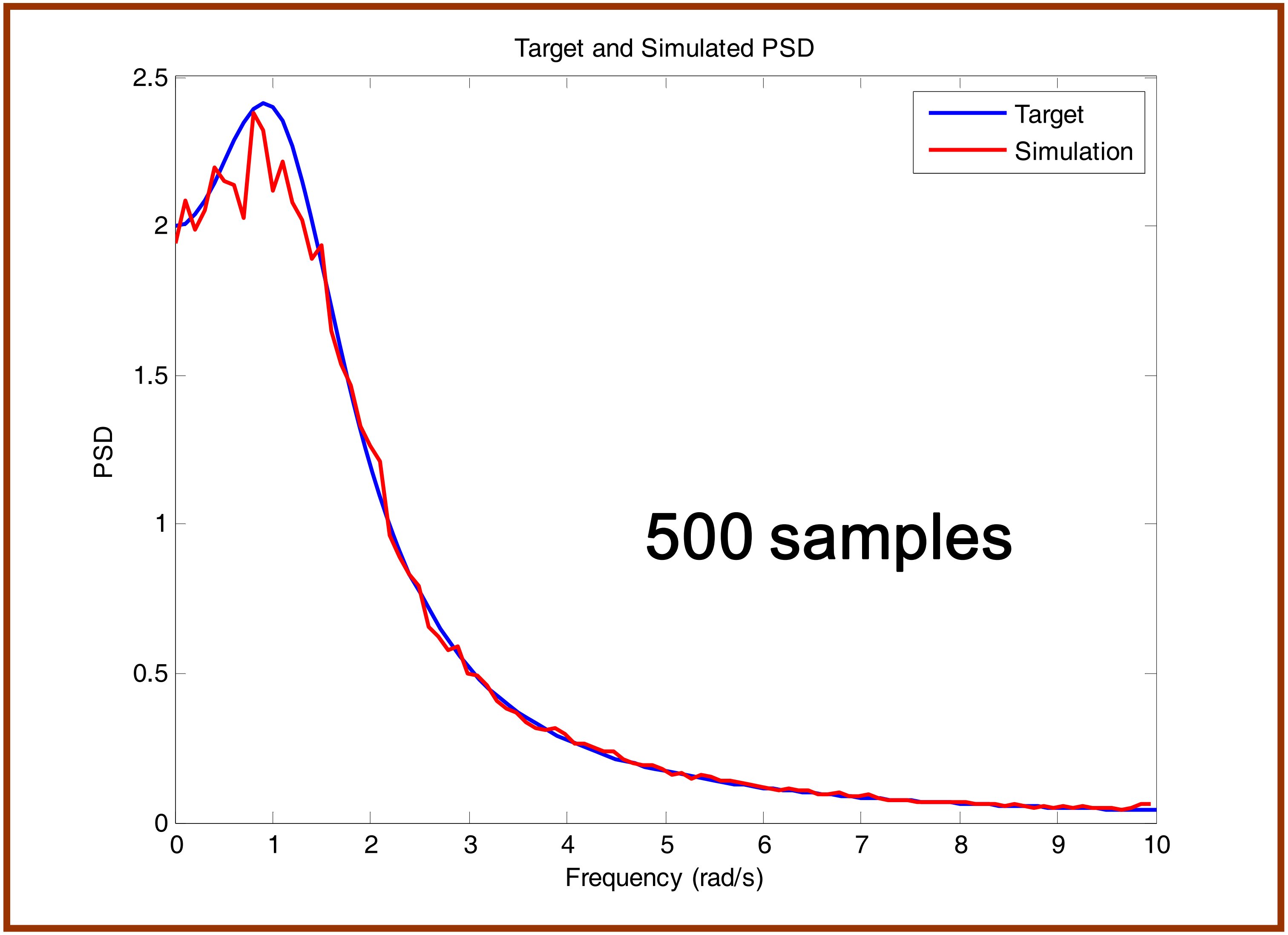


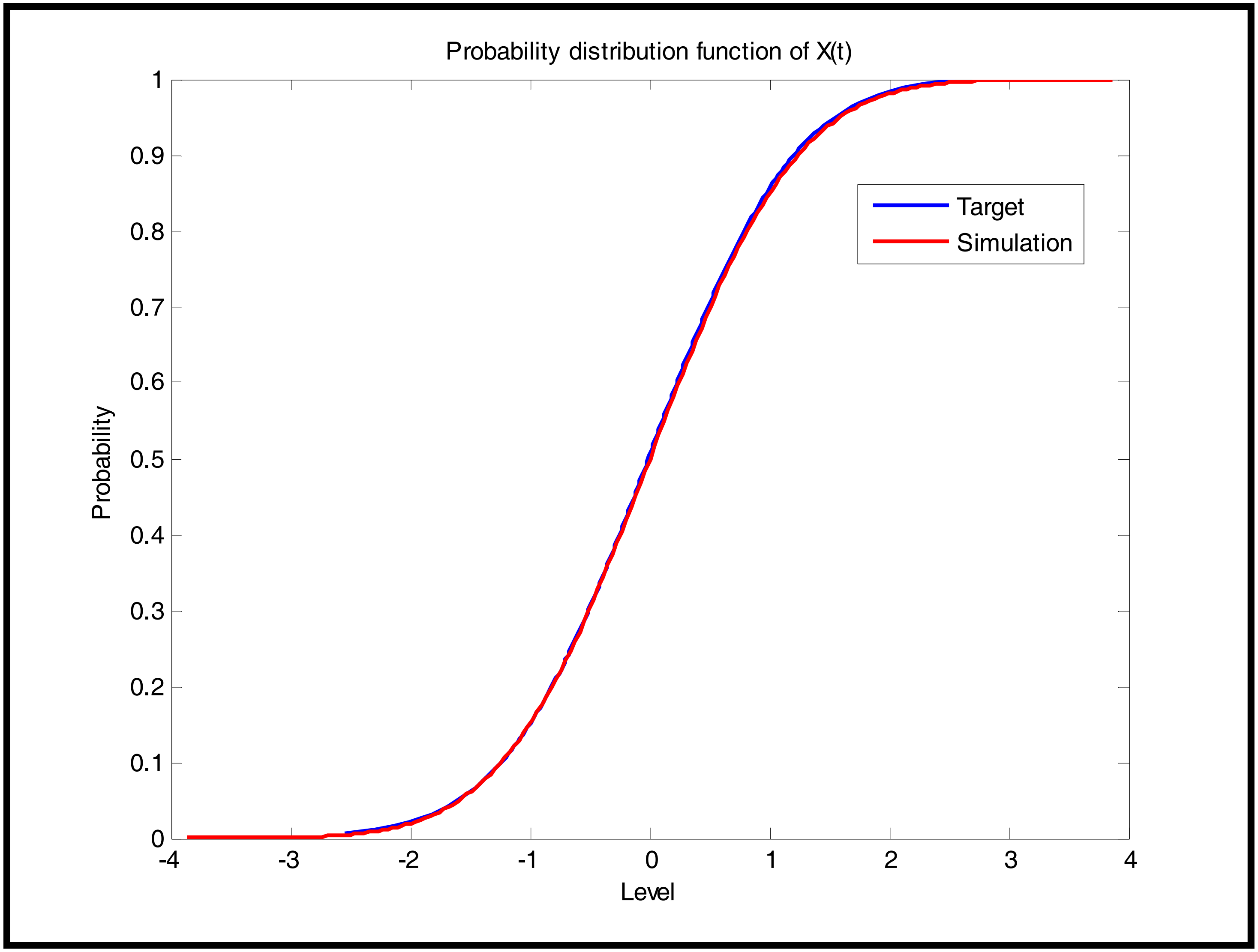
$$R(\tau) = \sigma^2 \exp\left(-\alpha\left|\tau\right|\right)\cos\beta\tau;$$

