Stochastic Structural Dynamics

Lecture-23

Markov Vector Approach-3

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General : n - dimensional Ito SDE

$$dX(t) = f\left[t, X(t)\right]dt + G\left[t, X(t)\right]dB(t); t \geq 0; X(0) = X_0$$

$$X(t), f\left[t, X(t)\right] \sim n \times 1$$

$$G\left[t, X(t)\right] \sim n \times m$$

$$dB(t) \sim m \times 1$$

$$\left\langle dB(t)\right\rangle = 0; \left\langle \Delta B_i(t)\Delta B_j(t+\tau)\right\rangle = 2D_{ij}\delta(\tau)$$

$$\alpha_j = f_j[t, \tilde{x}]; j = 1, 2, \cdots, n$$

$$\alpha_{ij} = 2\left[GDG^t\right]_{ij}(\tilde{x}); i, j = 1, 2, \cdots, m$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[\alpha_j p\right] + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{\partial^2}{\partial x_j \partial x_k}\left[\alpha_{jk} p\right]$$

$$p(\tilde{x}; 0 \mid \tilde{x}_0; 0) = \prod_{i=1}^{n} \delta(x_i - x_{i0}) + \text{BCS}$$

Remarks

- The application of Markov vector approach requires
- The excitations to be modeled as white noises or as outputs dynamical systems to white noise inputs.

The equations to be represented in the state space form.

Further questions

- How to solve the FPK equations?
  - A few selected examples for which exact solutions are possible
- Can we derive equations governing the moments?

Recall : Lagrange's method for solving linear PDE-s.

Consider the PDE of the form

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z)\cdots(1)$$

To obtain an integral of the above equation we consider the auxiliary equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Let two independent solutions of this equation be written as $u(x,y,z)=a \,\&\, v(x,y,z)=b$ where a and b are constants.

Then $\phi(u,v)=0$ is a solution of (1).

Alternatively, $u = f(v)$ is also a solution.

Illustration

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = xy$$

Auxiliary equation: $\dfrac{dx}{xz} = \dfrac{dy}{yz} = \dfrac{dz}{xy}$

Consider $\dfrac{dx}{xz} = \dfrac{dy}{yz} \Rightarrow \dfrac{y}{x} = a$

Consider $\dfrac{dx}{xz} = \dfrac{dz}{xy} \Rightarrow z^2 - xy = b$

General solution: $\phi\left(\dfrac{y}{x}, z^2 - xy\right) = 0$

Example : Linear MDOF systems

$$M\ddot{X}+C\dot{X}+KX=\Gamma W(t);t\geq 0;X(0)=X_0;\dot{X}(0)=\dot{X}_0$$

$$X(t)\sim N\times 1;\ \Gamma\sim n\times m;\ W(t)\sim m\times 1$$

$$\langle W(t)\rangle=0;\langle W(t)W^t(t+\tau)\rangle=\left[2D_{ij}\right]_{m\times m}\delta(\tau)$$

$$\ddot{X}+M^{-1}C\dot{X}+M^{-1}KX=M^{-1}\Gamma W(t)$$

$$Y=\begin{Bmatrix}Y_I\\Y_{II}\end{Bmatrix}=\begin{Bmatrix}X\\\dot{X}\end{Bmatrix}$$

$$dY_I=Y_{II}dt$$

$$dY_{II}=-M^{-1}CY_{II}-M^{-1}KY_I+M^{-1}\Gamma dB(t)$$

$$dY(t)=-PYdt+QdB(t)t\geq 0;Y(0)=Y_0$$

$$P=\begin{bmatrix}0 & -I\\ M^{-1}K & M^{-1}C\end{bmatrix};Q=\begin{bmatrix}0\\ M^{-1}\Gamma\end{bmatrix}$$

$$\underbrace{\begin{Bmatrix} dY_I(t) \\ dY_{II}(t) \end{Bmatrix}}_{2N\times 1} = \underbrace{\begin{bmatrix} 0 & -I \\ M^{-1}K & M^{-1}C \end{bmatrix}}_{2N\times 2N} \underbrace{\begin{Bmatrix} Y_I(t) \\ Y_{II}(t) \end{Bmatrix}}_{2N\times 1} dt + \underbrace{\begin{bmatrix} 0 \\ M^{-1}\Gamma \end{bmatrix}}_{2N\times m} \underbrace{dB(t)}_{m\times 1}$$

$$dY(t) = -PYdt + QdB(t)t \geq 0; Y(0) = Y_0$$

Consider the eigenvalue problem

$$P\phi = \lambda\phi$$

Let Φ be the $2N \times 2N$ matrix of eigenvectors and Λ be the $2N \times 2N$ diagonal matrix of complete set of eigenvalues of P.

$$\Rightarrow \; P\Phi = \Phi\Lambda \Rightarrow \Phi^{-1}P\Phi = \Lambda$$

Introduce the transformation $Y(t) = \Phi Z(t)$

$$Y(t) = \Phi Z(t)$$

$$\Phi dZ(t) = -P\Phi Z(t)\,dt + Q\,dB(t)$$

$$\Rightarrow dZ(t) = -\Phi^{-1}P\Phi Z(t)\,dt + \Phi^{-1}Q\,dB(t)$$

$$\Rightarrow dZ(t) = -\Lambda Z(t)\,dt + G\,dB(t)$$

$$\Rightarrow \left\{\alpha_j\right\} = -\Lambda z; \quad \left[\alpha_{ij}\right] = \left[GDG^t\right]$$

$$\frac{\partial p}{\partial t} = \sum_{j=1}^{2N} \frac{\partial}{\partial z_j}\left[\lambda_j z_j p\right] + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} \alpha_{jk} \frac{\partial^2 p}{\partial z_j \partial z_k};$$

$$p \equiv p\left(\tilde{z};t \mid \tilde{z}_0;0\right)$$

$$\text{ICS: } p\left(z;0 \mid \tilde{z}_0;0\right) = \prod_{i=1}^{2N} \delta\left(z_i - z_{i0}\right)$$

$$BCS: \lim_{z_i \to \pm\infty} p\left(\tilde{z};t \mid \tilde{z}_0;0\right) \to 0 \; \forall i = 1, 2, \cdots, 2N$$

$$\frac{\partial p}{\partial t} = \sum_{j=1}^{2N} \frac{\partial}{\partial z_j}\left[\lambda_j z_j p\right] + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\alpha_{jk}\frac{\partial^2 p}{\partial z_j \partial z_k}$$