$$\alpha = 1; \sigma^2 = 1; \beta = 1$$

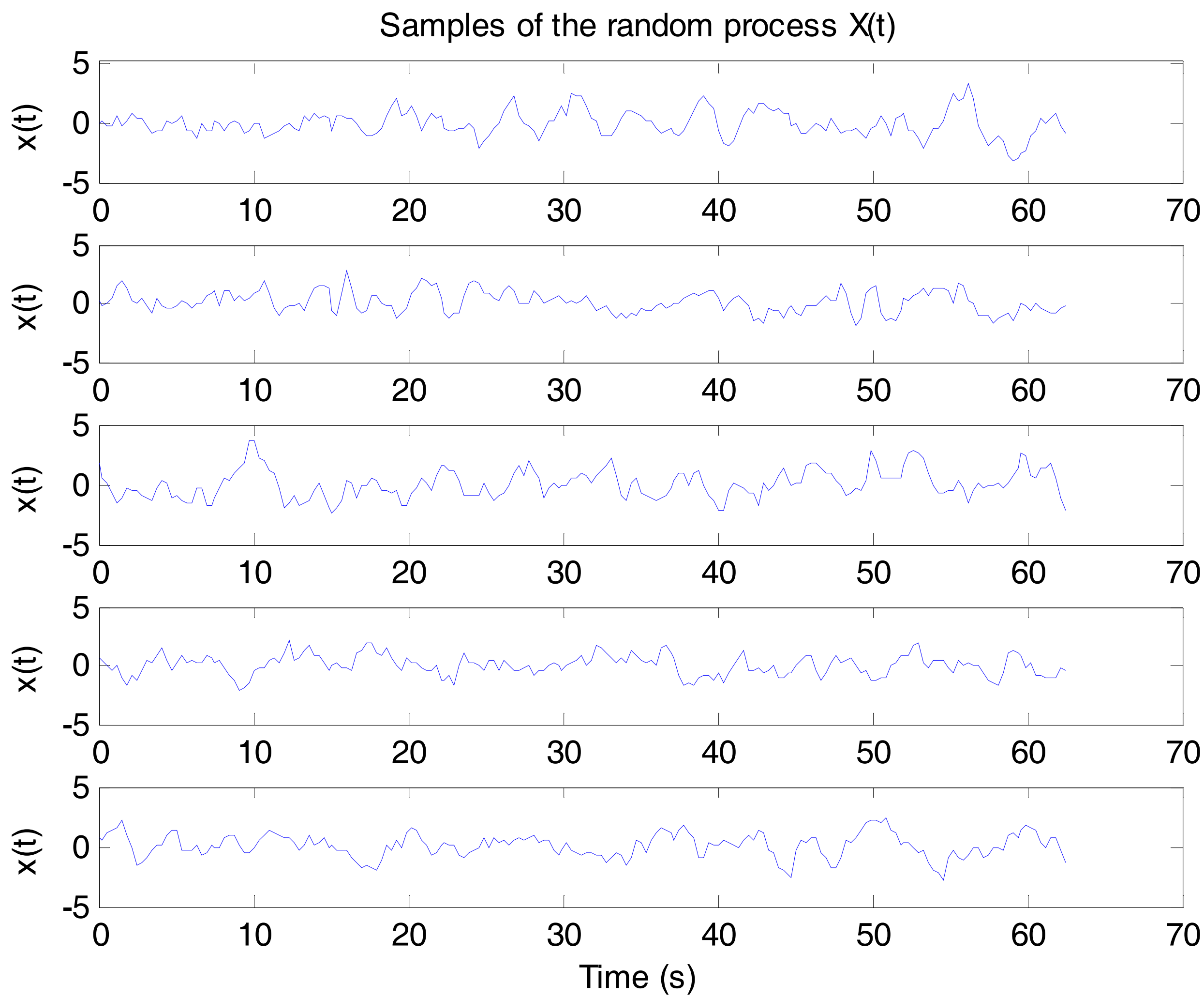
$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right);$$

$$-\infty < x < \infty$$


Target and Simulated PSD
Target
Simulation
500 samples
PSD
Frequency (rad/s)
0
0.5
1
1.5
2
2.5
0
1
2
3
4
5
6
7
8
9
10



Probability distribution function of X(t)
Probability
Level
Target
Simulation
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4


Samples of the random process X(t)
x(t)
5
0
-5
0
10
20
30
40
50
60
70
Time (s)

Simulation of partially specified non - Gaussian random processes : Nataf's transformation

Let $X(t)$ be a random process whose first order pdf and the ACF functions are available. No further information about the process is available.

$X(t)$ need not be stationary.

How to simulate samples of $X(t)$?

Define $Y(t)=\dfrac{X(t)-m_X(t)}{\sigma_X(t)}$ so that

$\langle Y(t)\rangle = 0$ & $\langle Y^2(t)\rangle = 1.$

Introduce a new random process Z(t) through the transformation

$\Phi\left[Z(t)\right]=P_Y\left[Y(t)\right]$

Here $\Phi[\bullet]$ = PDF of $\mathrm{N}(0,1)$ random variable.

$Z(t)$ is a zero mean Gaussian random process with an unknown covariance function.

$$\Phi\left[Z(t)\right] = P_Y\left[Y(t)\right]$$

$$Y(t) = P_Y^{-1}\left\{\Phi\left[Z(t)\right]\right\}$$

$$\left\langle Y(t_1)Y(t_2)\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_Y^{-1}\{\Phi[z_1]\}P_Y^{-1}\{\Phi[z_2]\}\phi\left(z_1, z_2; 0, \rho^*\right)dz_1dz_2$$

Known

$$\rho_{XX}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_Y^{-1}\{\Phi[z_1]\}P_Y^{-1}\{\Phi[z_2]\}\phi\left[z_1, z_2; 0, \rho^*(t_1,t_2)\right]dz_1dz_2$$

Unknown

Remarks

•RHS is known and $\rho^*(t_1,t_2)$ is not known

•$\left|\rho_{XX}(t_1,t_2)\right| \le 1 \,\&\, \left|\rho^*(t_1,t_2)\right| \le 1$

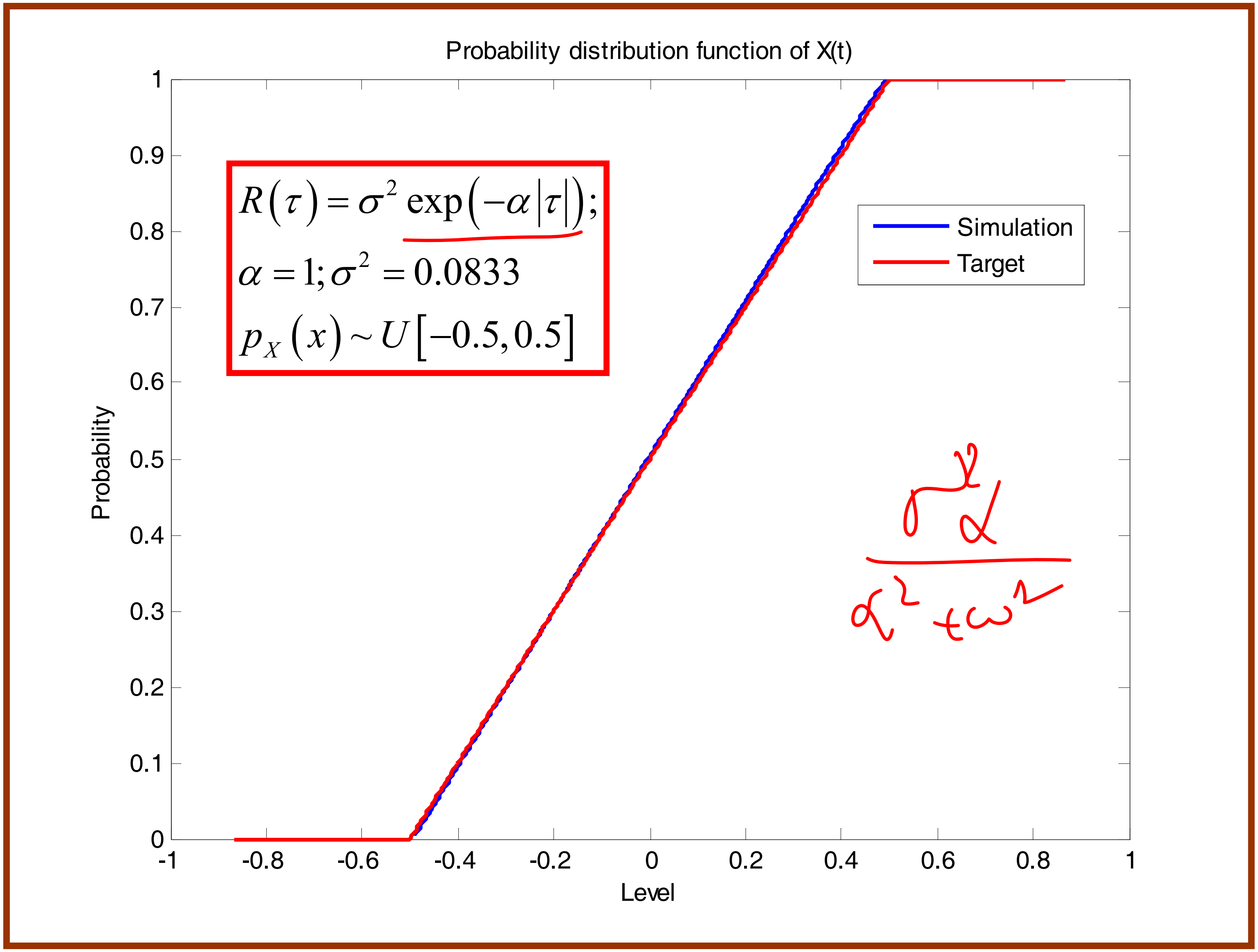
•$\phi\left[z_1, z_2; 0, \rho^*(t_1,t_2)\right]$

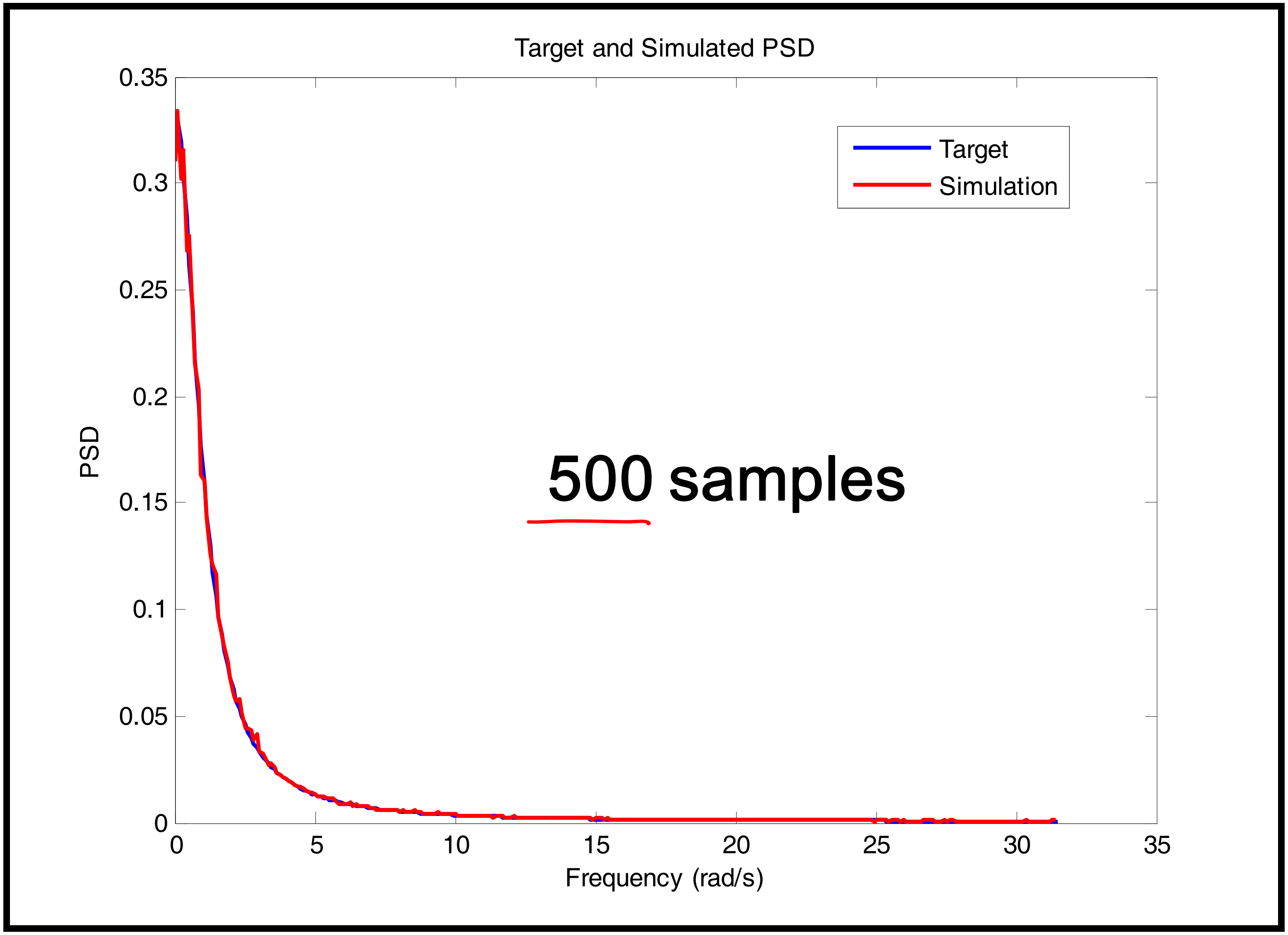
Steps

•Solve for $\rho^*(t_1,t_2)$

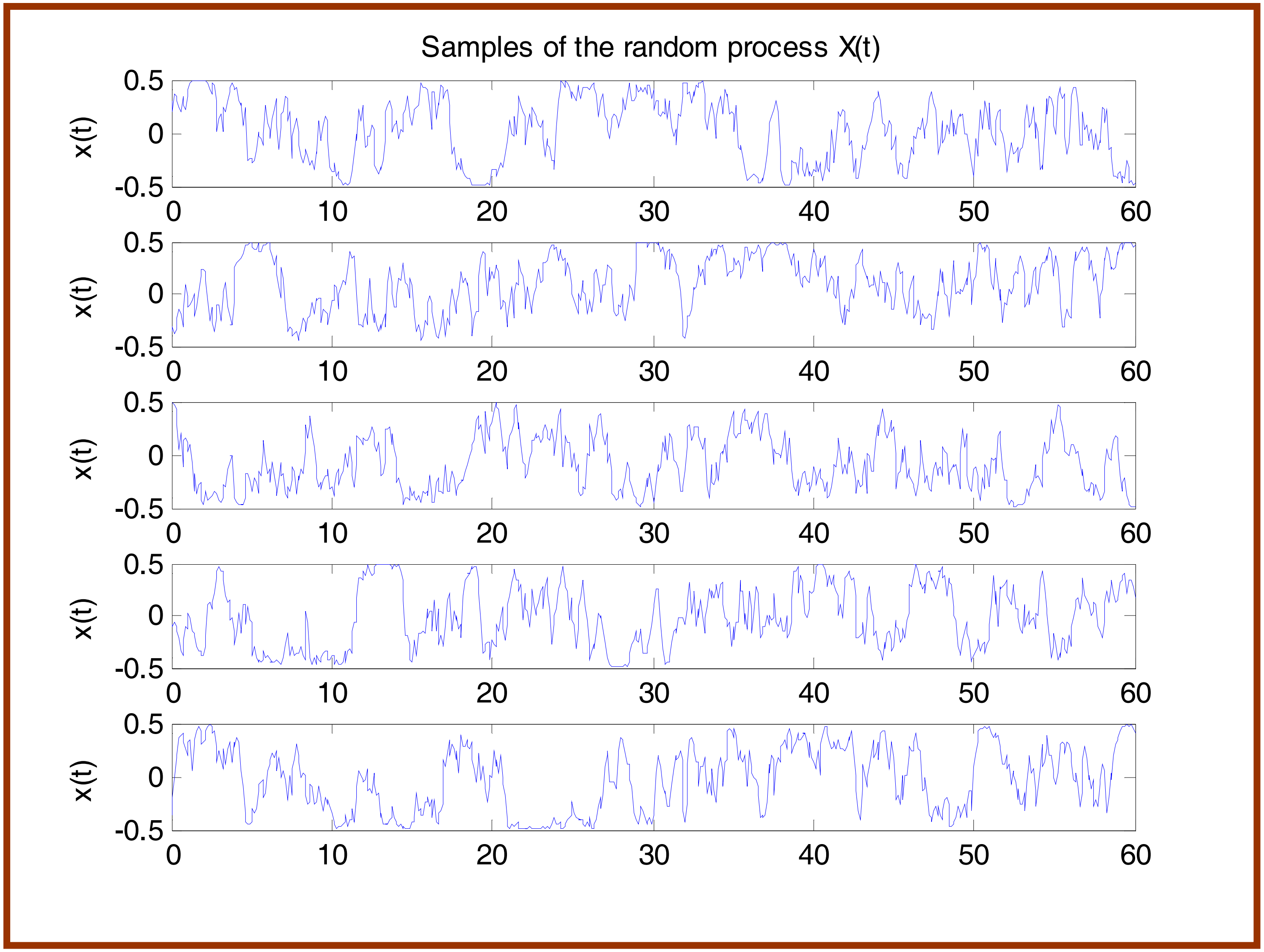
•Simulate $Z(t)$

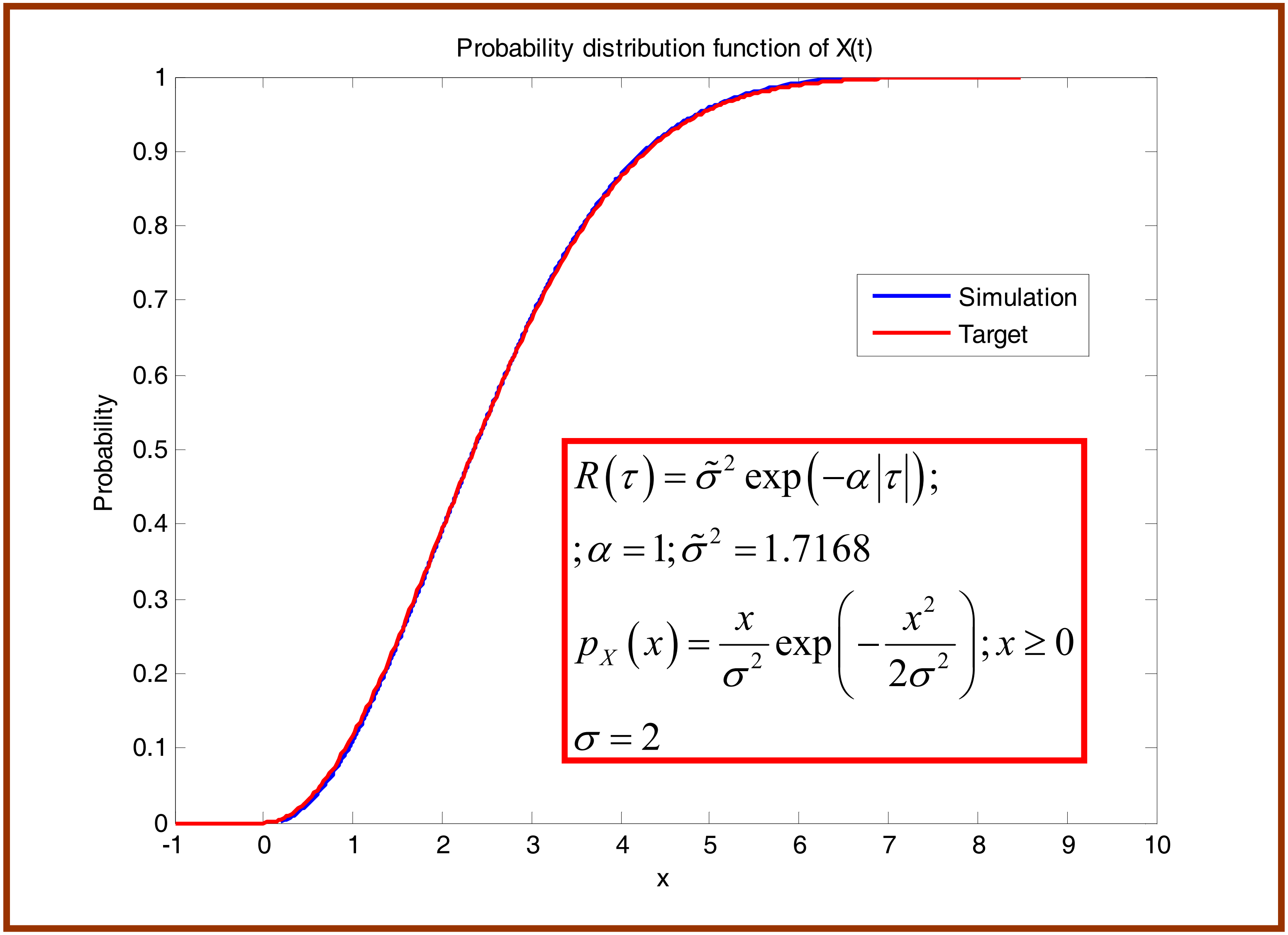
•Simulate $Y(t)$ and hence $X(t)$.

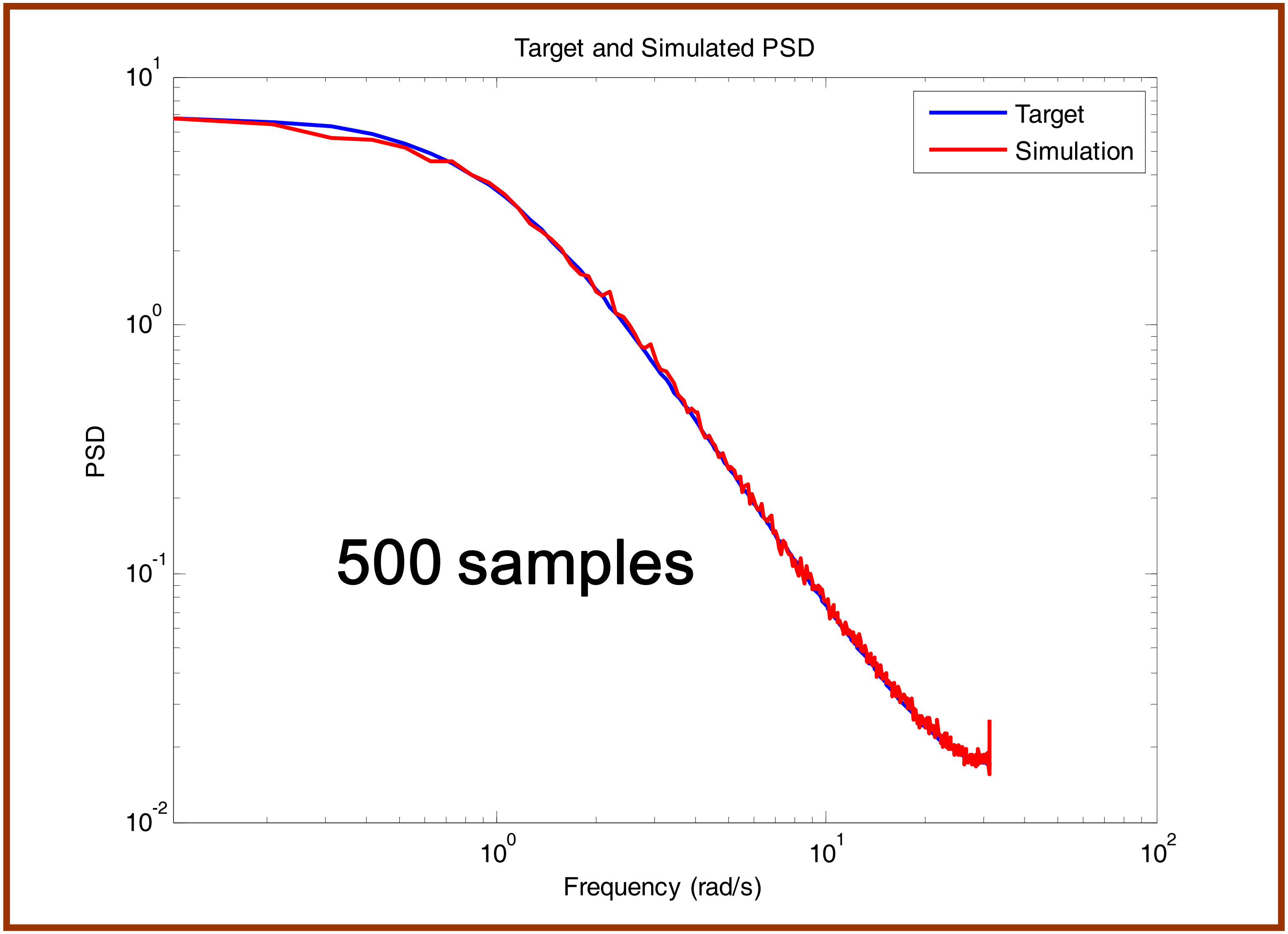

Probability distribution function of X(t)
$R(\tau) = \sigma^2 \exp(-\alpha|\tau|);$
$\alpha = 1; \sigma^2 = 0.0833$
$p_X(x) \sim U[-0.5, 0.5]$
Simulation
Target
Probability
Level
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
$\frac{\sigma^2 \alpha}{\alpha^2 + \omega^2}$



Target and Simulated PSD
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
PSD
0
5
10
15
20
25
30
35
Frequency (rad/s)
Target
Simulation
500 samples


Samples of the random process X(t)
x(t)
0.5
0
-0.5
0
10
20
30
40
50
60


Probability distribution function of X(t)
Probability
x
Simulation
Target
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-1
0
1
2
3
4
5
6
7
8
9
10
$R(\tau) = \tilde{\sigma}^2 \exp(-\alpha|\tau|);$
$;\alpha = 1; \tilde{\sigma}^2 = 1.7168$
$p_X(x) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right); x \geq 0$
$\sigma = 2$


Target and Simulated PSD
Target
Simulation
500 samples
PSD
Frequency (rad/s)
10^1
10^0
10^-1
10^-2
10^0
10^1
10^2