Define the conditional characteristic function

$$M\left[\theta_1, \theta_2, \cdots, \theta_{2N}; t \mid Z(0) = z_0\right] = M\left(\tilde{\theta}; t\right) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p\left(\tilde{z}; t \mid \tilde{z}_0; 0\right)\exp\left(i\sum_{j=1}^{2N}\theta_j z_j\right) dz_1 dz_2 \cdots dz_{2N}$$

$$= \int_{-\infty}^{\infty} p\left(\tilde{z}; t \mid \tilde{z}_0; 0\right)\exp\left(i\sum_{j=1}^{2N}\theta_j z_j\right) d\tilde{z}$$

$$\frac{\partial M\left(\tilde{\theta}; t\right)}{\partial \theta_k} = \int_{-\infty}^{\infty} z_k\, p\left(\tilde{z}; t \mid \tilde{z}_0; 0\right)\exp\left(i\sum_{j=1}^{2N}\theta_j z_j\right) d\tilde{z}$$

$$M\left(\tilde{\theta};t\right)=\int_{-\infty}^{\infty} p\left(\tilde{z};t\mid\tilde{z}_0;0\right)\exp\left(i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{z}$$

$$\Rightarrow p\left(\tilde{z};t\mid\tilde{z}_0;0\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty} M\left(\tilde{\theta};t\right)\exp\left(-i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{\theta}$$

$$\frac{\partial M\left(\tilde{\theta};t\right)}{\partial\theta_k}=\int_{-\infty}^{\infty} iz_k p\left(\tilde{z};t\mid\tilde{z}_0;0\right)\exp\left(i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{z}$$

$$\Rightarrow z_k p\left(\tilde{z};t\mid\tilde{z}_0;0\right)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial M\left(\tilde{\theta};t\right)}{\partial\theta_k}\exp\left(-i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{\theta}$$

$$\frac{\partial}{\partial z_k}\left[z_k p\left(\tilde{z};t\mid\tilde{z}_0;0\right)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial M\left(\tilde{\theta};t\right)}{\partial\theta_k}\left(-i\theta_k\right)\exp\left(-i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{\theta}$$

$$\frac{\partial p}{\partial t} = \sum_{j=1}^{2N} \frac{\partial}{\partial z_j}\left[\lambda_j z_j p\right] + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\alpha_{jk}\frac{\partial^2 p}{\partial z_j \partial z_k}$$

$$p\left(\tilde{z};t \mid \tilde{z}_0;0\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M\left(\tilde{\theta};t\right)\exp\left(-i\sum_{l=1}^{2N}\theta_l z_l\right)d\tilde{\theta}$$

$$\frac{\partial p}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial M\left(\tilde{\theta};t\right)}{\partial t}\exp\left(-i\sum_{l=1}^{2N}\theta_l z_l\right)d\tilde{\theta}$$

$$\frac{\partial}{\partial z_j}\left[z_j p\right] = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial M\left(\tilde{\theta};t\right)}{\partial \theta_j}\left(-i\theta_j\right)\exp\left(-i\sum_{l=1}^{2N}\theta_l z_l\right)d\tilde{\theta}$$

$$\frac{\partial^2 p}{\partial z_j \partial z_k} = \frac{1}{2\pi}\int_{-\infty}^{\infty} M\left(\tilde{\theta};t\right)\left(-i\theta_j\right)\left(-i\theta_k\right)\exp\left(-i\sum_{l=1}^{2N}\theta_l z_l\right)d\tilde{\theta}$$

$$\frac{\partial p}{\partial t}=\sum_{j=1}^{2N}\frac{\partial}{\partial z_j}\left[\lambda_j z_j p\right]+\frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m}\alpha_{jk}\frac{\partial^2 p}{\partial z_j \partial z_k}$$

$$\Rightarrow \frac{\partial M}{\partial t}=-\sum_{j=1}^{2N}\lambda_j\theta_j\frac{\partial M}{\partial \theta_j}-\frac{1}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}\alpha_{jk}\theta_j\theta_k M$$

$$\Rightarrow$$

$$\frac{dt}{1}=\frac{d\theta_1}{\lambda_1\theta_1}=\frac{d\theta_2}{\lambda_2\theta_2}=\cdots=\frac{d\theta_{2N}}{\lambda_{2N}\theta_{2N}}=\frac{dM}{-\frac{1}{2}\sum_{j=1}^{2N}\sum_{k=1}^{2N}\alpha_{jk}\theta_j\theta_k M}$$

$$\frac{dt}{1}=\frac{d\theta_i}{\lambda_i\theta_i};i=1,2,\cdots,2N\Rightarrow\theta_i\left(t\right)=\theta_{i0}\exp\left(\lambda_i t\right);i=1,2,\cdots,2N$$

$$\Rightarrow\left\{\theta_{i0}\right\}=\left[\Omega\right]\left\{\theta_i\left(t\right)\right\}$$

$$\Omega=\text{diagonal matrix with entries }\exp\left(-\lambda_i t\right)$$

Consider

$$\frac{d\theta_i}{\lambda_i\theta_i} = -\frac{dM}{\left(\frac{1}{2}\right)\sum_{j=1}^{2N}\sum_{k=1}^{2N}\alpha_{jk}\theta_j\theta_k M}\cdots(a)\ \&\ \frac{d\theta_l}{\lambda_l\theta_l} = -\frac{dM}{\left(\frac{1}{2}\right)\sum_{j=1}^{2N}\sum_{k=1}^{2N}\alpha_{jk}\theta_j\theta_k M}\cdots(b)$$

$$(a)\lambda_i\theta_i\theta_l + (b)\lambda_l\theta_i\theta_l \Rightarrow \theta_l d\theta_i + \theta_i d\theta_l = -\frac{(\lambda_i+\lambda_l)\theta_l\theta_i dM}{\left(\frac{1}{2}\right)\sum_{j=1}^{2N}\sum_{k=1}^{2N}\alpha_{jk}\theta_j\theta_k M}$$

$$\frac{d(\theta_l\theta_i)}{(\lambda_i+\lambda_l)} = -\frac{\theta_l\theta_i dM}{\left(\frac{1}{2}\right)\sum_{j=1}^{2N}\sum_{k=1}^{2N}\alpha_{jk}\theta_j\theta_k M}$$

Multiply both sides by $\frac{\alpha_{il}}{2}$ and sum over i and l

$$\frac{dM}{M} = -\sum_{i=1}^{2N}\sum_{l=1}^{2N}\frac{1}{2}\frac{\alpha_{il}}{(\lambda_i+\lambda_l)}d(\theta_l\theta_i)$$

$$\frac{dM}{M} = -\sum_{i=1}^{2N}\sum_{l=1}^{2N}\frac{1}{2}\frac{\alpha_{il}}{(\lambda_i+\lambda_l)}d(\theta_l\theta_i)$$

$$\Rightarrow M(\tilde{\theta},t) = M_0\exp\left[-\sum_{i=1}^{2N}\sum_{l=1}^{2N}\frac{1}{2}\frac{\alpha_{il}\theta_l\theta_i}{(\lambda_i+\lambda_l)}\right]$$

Define $\Xi = \left[\frac{\alpha_{il}}{(\lambda_i+\lambda_l)}\right] \Rightarrow$

$$M(\tilde{\theta},t) = M_0\exp\left[-\frac{1}{2}\tilde{\theta}^t\Xi\tilde{\theta}\right]$$

We have $p(\tilde{z};t\,|\,z_0;0) = \prod_{i=1}^{2N}\delta(z_i - z_{i0})$ & $\{\theta_{i0}\} = [\Omega]\{\theta_i(t)\}$

$$M(\tilde{\theta};0) = \int_{-\infty}^{\infty}\prod_{i=1}^{2N}\delta(z_i - z_{i0})\exp\left(i\sum_{j=1}^{2N}\theta_j z_j\right)d\tilde{z} = \exp\left(i\theta_0^t z_0\right)$$

$$\Rightarrow M_0 \exp\left[-\frac{1}{2}\tilde{\theta}_0^t \Xi \tilde{\theta}_0\right] = \exp\left(i\theta_0^t z_0\right)$$

$$M_0 = \exp\left(i\theta_0^t z_0 + \frac{1}{2}\tilde{\theta}_0^t \Xi \tilde{\theta}_0\right)$$

$$M\left(\tilde{\theta}, t\right) = \exp\left[i\theta_0^t z_0 + \frac{1}{2}\tilde{\theta}^t \Omega^t \Xi \Omega \tilde{\theta} - \frac{1}{2}\tilde{\theta}^t \Xi \tilde{\theta}\right]$$

$$= \exp\left[i\theta_0^t z_0 - \frac{1}{2}\tilde{\theta}^t \left(\Xi - \Omega^t \Xi \Omega\right)\tilde{\theta}\right]$$

This is the characteristic function of a multivariate Gaussian PDF.