Samples of the random process X(t)
x(t)
10
5
0
0
10
20
30
40
50
60

Simulation of vector Gaussian random process

Let $X(t)$ and $\mathrm{Y}(t)$ be two jointly stationary, zero mean Gaussian processes with

$$\langle X(t)X(t+\tau)\rangle = R_{XX}(\tau)$$

$$\langle Y(t)Y(t+\tau)\rangle = R_{YY}(\tau)$$

$$\langle X(t)Y(t+\tau)\rangle = R_{XY}(\tau)$$

Note:

$$R_{XY}(\tau) = \langle X(t)Y(t+\tau)\rangle = \langle Y(t+\tau)X(t)\rangle = R_{YX}(-\tau)$$

$$R_{XY}(\tau) = \langle X(t)Y(t+\tau)\rangle \neq \langle Y(t)X(t+\tau)\rangle = R_{YX}(\tau)$$

$$R(\tau) = \begin{bmatrix} R_{XX}(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_{YY}(\tau) \end{bmatrix}$$

$$R_{XX}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega; S_{XX}(\omega)=\int_{-\infty}^{\infty}R_{XX}(\tau)\exp(i\omega\tau)\,d\tau$$

$$R_{YY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega; S_{YY}(\omega)=\int_{-\infty}^{\infty}R_{YY}(\tau)\exp(i\omega\tau)\,d\tau$$

$$R_{XY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XY}(\omega)\exp(-i\omega\tau)\,d\omega; S_{XY}(\omega)=\int_{-\infty}^{\infty}R_{XY}(\tau)\exp(i\omega\tau)\,d\tau$$

$$R_{YX}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{YX}(\omega)\exp(-i\omega\tau)\,d\omega; S_{YX}(\omega)=\int_{-\infty}^{\infty}R_{YX}(\tau)\exp(i\omega\tau)\,d\tau$$

$$S_{XY}(\omega)=\Gamma_{XY}(\omega)+i\Delta_{XY}(\omega);$$

$$S_{XY}(\omega)=\int_{-\infty}^{\infty}R_{XY}(\tau)\exp(i\omega\tau)\,d\tau=\int_{-\infty}^{\infty}R_{YX}(-\tau)\exp(i\omega\tau)\,d\tau=S_{YX}^{*}(\omega)$$

$$R_{XY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$S_{XY}(\omega)=\Gamma_{XY}(\omega)+i\Delta_{XY}(\omega);$$

$$R_{XY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Gamma_{XY}(\omega)+i\Delta_{XY}(\omega)\right]\left[\cos\omega\tau-i\sin\omega\tau\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Gamma_{XY}(\omega)\cos\omega\tau+\Delta_{XY}(\omega)\sin\omega\tau\right]d\omega$$

$$-i\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Gamma_{XY}(\omega)\sin\omega\tau-\Delta_{XY}(\omega)\cos\omega\tau\right]d\omega$$

$R_{XY}(\tau)$ is real valued $\Rightarrow$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Gamma_{XY}(\omega)\sin\omega\tau-\Delta_{XY}(\omega)\cos\omega\tau\right]d\omega=0$$

The above is true if

$$\Gamma_{XY}(-\omega)=\Gamma_{XY}(\omega)\ \&\ \Delta_{XY}(-\omega)=-\Delta_{XY}(\omega)$$

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Gamma_{XY}(\omega)\cos\omega\tau + \Delta_{XY}(\omega)\sin\omega\tau\right]d\omega$$

$$\Gamma_{XY}(-\omega) = \Gamma_{XY}(\omega)\ \&\ \Delta_{XY}(-\omega) = -\Delta_{XY}(\omega)$$

$$\Rightarrow R_{XY}(\tau) = \frac{1}{\pi}\int_{0}^{\infty}\left[\Gamma_{XY}(\omega)\cos\omega\tau + \Delta_{XY}(\omega)\sin\omega\tau\right]d\omega$$

$$S_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T} \left\langle X_T(\omega) X_T^*(\omega) \right\rangle$$

$$S_{YY}(\tau) = \lim_{T\to\infty} \frac{1}{T} \left\langle Y_T(\omega) Y_T^*(\omega) \right\rangle$$

$$S_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{T} \left\langle X_T(\omega) Y_T^*(\omega) \right\rangle$$

$$S_{YX}(\tau) = \lim_{T\to\infty} \frac{1}{T} \left\langle Y_T(\omega) X_T^*(\omega) \right\rangle$$

$$S(\omega) = \begin{bmatrix} S_{XX}(\omega) & S_{XY}(\omega) \\ S_{YX}(\omega) & S_{YY}(\omega) \end{bmatrix}$$

Fourier representations

$$X(t) = \sum_{n=1}^{N} a_n \cos\omega_n t + b_n \sin\omega_n t; \;\; \omega_n = n\omega_0$$

$$Y(t) = \sum_{n=1}^{N} c_k \cos\omega_k t + d_k \sin\omega_k t; \;\; \omega_k = k\omega_0$$

Assumptions

$$a_n \sim N(0, \sigma_{Xn}), b_n \sim N(0, \sigma_{Xn}),$$

$$\langle a_n a_k \rangle = 0 \forall n \neq k, \langle b_n b_k \rangle = 0 \forall n \neq k, \langle a_n b_k \rangle = 0 \forall n, k = 1, 2, \cdots, N$$

$$\Rightarrow \langle X(t) \rangle = \sum_{n=1}^{N} \left\{ \langle a_n \rangle \cos\omega_n t + \langle b_n \rangle \sin\omega_n t \right\} = 0$$

$$\langle X(t) X(t+\tau) \rangle = \sum_{n=1}^{N} \sigma_{Xn}^2 \cos(\omega_n \tau)$$

$$Y(t) = \sum_{n=1}^{N} c_k \cos\omega_k t + d_k \sin\omega_k t;\ \ \omega_k = k\omega_0$$

Assumptions

$$c_n \sim N(0, \sigma_{Yn}), d_n \sim N(0, \sigma_{Yn}),$$

$$\langle c_n c_k \rangle = 0 \forall n \neq k, \langle d_n d_k \rangle = 0 \forall n \neq k, \langle c_n d_k \rangle = 0 \forall n, k = 1, 2, \cdots, N$$

$$\Rightarrow \langle Y(t) \rangle = \sum_{n=1}^{N} \left\{ \langle c_n \rangle \cos\omega_n t + \langle d_n \rangle \sin\omega_n t \right\} = 0$$

$$\langle Y(t) Y(t+\tau) \rangle = \sum_{n=1}^{N} \sigma_{Yn}^2 \cos(\omega_n \tau)$$

$$X(t)=\sum_{n=1}^{N} a_n \cos\omega_n t + b_n \sin\omega_n t;\ \ \omega_n = n\omega_0$$

$$Y(t)=\sum_{n=1}^{N} c_k \cos\omega_k t + d_k \sin\omega_k t;\ \ \omega_k = k\omega_0$$

$$\langle X(t)Y(t+\tau)\rangle =$$

$$\left\langle \left\{\sum_{n=1}^{N} a_n \cos\omega_n t + b_n \sin\omega_n t\right\}\left\{\sum_{n=1}^{N} c_k \cos\omega_k (t+\tau) + d_k \sin\omega_k (t+\tau)\right\}\right\rangle$$

$$=\left\langle \sum_{n=1}^{N}\sum_{k=1}^{N}\left[a_n \cos\omega_n t + b_n \sin\omega_n t\right]\left[c_k \cos\omega_k (t+\tau) + d_k \sin\omega_k (t+\tau)\right]\right\rangle$$

$$=\sum_{n=1}^{N}\sum_{k=1}^{N}\langle a_n c_k\rangle \cos\omega_n t \cos\omega_k (t+\tau) + \langle a_n d_k\rangle \cos\omega_n t \sin\omega_k (t+\tau)$$

$$+\langle b_n c_k\rangle \sin\omega_n t \cos\omega_k (t+\tau) + \langle b_n d_k\rangle \sin\omega_n t \sin\omega_k (t+\tau)$$

$$\left\langle X(t)Y(t+\tau)\right\rangle =$$

$$\sum_{n=1}^{N}\sum_{k=1}^{N}\left\langle a_n c_k\right\rangle \cos\omega_n t\cos\omega_k(t+\tau)+\left\langle a_n d_k\right\rangle \cos\omega_n t\sin\omega_k(t+\tau)$$

$$+\left\langle b_n c_k\right\rangle \sin\omega_n t\cos\omega_k(t+\tau)+\left\langle b_n d_k\right\rangle \sin\omega_n t\sin\omega_k(t+\tau)$$

Take

$$\left\langle a_n c_k\right\rangle=\sigma_{acn}\delta_{nk};\left\langle a_n d_k\right\rangle=\sigma_{adn}\delta_{nk};\left\langle b_n c_k\right\rangle=\sigma_{bcn}\delta_{nk};\left\langle b_n d_k\right\rangle=\sigma_{bdn}\delta_{nk}$$

$$\Rightarrow$$

$$R_{XY}(\tau)=\sum_{n=1}^{N}\sigma_{acn}\cos\omega_n t\cos\omega_n(t+\tau)+\sigma_{adn}\cos\omega_n t\sin\omega_k(t+\tau)$$

$$+\sigma_{bcn}\sin\omega_n t\cos\omega_k(t+\tau)+\sigma_{bdn}\sin\omega_n t\sin\omega_k(t+\tau)$$