The mean vector and covariance matrix can be evaluated from the charateristic function.

The PDF in the original coordinate system can be obtained by using the transformation $Y(t) = \Phi Z(t)$.

Remarks

- For linear systems, the exact solution can also be obtained using convolution integral approach discussed earlier in this course. The Markov vector approach does not offer any special advantage here.
- The above formulation is also valid when excitations are modeled as filtered white noise excitations.

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = f(t); t \geq 0; x(0) = x_0; \dot{x}(0) = \dot{x}_0$$

$$\ddot{f} + 2\xi\lambda\dot{f} + \lambda^2 f = w(t); t \geq 0; f(0) = f_0; \dot{f}(0) = \dot{f}_0$$

$$\langle w(t) \rangle = 0; \langle w(t) w(t+\tau) \rangle = 2D\delta(\tau)$$

$$X(t) = \{ x(t) \quad \dot{x}(t) \quad f(t) \quad \dot{f}(t) \}^t \text{ is Markov}$$

$$dX(t) = -PXdt + QdB(t) t \geq 0; X(0) = X_0$$

Example: First order nonlinear systems

$$\dot{x} + \beta(x) = w(t); t \geq 0 \;\&\; x(0) = x_0$$

$$\langle w(t) \rangle = 0; \langle w(t_1) w(t_2) \rangle = 2D\delta(t_1 - t_2)$$

$$dx = -\beta\left[x(t)\right]dt + dB(t)$$

$$\frac{\partial p}{\partial t} = \frac{\partial\left[\beta(x)p\right]}{\partial x} + D\frac{\partial^2 p}{\partial x^2}$$

Transient solutions by eigenfunction expansion.

As $t \to \infty, \dfrac{\partial p}{\partial t} \to 0 \Rightarrow$

Stationary solution : $\dfrac{d\left[\beta(x)p\right]}{dx} + D\dfrac{d^2 p}{dx^2} = 0$

$$\frac{d\left[\beta(x)p\right]}{dx}+D\frac{d^2p}{dx^2}=0;\ \lim_{x\to\pm\infty}p(x)\to 0$$

$$D\frac{dp}{dx}=-\beta(x)p$$

$$\Rightarrow\ p(x)=C\exp\left[-\frac{1}{D}\int_0^x\beta(s)ds\right];-\infty<x<\infty$$

Select C such that $\int_{-\infty}^{\infty}p(x)dx=1$

Example: $\beta(x)=ax+bx^3$

$$p(x)=C\exp\left[-\frac{1}{D}\left(\frac{ax^2}{2}+\frac{bx^4}{4}\right)\right];-\infty<x<\infty$$

• $b=0\Rightarrow$ pdf is Gaussian, as it should be.

Example : sdof system with nonlinear damping and nonlinear stiffness

$$\ddot{x} + \dot{x} f\left(H\right) + g\left(x\right) = w\left(t\right); t \geq 0; x\left(0\right) = x_0; \dot{x}\left(0\right) = \dot{x}_0.$$

$$\left\langle w\left(t\right)\right\rangle = 0; \left\langle w\left(t\right) w\left(t+\tau\right)\right\rangle = 2D\delta\left(\tau\right)$$

$$H = \frac{\dot{x}^2}{2} + \int_0^x g\left(u\right) du = \text{Total energy}$$

$$\begin{Bmatrix} X_1\left(t\right) \\ X_2\left(t\right) \end{Bmatrix} = \begin{Bmatrix} x\left(t\right) \\ \dot{x}\left(t\right) \end{Bmatrix}$$

$$dX_1\left(t\right) = X_2\left(t\right) dt$$

$$dX_2\left(t\right) = \left[-X_2 g\left(H\right) - g\left(X_1\right)\right] dt + dB\left(t\right)$$

$$H = \frac{X_2^2}{2} + \int_0^{X_1} g\left(u\right) du$$

$$\frac{\partial p}{\partial t} = -x_2 \frac{\partial p}{\partial x_1} - \frac{\partial}{\partial x_2}\left[\left\{-x_2 F\left(H\right) - g\left(x_1\right)\right\} p\right] + D\frac{\partial^2 p}{\partial x_2^2}$$

$$p \equiv p\left(x_1, x_2; t \mid x_0, \dot{x}_0\right) = tpdf$$

$$p\left(x_1, x_2; 0 \mid x_0, \dot{x}_0\right) = \delta\left(x_1 - x_0\right)\delta\left(x_2 - \dot{x}_0\right)$$

$$p\left(\pm\infty, x_2; t \mid x_0, \dot{x}_0\right) = p\left(x_1, \pm\infty; t \mid x_0, \dot{x}_0\right) = 0$$

Steady state

$$-x_2 \frac{\partial p}{\partial x_1} - \frac{\partial}{\partial x_2}\left[\left\{-x_2 F\left(H\right) - g\left(x_1\right)\right\} p\right] + D\frac{\partial^2 p}{\partial x_2^2} = 0$$

$$-x_2 \frac{\partial p}{\partial x_1} - \frac{\partial\left[-x_2 F\left(H\right) p\right]}{\partial x_2} + g\left(x_1\right)\frac{\partial p}{\partial x_2} + D\frac{\partial^2 p}{\partial x_2^2} = 0$$

$$-x_2 \frac{\partial p}{\partial x_1} - \frac{\partial\left[-x_2 F(H)\, p\right]}{\partial x_2} + g(x_1)\frac{\partial p}{\partial x_2} + D\frac{\partial^2 p}{\partial x_2^2} = 0$$

Look for solutions such that

$$-x_2 \frac{\partial p}{\partial x_1} + g(x_1)\frac{\partial p}{\partial x_2} = 0$$

$$-\frac{\partial\left[-x_2 F(H)\, p\right]}{\partial x_2} + D\frac{\partial^2 p}{\partial x_2^2} = 0$$

Consider $-x_2 \frac{\partial p}{\partial x_1} + g(x_1)\frac{\partial p}{\partial x_2} = 0$

Solve this equation using Lagrange's method.

Auxiliary equation: $\frac{dx_1}{-x_2} = \frac{dx_2}{g(x_1)} = \frac{dp}{0}$

$$\frac{dx_1}{-x_2} = \frac{dx_2}{g\left(x_1\right)} = \frac{dp}{0}$$

$$\frac{dx_1}{-x_2} = \frac{dx_2}{g\left(x_1\right)} \Rightarrow \frac{x_2^2}{2} + \int_0^{x_1} g\left(u\right) du = a$$

$$\frac{dx_2}{g\left(x_1\right)} = \frac{dp}{0} \Rightarrow p = b$$

$$p = \Phi\left(\frac{x_2^2}{2} + \int_0^{x_1} g\left(u\right) du\right) = \Phi\left(H\right)$$

Now consider the equation $-\frac{\partial\left[-x_2 F\left(H\right) p\right]}{\partial x_2} + D\frac{\partial^2 p}{\partial x_2^2} = 0$

$$\frac{\partial}{\partial x_2}\left\{-x_2 F\left(H\right) p + D\frac{\partial p}{\partial x_2}\right\} = 0$$

$$\frac{\partial}{\partial x_2}\left\{-x_2 F(H)\,p + D\frac{\partial p}{\partial x_2}\right\} = 0$$

$$\Rightarrow x_2 F(H)\Phi(H) + D\frac{\partial \Phi}{\partial H}x_2 = 0$$

$$\Rightarrow \frac{d\Phi}{dH} = -\frac{1}{D}F(H)\Phi(H)$$

$$\Rightarrow \Phi(H) = \Phi_0 \exp\left[-\frac{1}{D}\int_0^H F(\xi)\,d\xi\right]$$

$$p(x_1, x_2; t) = \Phi_0 \exp\left[-\frac{1}{D}\int_0^H F(\xi)\,d\xi\right]; -\infty < x_1, x_2 < \infty$$

$$H = \frac{x_2^2}{2} + \int_0^{x_1} g(u)\,du$$

Case - 1 $F(H) = 2\eta\omega$

$$\ddot{x} + 2\eta\omega\dot{x} + g(x) = w(t); t \geq 0; x(0) = x_0; \dot{x}(0) = \dot{x}_0.$$

$$\langle w(t) \rangle = 0; \langle w(t) w(t+\tau) \rangle = 2D\delta(\tau)$$

$$H = \frac{\dot{x}^2}{2} + \int_0^x g(u)\,du = \text{Total energy}$$

$$p(x_1, x_2; t) = \Phi_0 \exp\left[-\frac{1}{D}\int_0^H F(\xi)\,d\xi\right]; -\infty < x_1, x_2 < \infty$$

$$= \Phi_0 \exp\left[-\frac{1}{D}(2\eta\omega H)\right]$$

$$p(x, \dot{x}; t) = \Phi_0 \exp\left[-\frac{2\eta\omega}{D}\left(\frac{\dot{x}^2}{2} + \int_0^x g(u)\,du\right)\right]; -\infty < x, \dot{x} < \infty$$

Let $g(x) = \omega^2 x + \alpha x^3 \Rightarrow$

$$p(x, \dot{x}; t) = \Phi_0 \exp\left[-\frac{2\eta\omega}{D}\left(\frac{\dot{x}^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\alpha x^4}{4}\right)\right]; -\infty < x, \dot{x} < \infty$$

Φ_0 to be selected such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, \dot{x}; t)\, dx\, d\dot{x} = 1.$

Remarks

- This is an exact solution
- If α=0, the result for linear sdof system under white noise is recovered.
- It can be verified that $p(x_1, x_2; t) = p(x_1; t)\, p(x_2; t)$
- We know that $X(t)$ and $\dot{X}(t)$ are uncorrelated and it turns out that $X(t)$ and $\dot{X}(t)$ are independent

Remarks (continued)

- It can be verified that

$$p(x;t) = \Phi_{01} \exp\left[-\frac{2\eta\omega}{D}\left(\frac{\omega^2 x^2}{2} + \frac{\alpha x^4}{4} \right) \right]; -\infty < x < \infty$$

Non-G

$$p(\dot{x};t) = \Phi_{02} \exp\left[-\frac{2\eta\omega}{D}\frac{\dot{x}^2}{2} \right]; -\infty < \dot{x} < \infty$$

Gaussian

- The velocity is a gaussian random variable while displacement is a nongaussian random variable.
- The velocity is not a Gaussian random process (if it were so, displacement would also be Gaussian).

Thus velocity is a non-Gaussain random process with a first order pdf that is Gaussian.

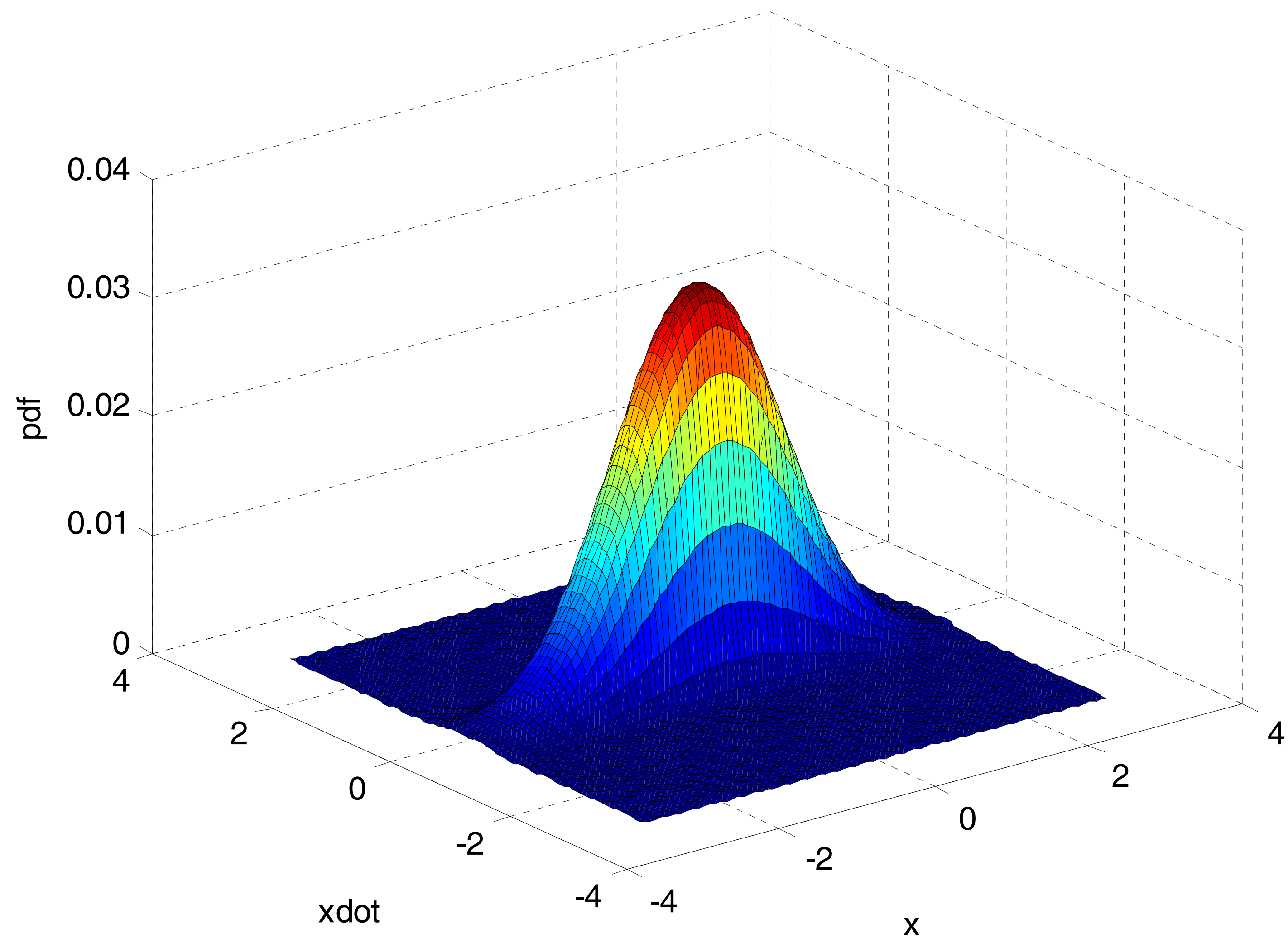
Remarks (continued)

$$\bullet H = \underbrace{\frac{\dot{x}^2}{2}}_{KE} + \underbrace{\frac{\omega^2 x^2}{2} + \frac{\alpha x^4}{4}}_{PE}$$ represents the total energy.