Furthermore, assume $\sigma_{acn}=-\sigma_{bdn}\ \&\ \sigma_{adn}=-\sigma_{bcn}$

$$\Rightarrow R_{XY}(\tau)=\sum_{n=1}^{N}\left(\sigma_{acn}\cos\omega_n\tau+\sigma_{adn}\sin\omega_n\tau\right)$$

$$R_{XX}(\tau)=\sum_{n=1}^{N}\sigma_{Xn}^{2}\cos(\omega_n\tau)\cdots(1)$$

$$R_{YY}(\tau)=\sum_{n=1}^{N}\sigma_{Yn}^{2}\cos(\omega_n\tau)\cdots(2)$$

$$R_{XY}(\tau)=\sum_{n=1}^{N}(\sigma_{acn}\cos\omega_n\tau+\sigma_{adn}\sin\omega_n\tau)\cdots(3)$$

Consider

$$S_{XX}(\omega)=\sum_{n=1}^{N}S_{XX}(\omega_n)\Delta\omega_n\delta(\omega-\omega_n)$$

$$\tilde{R}_{XX}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XX}(\omega)\cos\omega\tau\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{N}S_{XX}(\omega_n)\Delta\omega_n\delta(\omega-\omega_n)\cos\omega\tau\,d\omega$$

$$R_{XX}(\tau) = \sum_{n=1}^{N} \sigma_{Xn}^2 \cos(\omega_n \tau) \cdots (1)$$

$$\tilde{R}_{XX}(\tau) = \sum_{n=1}^{N} \left(\frac{S_{XX}(\omega_n)\Delta\omega_n}{2\pi} \right) \cos \omega_n \tau \cdots (4)$$

Select $\sigma_{Xn}^2 = \left(\frac{S_{XX}(\omega_n)\Delta\omega_n}{2\pi} \right)$ so that $R_{XX}(\tau) = \tilde{R}_{XX}(\tau)$

Similarly define

$$S_{YY}(\omega) = \sum_{n=1}^{N} S_{YY}(\omega_n)\Delta\omega_n \delta(\omega - \omega_n) \text{ and}$$

select $\sigma_{Yn}^2 = \left(\frac{S_{YY}(\omega_n)\Delta\omega_n}{2\pi} \right)$ so that $R_{YY}(\tau) = \tilde{R}_{YY}(\tau)$.

$$R_{XY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{XY}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Gamma_{XY}(\omega)+i\Delta_{XY}(\omega)\right]\left[\cos\omega\tau-i\sin\omega\tau\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\Gamma_{XY}(\omega)\cos\omega\tau+\Delta_{XY}(\omega)\sin\omega\tau\right]d\omega$$

Consider

$$\Gamma_{XY}(\omega)=\sum_{n=1}^{N}\Gamma_{XY}(\omega_n)\Delta\omega_n\delta(\omega-\omega_n)\ \&$$

$$\Delta_{XY}(\omega)=\sum_{n=1}^{N}\Delta_{XY}(\omega_n)\Delta\omega_n\delta(\omega-\omega_n)$$

$$\tilde{R}_{XY}(\tau) = \frac{1}{\pi}\int_0^\infty \sum_{n=1}^{N} \Gamma_{XY}(\omega_n)\Delta\omega_n \delta(\omega-\omega_n)\cos\omega\tau d\omega$$

$$+\frac{1}{\pi}\int_0^\infty \sum_{n=1}^{N} \Delta_{XY}(\omega_n)\Delta\omega_n \delta(\omega-\omega_n)\sin\omega\tau d\omega$$

$$= \sum_{n=1}^{N}\left[\frac{\Gamma_{XY}(\omega_n)\Delta\omega_n}{2\pi}\right]\cos\omega_n\tau + \left[\frac{\Delta_{XY}(\omega_n)\Delta\omega_n}{2\pi}\right]\sin\omega_n\tau$$

Compare this with

$$R_{XY}(\tau) = \sum_{n=1}^{N}\left(\sigma_{acn}\cos\omega_n\tau + \sigma_{adn}\sin\omega_n\tau\right)$$

If we select

$$\sigma_{acn} = \left[\frac{\Gamma_{XY}(\omega_n)}{2\pi}\right] \& \, \sigma_{adn} = \left[\frac{\Delta_{XY}(\omega_n)}{2\pi}\right]$$

$$\Rightarrow \tilde{R}_{XY}(\tau) = R_{XY}(\tau)$$

Summary

$$X(t)=\sum_{n=1}^{N} a_n \cos\omega_n t + b_n \sin\omega_n t;\ \ \omega_n = n\omega_0$$

$$Y(t)=\sum_{n=1}^{N} c_k \cos\omega_k t + d_k \sin\omega_k t;\ \ \omega_k = k\omega_0$$

$$a_n \sim N\left(0,\sigma_{Xn}\right), b_n \sim N\left(0,\sigma_{Xn}\right),$$

$$\left\langle a_n a_k \right\rangle = 0 \forall n \neq k, \left\langle b_n b_k \right\rangle = 0 \forall n \neq k, \left\langle a_n b_k \right\rangle = 0 \forall n, k = 1,2,\cdots,N$$

$$c_n \sim N\left(0,\sigma_{Yn}\right), d_n \sim N\left(0,\sigma_{Yn}\right),$$

$$\left\langle c_n c_k \right\rangle = 0 \forall n \neq k, \left\langle d_n d_k \right\rangle = 0 \forall n \neq k, \left\langle c_n d_k \right\rangle = 0 \forall n, k = 1,2,\cdots,N$$

$$\left\langle a_n c_k \right\rangle = \sigma_{acn}\delta_{nk}; \left\langle a_n d_k \right\rangle = \sigma_{adn}\delta_{nk}; \left\langle b_n c_k \right\rangle = \sigma_{bcn}\delta_{nk}; \left\langle b_n d_k \right\rangle = \sigma_{bdn}\delta_{nk}$$

$$\sigma_{acn} = -\sigma_{bdn}\ \&\ \sigma_{adn} = -\sigma_{bcn}$$

Summary (continued)

$$\begin{Bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{Bmatrix} \sim N\left[\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \begin{pmatrix} \sigma_{X_n}^2 & 0 & \sigma_{acn} & -\sigma_{bcn} \\ 0 & \sigma_{X_n}^2 & \sigma_{bcn} & -\sigma_{acn} \\ \sigma_{acn} & \sigma_{bcn} & \sigma_{Y_n}^2 & 0 \\ -\sigma_{bcn} & \sigma_{acn} & 0 & \sigma_{Y_n}^2 \end{pmatrix}\right]$$

$$\sigma_{Xn}^2 = \left(\frac{S_{XX}(\omega_n)\Delta\omega_n}{2\pi}\right); \sigma_{Yn}^2 = \left(\frac{S_{YY}(\omega_n)\Delta\omega_n}{2\pi}\right);$$

$$\sigma_{acn} = \left[\frac{\Gamma_{XY}(\omega_n)}{2\pi}\right] \& \sigma_{bcn} = -\left[\frac{\Delta_{XY}(\omega_n)}{2\pi}\right]$$

Summary (continued)

$$R_{XX}(\tau) = \sum_{n=1}^{N} \sigma_{Xn}^2 \cos(\omega_n \tau)$$

$$R_{YY}(\tau) = \sum_{n=1}^{N} \sigma_{Yn}^2 \cos(\omega_n \tau)$$

$$R_{XY}(\tau) = \sum_{n=1}^{N} \left(\sigma_{acn} \cos \omega_n \tau + \sigma_{adn} \sin \omega_n \tau \right)$$

Notes

$$\left\langle X(t)Y(t+\tau)\right\rangle = R_{XY}(\tau)$$

$$\left\langle Y(t+\tau)X(t)\right\rangle = R_{YX}(-\tau) = R_{XY}(\tau)$$

$$\left\langle Y(t-\tau)X(t)\right\rangle = R_{YX}(\tau) = R_{XY}(-\tau)$$

$$R_{XY}(\tau) = \sum_{n=1}^{N}\left(\sigma_{acn}\cos\omega_n\tau + \sigma_{adn}\sin\omega_n\tau\right)$$

$$\neq R_{YX}(\tau) =$$

$$R_{XY}(-\tau) = \sum_{n=1}^{N}\left(\sigma_{acn}\cos\omega_n\tau - \sigma_{adn}\sin\omega_n\tau\right)$$

Exercise : Simulation of spatially varying earthquake ground acceleration

Consider $X(t)$ and $Y(t)$ to be two random processes representing ground accelerations in the horizontal direction at two stations separated by a distance d_{XY}.