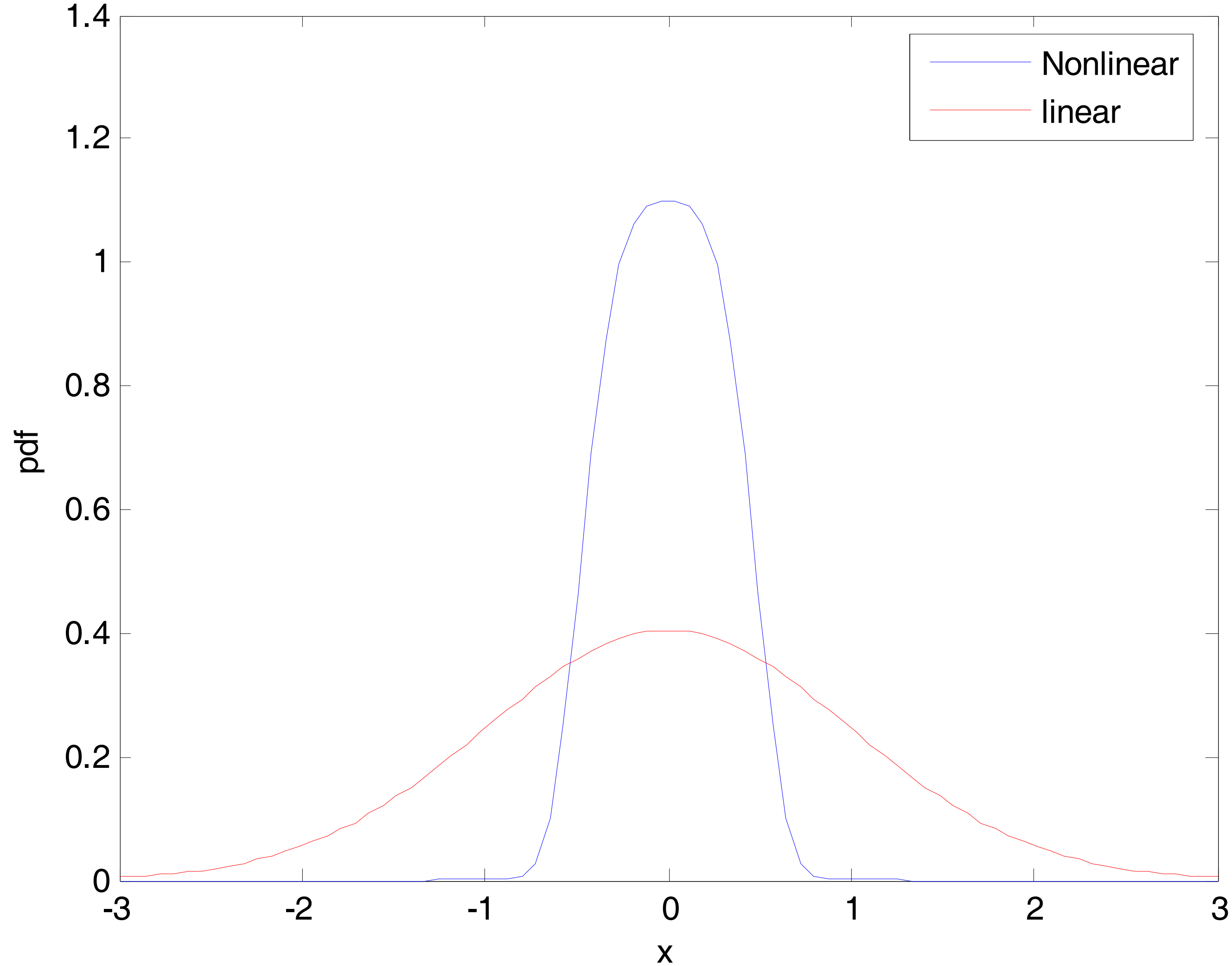
$$p(x_1, x_2; t) = \Phi_0 \exp\left[-\frac{2\eta\omega}{D}\left(\frac{\dot{x}^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\alpha x^4}{4}\right)\right]; -\infty < x, \dot{x} < \infty$$

$\Rightarrow H$ is exponentially distributed

$$p_H(h;t) = \Phi_0 \exp\left(-\frac{2\eta\omega}{D}h\right); 0 < h < \infty$$


0.04
0.03
0.02
0.01
0
pdf
4
2
0
-2
-4
xdot
-4
-2
0
2
4
x

pdf of displacement



Remarks (continued)

- We have determined

$$p\left(x,\dot{x};t\right)=\Phi_0\exp\left[-\frac{2\eta\omega}{D}\left(\frac{\dot{x}^2}{2}+\frac{\omega^2x^2}{2}+\frac{\alpha x^4}{4}\right)\right];-\infty<x,\dot{x}<\infty$$

Therefore we are in a position to study

◦Number of times a level ς is crossed in an interval 0 to T

$$N(T)=\int_0^T\left|\dot{X}\left(t\right)\right|\delta\left[X\left(t\right)-\varsigma\right]dt$$

◦PDF of first passge times

$$P\left[T_f\left(\varsigma\right)>t\right]=P\left[N^+\left(\varsigma,0,T\right)=0\right]$$

◦PDF of extremes of $X(t)$ over an interval 0 to T

$$P_{X_m}\left(\varsigma\right)=P\left[X_m\le\varsigma\right]=P\left[T_f\left(\varsigma\right)>T\right]$$

$$\left\langle n^{+}(\alpha,t)\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{y}|\,\delta[y-\alpha]\,p\left(y,\dot{y}\right)dy\,d\dot{y}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{y}|\,\delta[y-\alpha]\,p\left(y\right)p\left(\dot{y}\right)dy\,d\dot{y}$$

$$= p_{Y}(\alpha)\int_{-\infty}^{\infty} |\dot{y}|\,p_{\dot{Y}}(\dot{y})\,d\dot{y}$$

$$\textbf{Case - 2}\; F(H) = -\left(1 - x^2 - \dot{x}^2\right); g(x) = x$$

$$\ddot{x} - \dot{x}\left(1 - x^2 - \dot{x}^2\right) + x = w(t); t \geq 0; x(0) = x_0; \dot{x}(0) = \dot{x}_0.$$

$$\langle w(t) \rangle = 0; \langle w(t) w(t+\tau) \rangle = 2D\delta(\tau)$$

$$H = \frac{\dot{x}^2}{2} + \frac{x^2}{2} = \text{Total energy} \Rightarrow \text{F}(H) = -(1 - 2H)$$

$$p(x_1, x_2; t) = \Phi_0 \exp\left[-\frac{1}{D}\int_0^H F(\xi)\, d\xi\right]; -\infty < x_1, x_2 < \infty$$

$$= \Phi_0 \exp\left[\frac{1}{D}\int_0^H (1 - 2\xi)\, d\xi\right] = \Phi_0 \exp\left[\frac{1}{D}\left(H - H^2\right)\right]$$

$$p(x, \dot{x}; t) = \Phi_0 \exp\left[\frac{1}{D}\left\{\left(\frac{\dot{x}^2}{2} + \frac{x^2}{2}\right) - \left(\frac{\dot{x}^2}{2} + \frac{x^2}{2}\right)^2\right\}\right]; -\infty < x_1, x_2 < \infty$$

$$p\left(x,\dot{x};t\right)=\Phi_0\exp\left[\frac{1}{D}\left\{\left(\frac{\dot{x}^2}{2}+\frac{x^2}{2}\right)-\left(\frac{\dot{x}^2}{2}+\frac{x^2}{2}\right)^2\right\}\right];-\infty<x,\dot{x}<\infty$$

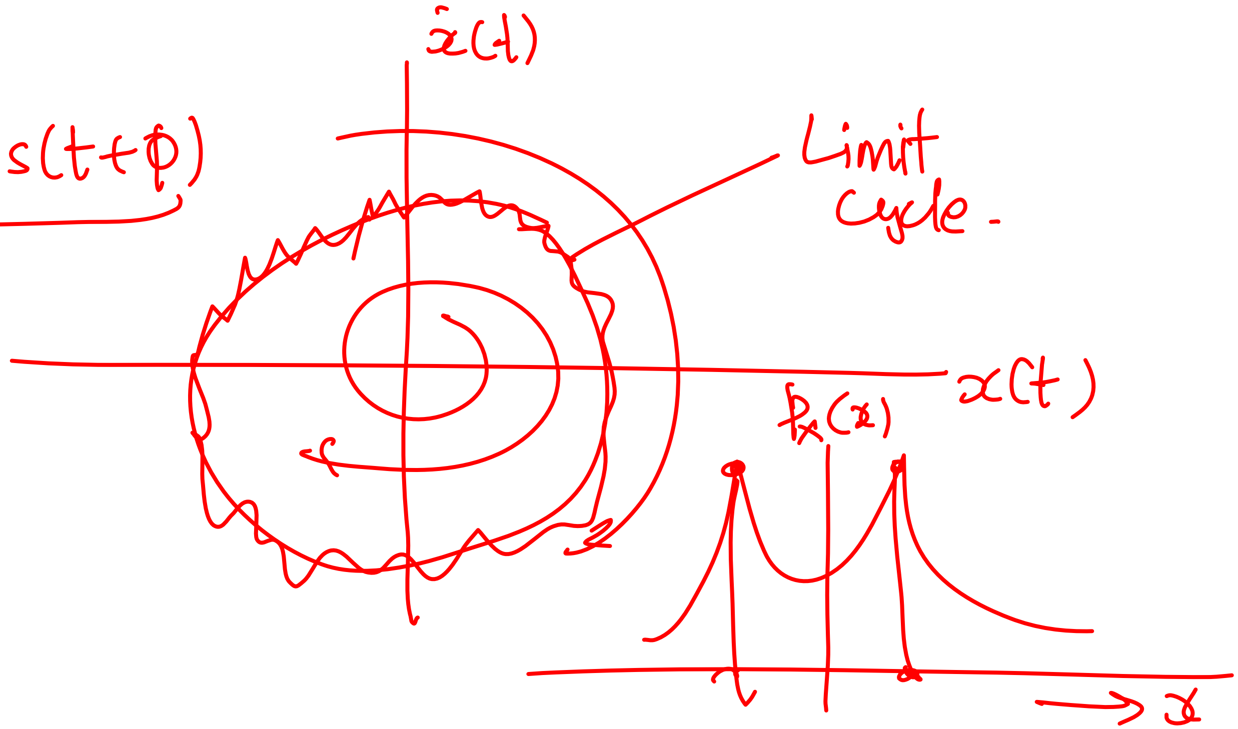
Remarks

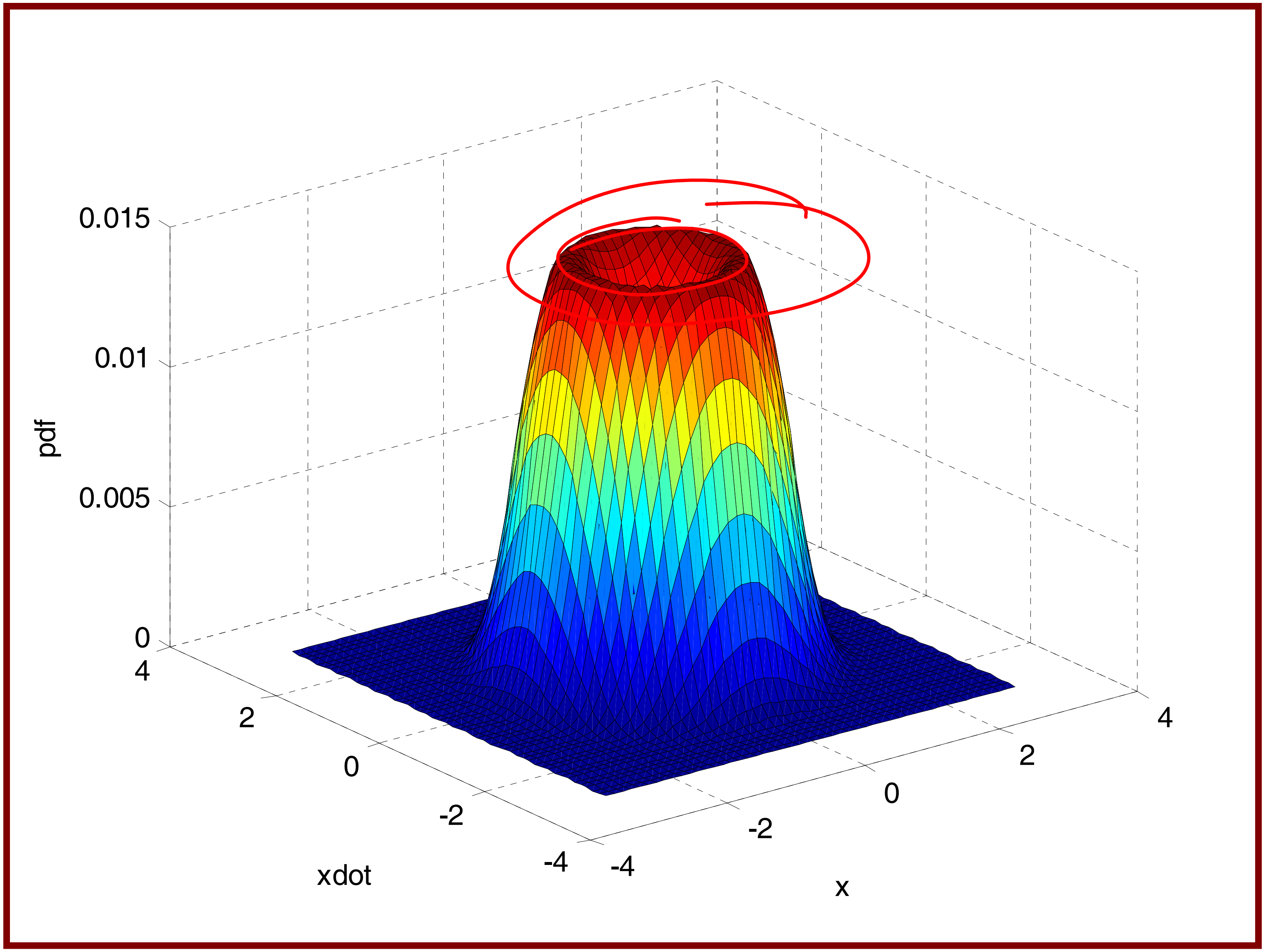
- Displacement and velocity here are non-Gaussian and dependent (although they are uncorrelated)
- In the absence of noise, the system has a stable limit cycle.
- The jpdf has a modal-line in the neighbourhood of the limit cycle.
- The pdf of dispalcement and velocity are not uni-modal.