$X(t)$ and $Y(t)$ can be taken to be jointly stationary, Gaussian and zero mean random processes. The auto-psd functions of $X(t)$ and $Y(t)$ may be taken to be of the form

$$S(\omega) = I\left|H_1(\omega)\right|^2 \left|H_2(\omega)\right|^2$$

$$S(\omega) = I\left|H_1(\omega)\right|^2 \left|H_2(\omega)\right|^2$$

with

$$H_1(\omega) = \frac{1 + i2\eta\left(\dfrac{\omega}{\omega_g}\right)}{\left(1 - \left(\dfrac{\omega}{\omega_g}\right)^2 + i2\eta\left(\dfrac{\omega}{\omega_g}\right)\right)} \;\&\; H_2(\omega) = \frac{\left(\dfrac{\omega}{\omega_f}\right)}{\left(1 - \left(\dfrac{\omega}{\omega_f}\right)^2 + i2\xi\left(\dfrac{\omega}{\omega_f}\right)\right)}$$

The above psd is fashioned after the Kanai-Tajimi psd function in which an additonal filter $H_2(\omega)$ is introduced to ensure that the ground displacement is well behaved at low frequencies.

The coherency function can be taken to be given by

$$\gamma_{XY}(\omega) = \exp\left[-\alpha\left|d_{XY}\right|\right]\exp\left[-i\omega\frac{d_{XY}}{V}\right]$$

Develop a code to simulate samples of $X(t)$ and $Y(t)$.

The coherency function can be taken to be given by

$$\gamma_{XY}(\omega) = \exp\left[-\alpha \left| d_{XY} \right|\right] \exp\left[-i\omega \frac{d_{XY}}{V}\right]$$

Develop a code to simulate samples of $X(t)$ and $Y(t)$.

Assume $\omega_g = 15.6$ rad/s, η=0.6, I=1 (m/s/s),

$\omega_f = 0.8$ rads/, ξ=0.5, $d_{XY} = 100$ m, α=0.1 /m,

and V=250 m/s.

From the ensemble of time histories generated estimate the psd and cross psd functions and compare them with the target values.

Generalization

$$X\left(t\right)=\sum_{n=1}^{N}a_n\cos\omega_n t+b_n\sin\omega_n t;\ \omega_n=n\omega_0$$

$$Y\left(t\right)=\sum_{k=1}^{N}c_k\cos\omega_k t+d_k\sin\omega_k t;\ \omega_k=k\omega_0$$

$$Z\left(t\right)=\sum_{m=1}^{N}e_m\cos\omega_m t+f_m\sin\omega_m t;\ \omega_m=m\omega_0$$

$$a_n\sim N\left(0,\sigma_{Xn}\right),b_n\sim N\left(0,\sigma_{Xn}\right),$$

$$\left\langle a_n a_k\right\rangle=0\forall n\neq k,\left\langle b_n b_k\right\rangle=0\forall n\neq k,\left\langle a_n b_k\right\rangle=0\forall n,k=1,2,\cdots,N$$

$$c_n\sim N\left(0,\sigma_{Yn}\right),d_n\sim N\left(0,\sigma_{Yn}\right),$$

$$\left\langle c_n c_k\right\rangle=0\forall n\neq k,\left\langle d_n d_k\right\rangle=0\forall n\neq k,\left\langle c_n d_k\right\rangle=0\forall n,k=1,2,\cdots,N$$

$$e_n\sim N\left(0,\sigma_{Zn}\right),f_n\sim N\left(0,\sigma_{Zn}\right),$$

$$\left\langle e_n e_k\right\rangle=0\forall n\neq k,\left\langle f_n f_k\right\rangle=0\forall n\neq k,\left\langle e_n f_k\right\rangle=0\forall n,k=1,2,\cdots,N$$

Generalization (continued)

$$\langle a_n c_k \rangle = \sigma_{acn}\delta_{nk};\langle a_n d_k \rangle = \sigma_{adn}\delta_{nk};\langle b_n c_k \rangle = \sigma_{bcn}\delta_{nk};\langle b_n d_k \rangle = \sigma_{bdn}\delta_{nk}$$

$$\langle a_n e_k \rangle = \sigma_{aen}\delta_{nk};\langle a_n f_k \rangle = \sigma_{afn}\delta_{nk};\langle b_n e_k \rangle = \sigma_{ben}\delta_{nk};\langle b_n f_k \rangle = \sigma_{bfn}\delta_{nk}$$

$$\langle c_n e_k \rangle = \sigma_{cen}\delta_{nk};\langle c_n f_k \rangle = \sigma_{cfn}\delta_{nk};\langle d_n e_k \rangle = \sigma_{den}\delta_{nk};\langle d_n f_k \rangle = \sigma_{dfn}\delta_{nk}$$

$$\sigma_{acn} = -\sigma_{bdn} \;\&\; \sigma_{adn} = -\sigma_{bcn}$$

$$\sigma_{aen} = -\sigma_{bfn} \;\&\; \sigma_{afn} = -\sigma_{ben}$$

$$\sigma_{cen} = -\sigma_{dfn} \;\&\; \sigma_{cfn} = -\sigma_{den}$$

Simulation of a multi - parameter random process

Let $f(x,t)$ be a random process evolving in x and t.

Let $\langle f(x,t)\rangle = 0$ &

$$\langle f(x,t) f(x+\xi, t+\tau)\rangle = R_{ff}(\xi,\tau)$$

$$R_{ff}(\xi,\tau) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{ff}(\lambda,\omega)\exp\left[-i(\lambda\xi+\omega\tau)\right]d\xi d\omega$$

$$f(x,t) = \sum_{n=1}^{N}\sum_{k=1}^{N} A_{nk}\cos\omega_n t\cos\lambda_n x + B_{nk}\cos\omega_n t\sin\lambda_n x$$
$$+C_{nk}\sin\omega_n t\cos\lambda_n x + D_{nk}\sin\omega_n t\sin\lambda_n x$$

$\{A_{nk}, B_{nk}, C_{nk}, D_{nk}\}_{n,k=1}^{N}$ = zero mean Gaussian random variables.

$$\left\{A_{nk}, B_{nk}, C_{nk}, D_{nk}\right\}_{n,k=1}^{N} = \text{zero mean Gaussian random variables.}$$

$$\left\langle A_{nk} A_{rs} \right\rangle = \sigma_{Ank} \delta_{nr} \delta_{ks}$$

$$\left\langle B_{nk} B_{rs} \right\rangle = \sigma_{Bnk} \delta_{nr} \delta_{ks}$$

$$\left\langle C_{nk} C_{rs} \right\rangle = \sigma_{Cnk} \delta_{nr} \delta_{ks}$$

$$\left\langle D_{nk} D_{rs} \right\rangle = \sigma_{Dnk} \delta_{nr} \delta_{ks}$$

$$\left\langle A_{nk} B_{rs} \right\rangle = \left\langle A_{nk} C_{rs} \right\rangle = \left\langle A_{nk} D_{rs} \right\rangle = 0 \forall n, k, r, s \in [1, N]$$

$$\left\langle B_{nk} C_{rs} \right\rangle = \left\langle B_{nk} D_{rs} \right\rangle = 0 \forall n, k, r, s \in [1, N]$$

$$\left\langle C_{nk} D_{rs} \right\rangle = 0 \forall n, k, r, s \in [1, N]$$

$$\left\langle f\left(x,t\right) f\left(x+\xi,t+\tau\right)\right\rangle = <[\sum_{n=1}^{N}\sum_{k=1}^{N} A_{nk}\cos\omega_n t\cos\lambda_n x + B_{nk}\cos\omega_n t\sin\lambda_n x$$

$$+C_{nk}\sin\omega_n t\cos\lambda_n x + D_{nk}\sin\omega_n t\sin\lambda_n x]$$

$$[\sum_{r=1}^{N}\sum_{s=1}^{N} A_{rs}\cos\omega_r\left(t+\tau\right)\cos\lambda_s\left(x+\xi\right) + B_{rs}\cos\omega_r\left(t+\tau\right)\sin\lambda_s\left(x+\xi\right)$$

$$+C_{rs}\sin\omega_r t\cos\lambda_s x + D_{rs}\sin\omega_r t\sin\lambda_s x]>$$

$$\Rightarrow$$

This can be reduced to the form

$$R_{ff}\left(\xi,x\right) = \sum_{n=1}^{N}\sum_{k=1}^{N}\Lambda_{nk}\cos\omega_n\tau\cos\lambda_n\xi$$

with

$$\Lambda_{nk} = \left[\frac{S\left(\omega_n,\lambda_k\right)}{4\pi^2}\Delta\omega_n\Delta\xi_n\right]^{\frac{1}{2}}$$

Markov process approach for simulation of non - Gaussian random processes

Let $X(t)$ be a stationary Gaussian random process defined on $[x_l, x_r]$.

Let $p_X(x)$ be the first order density function and the psd function (for the purpose of illustration) be

$$S_{XX}(\omega) = \frac{\alpha\sigma^2}{\pi(\omega^2 + \alpha^2)}; \alpha > 0$$

Here σ^2 is the meansquare value of $X(t)$.

Consider the SDE

$$dX(t) = -\alpha X dt + D(X)dB(t)$$

The psd of response in the steady state is obtainable.

The steady state solution to the govenring FPK equation is obtainable.

Demand that these solutions match with the target psd and pdf.

Determine drift and diffusion coefficients so that this becomes possible.

Multiply the above equation by $X(t-\tau)$ and take the ensemble average:

$$dX(t) = -\alpha X dt + D(X)\, dB(t)$$

$$\langle X(t-\tau)\, dX(t)\rangle = -\alpha \langle X(t-\tau) X(t)\rangle dt + \langle X(t-\tau) D[X(t)]\, dB(t)\rangle$$

$$\Rightarrow \frac{dR(\tau)}{d\tau} = -\alpha R(\tau)$$

$$\Rightarrow R(\tau) = A \exp(-\alpha|\tau|)$$

Select $A = \sigma^2$.

$$\sigma^2 \exp(-\alpha|\tau|) \Leftrightarrow \frac{\alpha\sigma^2}{\pi(\omega^2 + \alpha^2)}$$

Remark

$D[X(t)]$ has no influence on psd of $X(t)$.

$$dX(t) = -\alpha X dt + D(X)\, dB(t)$$

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\alpha x p(x)\right] - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[D^2(x)\, p(x)\right]$$

Steady state

$$-\frac{d}{dx}\left[\alpha x p(x)\right] - \frac{1}{2}\frac{d^2}{dx^2}\left[D^2(x)\, p(x)\right] = 0$$

$$\Rightarrow \alpha x p(x) + \frac{1}{2}\frac{d}{dx}\left[D^2(x)\, p(x)\right] = 0$$

Note:

Since $p(x)$ is specified, the above equation needs to be viewed as an equation for the unknown $D(x)$.

$$\alpha x p(x) + \frac{1}{2}\frac{d}{dx}\left[D^2(x)p(x)\right] = 0$$

$$\Rightarrow D^2(x) = -\frac{2\alpha}{p(x)}\int_{x_l}^{x} u p(u)\,du$$

Summary

To generate samples of $X(t)$ with pdf $p(x)$ and psd

$S(\omega) = \dfrac{\alpha\sigma^2}{\pi(\omega^2 + \alpha^2)}$, obtain steady state solutions of the SDE

$$dX(t) = -\alpha X dt + D(X)\,dB(t)$$

with $D^2(x) = -\dfrac{2\alpha}{p(x)}\displaystyle\int_{x_l}^{x} u p(u)\,du.$

VARIANCE REDUCTION

$$P_f = \int_{g(x)<0} p_X(x)\,dx = \int_{-\infty}^{\infty} I[g(x)]p_X(x)\,dx = \langle I[g(X)]\rangle$$

$$\Theta = \sum_{i=1}^{n} a_i I[g(X_i)]$$

$$\langle\Theta\rangle = \sum_{i=1}^{n} a_i \langle I[g(X_i)]\rangle = P_F \sum_{i=1}^{n} a_i$$

Select $\sum_{i=1}^{n} a_i = 1 \Rightarrow \Theta$ is an unbiased estimator

$$Var(\Theta) = \sum_{i=1}^{n} a_i^2 Var[I(g(X_i)]$$

$$\left\langle [I(g(X)]^2 \right\rangle = 1^2 P\{g(X)<0\} + 0P\{g(X)>0\} = P_F$$

$$\mathrm{Var}\left[I\left\{\left(g(X)\right)\right\}\right] = P_F - P_F^2$$

$$\mathrm{Var}\left(\Theta\right) = \sum_{i=1}^{n} a_i^2 P_F (1 - P_F)$$

Select $\left\{a_i\right\}_{i=1}^{n} \ni \mathrm{Var}\left(\Theta\right)$

is Minimized subject to $\sum_{i=1}^{n} a_i = 1.$

$$L = \sum_{i=1}^{n} a_i^2 P_F (1 - P_F) + \lambda \left[\sum_{i=1}^{n} a_i - 1\right]$$

$$\frac{\partial L}{\partial a_k} = 0 \Rightarrow 2a_k P_F (1 - P_F) + \lambda = 0; k = 1, 2, \cdots, n$$

$$\Rightarrow a_k = -\frac{\lambda}{2P_F (1 - P_F)}$$

$$\sum_{i=1}^{n} a_i = 1 \Rightarrow 1 = -\frac{\lambda n}{2P_F(1-P_F)}$$

$$\Rightarrow \lambda = -\frac{2P_F(1-P_F)}{n}$$

$$a_k = -\frac{2P_F(1-P_F)}{n}\left(-\frac{1}{2P_F(1-P_F)}\right) = \frac{1}{n}$$

$$\mathrm{Var}(\Theta) = \sum_{i=1}^{n} \frac{1}{n^2} P_F(1-P_F) = \frac{P_F(1-P_F)}{n}$$

$$\Rightarrow \sigma_\Theta = \sqrt{\frac{P_F(1-P_F)}{n}}$$

Illustration

$$\sigma = \sqrt{\frac{\mathrm{P_F}\left(1-P_F\right)}{n}} \Rightarrow$$

Coefficient of variation $\zeta = \frac{\sigma}{m} = \frac{1}{P_F}\sqrt{\frac{P_F\left(1-P_F\right)}{n}}$

$$\Rightarrow \zeta = \sqrt{\frac{\left(1-P_F\right)}{P_F n}} \approx \frac{1}{\sqrt{P_F n}} \text{ (for small } P_F\text{)}$$

$\Rightarrow$ Suppose $\zeta = 0.10 \& P_F \approx 10^{-5}$

$\Rightarrow$ Number of samples needed $n \approx 10^7$.

Similarly, for $\zeta = 0.01$, $P_F \approx 10^{-5}$

$\Rightarrow$ Number of samples needed $n \approx 10^9$.

Remarks

(1) Variance of estimator $\left(= \dfrac{P_F(1-P_F)}{n} \right)$ is independent of size of basic random variable vector X.

(2) If this variance is large, the utility of estimator becomes questionable.

(3) It appears that, in order to reduce the variance of the estimator we need to increase sample size n.

(4) Question: Can we reduce the variance of the estimator without increasing n?

$\Rightarrow$ **Variance reduction techniques**.

Problem of variance reduction: how to reduce $\mathrm{Var}(\Theta)$ without increasing sample size?

$$P_F = \int_{-\infty}^{\infty} I\{g(x) \le 0\}\, p_X(x)\, dx$$

This is re-written as

$$P_F = \int_{-\infty}^{\infty} \frac{I\{g(x) \le 0\}\, p_X(x)}{h_V(x)}\, h_V(x)\, dx$$

where $h_V(x)$ is a valid pdf and satisfies the condition

$p_X(x) > 0 \Rightarrow h_V(x) > 0.$

$$\Rightarrow P_F = \int_{-\infty}^{\infty} F(x) h_V(x) dx \text{ where}$$

$$F(x) = \frac{I\{g(x) \le 0\}\, p_X(x)}{h_V(x)}.$$

$$\Rightarrow P_F = \langle F(X) \rangle_h$$

$\langle \bullet \rangle_h$ = Expectation defined with respect to the pdf $h_V(x)$.

Note: at this stage the function $h_V(x)$ is yet undefined and needs to be suitably selected.

Let $J = \frac{1}{N}\sum_{i=1}^{N} F(V_i)$ where $\{\mathrm{V_i}\}_{i=1}^{N}$ are drawn from $h_V(x)$.

We have shown that J is an unbiased estimator for P_F which minimzes the sampling variance with the lowest sampling variance being

$$\mathrm{Var}(J) = \frac{\mathrm{Var}[F(V)]}{N}.$$

$$\mathrm{Var}[F(V)] = \left\langle \left\{ \frac{I[g(V) \leq 0]\, p_X(V)}{h_V(V)} - P_F \right\}^2 \right\rangle$$

We now select $h_V(v)$ such that $\text{Var}\left[F(V)\right]$ is minimized. Clearly if we select

$$h_V(v) = \frac{I\left[g(v) \le 0\right] p_X(v)}{P_F}$$

it follows $\text{Var}\left[F(V)\right] = 0.$

This would mean that even with one sample we will get the exact estimate of P_F.

The pdf $h_V(v)$ is called the **ideal importance sampling density function (ispdf)**.