Limit cycle oscillations

$$\ddot{x} - \dot{x}\left(1 - x^2 - \dot{x}^2\right) + x = 0; x\left(0\right) = x_0; \dot{x}\left(0\right) = \dot{x}_0$$

$x(t) = \cos(t + \phi)$

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0.01
0.005
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0
-2
-4
xdot
-4
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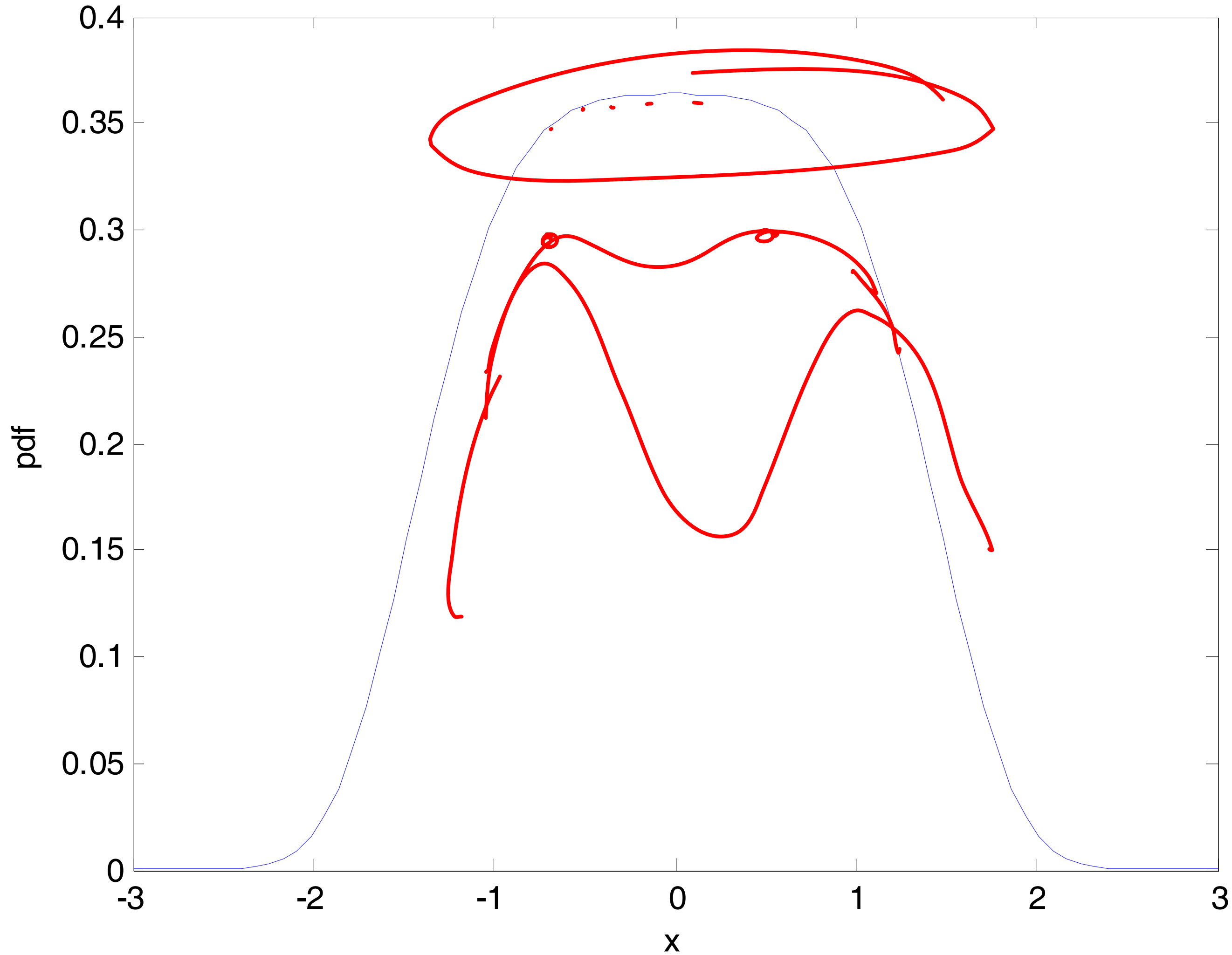
4
x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1
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3
x

Example: Nonlinear *n*-dof systems

$$m_j \ddot{X}_j + m_j \beta_j \dot{X}_j + \frac{\partial U}{\partial X_j} = W_j(t); t \geq 0;$$

$$X_j(0) = x_{j0} \; \& \; \dot{X}_j(0) = \dot{x}_{j0}$$

$$U = \frac{1}{2} X^t K X = \text{Potential energy of the structure}$$

$$\langle W_j(t) \rangle = 0; \langle W_j(t) W_k(t+\tau) \rangle = 2D_{jk} \delta_{jk} \delta(\tau);$$

$$\frac{m_j \beta_j}{2D_{jj}} = \gamma \forall j = 1, 2, \cdots, n$$

$$dY_j = Y_{j+n} dt$$

$$dY_{j+n} = \left(-\beta_j Y_{j+n} - \frac{1}{m_j} \frac{\partial U}{\partial Y_j} \right) dt + \frac{1}{m_j} dB_j(t)$$

$$\alpha_j = y_{j+n};$$

$$\alpha_{j+n} = \left(-\beta_j y_{j+n} - \frac{1}{m_j} \frac{\partial U}{\partial y_j} \right)$$

$$\alpha_{ij} = 0 \forall i \neq j; \alpha_{jj} = 0 \forall j < n;$$

$$\alpha_{j+n,j+n} = \frac{2D_{jj}}{m_j^2}$$

Exercise : show that the steady state pdf is given by

$$p(\tilde{y};t) = C \exp\left\{ -\frac{\gamma}{\pi} \left[\frac{1}{2} \sum_{j=1}^{n} m_j^2 y_{j+n}^2 + U(y_1, y_2, \cdots, y_n) \right] \right\};$$

$$-\infty < y_i < \infty \forall i = 1, 2, \cdots, 2n$$

Moment equations

$$dX(t) = f\left[X(t),t\right]dt + G\left[X(t),t\right]dB(t); t \geq 0; X(0) = X_0$$

$$X(t), f \sim n\times 1; dB(t) \sim m\times 1; G \sim n\times m$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[f(x,t)p\right] + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^t\right)p\right]$$

Denote

$$m_{k_1,k_2,\cdots,k_n}(t) = \left\langle X_1^{k_1}(t) X_2^{k_2}(t) \cdots X_n^{k_n}(t)\right\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}\, p\left(\tilde{x};t \mid \tilde{x}_0;0\right)p\left(x_0;0\right)d\tilde{x}dx_0$$

$$\dot{m}_{k_1,k_2,\cdots,k_n}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}\frac{\partial}{\partial t}p\left(\tilde{x};t \mid \tilde{x}_0;0\right)p\left(x_0;0\right)d\tilde{x}dx_0$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[f\left(x,t\right) p\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^t\right) p\right]$$

$$\dot{m}_{k_1,k_2,\cdots,k_n}\left(t\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \frac{\partial}{\partial t} p\left(\tilde{x};t \mid \tilde{x}_0;0\right) p\left(x_0;0\right) d\tilde{x}\, dx_0$$

$$\dot{m}_{k_1,k_2,\cdots,k_n}\left(t\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

$$-\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[f\left(x,t\right) p\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^t\right) p\right] p\left(x_0;0\right) d\tilde{x}\, dx_0$$

Integrate the terms on the RHS by parts.

General formulation

Let $h[X, t]$ be a function which is well behaved

That is, $\dfrac{\partial^2 h}{\partial x_i \partial x_j}$ & $\dfrac{\partial h}{\partial t}$ exist, continuous, bounded on any

finite interval of x and t.

Consider $\delta h = h(X + \delta X, t + \delta t) - h(X, t)$

Taylor's expansion

$$\delta h = \sum_{j=1}^{n} \delta X_j \frac{\partial h}{\partial X_j} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta X_i \delta X_j \frac{\partial^2 h}{\partial X_i \partial X_j} +$$

$$\delta t \frac{\partial h}{\partial t} + o\left(\delta X \delta X^t\right) + o(\delta t)$$

$$dX(t) = f\left[X(t),t\right]dt + G\left[X(t),t\right]dB(t); t \geq 0; X(0) = X_0$$

$$X(t), f \sim n\times 1; dB(t) \sim m\times 1; G \sim n\times m$$

$$\langle \delta h \mid X\rangle = \sum_{j=1}^{n} f_j(X,t)\frac{\partial h}{\partial X_j}\delta t + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(GDG^t\right)_{ij}\frac{\partial^2 h}{\partial X_i \partial X_j}\delta t +$$

$$\delta t\frac{\partial h}{\partial t} + o(\delta t)$$

Digress : Let X and Y be two random variables

$$\langle Y \mid X = x\rangle = \int_{-\infty}^{\infty} yp(y \mid x)dy$$

$$E\left[\langle Y \mid X = x\rangle\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yp(y \mid x)p(x)dy = \langle Y\rangle$$

$$\left\langle \delta h \mid X \right\rangle = \sum_{j=1}^{n} f_j\left(X,t\right)\frac{\partial h}{\partial X_j}\delta t + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(GDG^t\right)_{ij}\frac{\partial^2 h}{\partial X_i \partial X_j}\delta t + \delta t\left\langle \frac{\partial h}{\partial t}\right\rangle + o\left(\delta t\right)$$

$$E\left[\left\langle \delta h \mid X \right\rangle\right] = \left\langle \delta h \right\rangle$$

$$\left\langle \delta h \right\rangle = \sum_{j=1}^{n}\left\langle f_j\left(X,t\right)\frac{\partial h}{\partial X_j}\right\rangle \delta t + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle \left(GDG^t\right)_{ij}\frac{\partial^2 h}{\partial X_i \partial X_j}\right\rangle \delta t + \delta t\left\langle \frac{\partial h}{\partial t}\right\rangle + o\left(\delta t\right)$$

Divide by δt and consider the limit $\delta t \to 0 \Rightarrow$

$$\frac{d}{dt}\left\langle h\left[X\left(t\right),t\right]\right\rangle = \left\langle \frac{\partial h}{\partial t}\right\rangle + \sum_{j=1}^{n}\left\langle f_j\left(X,t\right)\frac{\partial h}{\partial X_j}\right\rangle + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle \left(GDG^t\right)_{ij}\frac{\partial^2 h}{\partial X_i \partial X_j}\right\rangle$$

Example

$$\dot{X} + \beta X = w(t); t \geq 0; X(0) = X_0$$

$$dX(t) = -\beta X dt + dB(t)$$

$$f = -\beta x$$

$$\frac{d}{dt}\left\langle h\left[X(t), t\right]\right\rangle = \left\langle \frac{\partial h}{\partial t}\right\rangle + \left\langle -\beta X \frac{\partial h}{\partial X_j}\right\rangle + D\left\langle \frac{\partial^2 h}{\partial X^2}\right\rangle$$

$$h = X^k; m_k = \left\langle X^k \right\rangle$$

$$\dot{m}_k = -\beta m_k + Dk(k-1)m_{k-2}$$

$$\dot{m}_1 = -\beta m_1$$

$$\dot{m}_2 = -\beta m_2 + 2D$$

$$\dot{m}_3 = -\beta m_3 + 6Dm_1$$

$$\dot{m}_4 = -\beta m_4 + 12Dm_2$$

$$\vdots$$

Steady state $\dot{m}_k = 0$

$$\dot{m}_1 = -\beta m_1 = 0 \Rightarrow m_1 = 0$$

$$\dot{m}_2 = -\beta m_2 + 2D = 0 \Rightarrow -\beta m_2 + 2D = 0; m_2 = \frac{2D}{\beta} = \sigma_X^2$$

$$\dot{m}_3 = -\beta m_3 + 6Dm_1 = 0 \Rightarrow m_3 = 0$$

$$\dot{m}_4 = -\beta m_4 + 12Dm_2 = 0 \Rightarrow m_4 = \frac{12D}{\beta} = 3\left(\frac{2D}{\beta}\right)^2 = 3\sigma_X^4$$

Remarks

- Moment equations are closed
- Moments in the steady state display Gaussian properties which is as it must be

Example

$$\ddot{X} + 2\eta\omega\dot{X} + \omega^2 X = w(t); t \geq 0; X(0) \,\&\, \dot{X}(0) \text{ specified.}$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left(-2\eta\omega X_2 - \omega^2 X_1\right) dt + dB(t)$$

$$\left\langle h\left(X_1, X_2\right)\right\rangle = \left\langle X_1^m X_2^n \right\rangle = m_{mn}$$

$$\frac{d}{dt}\left\langle h\left[X(t), t\right]\right\rangle = \left\langle \frac{\partial h}{\partial t} \right\rangle +$$

$$\sum_{j=1}^{n} \left\langle f_j(X, t) \frac{\partial h}{\partial X_j} \right\rangle + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle \left(GDG^t\right)_{ij} \frac{\partial^2 h}{\partial X_i \partial X_j} \right\rangle$$

$$\dot{m}_{10} = \frac{d}{dt}\langle X_1 \rangle = m_{01}$$

$$\dot{m}_{01} = \frac{d}{dt}\langle X_2 \rangle = -2\eta\omega m_{01} - \omega^2 m_{10}$$

$$\dot{m}_{20} = \langle X_2 2X_1 \rangle = 2m_{11}$$

$$\dot{m}_{11} = \langle X_2 X_2 \rangle + \left\langle \left(-2\eta\omega X_2 - \omega^2 X_1\right) X_1 \right\rangle = m_{02} - 2\eta\omega m_{11} - \omega^2 m_{20}$$

$$\dot{m}_{02} = \left\langle 2X_2\left(-2\eta\omega X_2 - \omega^2 X_1\right)\right\rangle + D = -4\eta\omega m_{02} - 2\omega^2 m_{11} + D$$

Remark

Moment equations are closed

Steady state

$$\dot{m}_{10} = 0 \Rightarrow m_{01} = 0$$

$$\dot{m}_{01} = 0 \Rightarrow -2\eta\omega m_{01} - \omega^2 m_{10} = 0 \Rightarrow m_{10}$$

$$\dot{m}_{20} = 0 \Rightarrow 2m_{11} = 0$$

$$\dot{m}_{11} = 0 \Rightarrow m_{02} - 2\eta\omega m_{11} - \omega^2 m_{20} = 0 \Rightarrow m_{02} = \omega^2 m_{20}$$

$$m_{02} = 0 \Rightarrow -4\eta\omega m_{02} - 2\omega^2 m_{11} + D = 0 \Rightarrow m_{02} = \frac{D}{4\eta\omega}$$

$$\sigma_X^2 = \frac{D}{4\eta\omega^3}; \sigma_{\dot{X}}^2 = \frac{D}{4\eta\omega} \;\&\; \left\langle X(t)\dot{X}(t)\right\rangle = 0$$

These results agree with the exact solutions obtained earlier using convolution integral approach