Stochastic Structural Dynamics

Lecture-22

Markov Vector Approach-2

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Recall

$$\underbrace{p\left(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n\right)}_{\text{Multi-dimensional jpdf}} = \underbrace{p(x_1; t_1)}_{\text{Initial pdf}} \underbrace{\prod_{\nu=2}^{n} p\left(x_\nu; t_\nu \mid x_{\nu-1}; t_{\nu-1}\right)}_{\text{Product of transistional pdfs}}$$

$$p\left(x_2, t_2 \mid x_1, t_1\right) = \int p\left(x_2, t_2 \mid x, \tau\right) p\left(x, \tau \mid x_1, t_1\right) dx$$

for all $t_1 < \tau < t_2$

$t_1 < \tau < t_2$

$t_1 \quad \tau \quad t_2$

Kinetic equation

$$\frac{\partial p}{\partial t} + \frac{\partial \lambda}{\partial x} = 0 \text{ \& BCS and IC}$$

$$\lambda\left(x, t\right) = -\sum_{n=1}^{\infty} \frac{\left(-1\right)^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \left[\alpha_n\left(x, t\right) p\left(x; t\right)\right]$$

$$\alpha_n\left(x, t\right) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[X\left(t + \Delta t\right) - X\left(t\right)\right]^n \mid X\left(t\right) = x \right\rangle; n = 1, 2, \cdots$$

Simple random walk

Let $\left\{X_i\right\}_{i=1}^{\infty}$ be an iid sequence of random variables with

$$\mathrm{P}\left(X = \Delta x\right) = p$$

$$\mathrm{P}\left(X = -\Delta x\right) = q$$

such that $p + q = 1$.

$$\left\langle X \right\rangle = \mathrm{P}\left(X = \Delta x\right)\left(\Delta x\right) + \mathrm{P}\left(X = -\Delta x\right)\left(-\Delta x\right)$$

$$= \Delta x\left(p - q\right)$$

$$\left\langle X^2 \right\rangle = \mathrm{P}\left(X = \Delta x\right)\left(\Delta x\right)^2 + \mathrm{P}\left(X = -\Delta x\right)\left(-\Delta x\right)^2$$

$$= \Delta x^2\left(p + q\right)$$

$$\mathrm{Var}\left(X\right) = \left\langle X^2 \right\rangle - \left\langle X^{\cancel{2}} \right\rangle^2$$

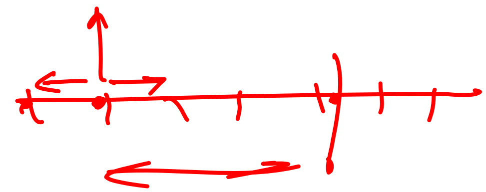
$$= \Delta x^2\left(p + q\right) - \Delta x^2\left(p - q\right)^2$$

$$= \Delta x^2\left(p + q\right)^2 - \Delta x^2\left(p - q\right)^2 \quad \left(\because p + q = 1\right)$$

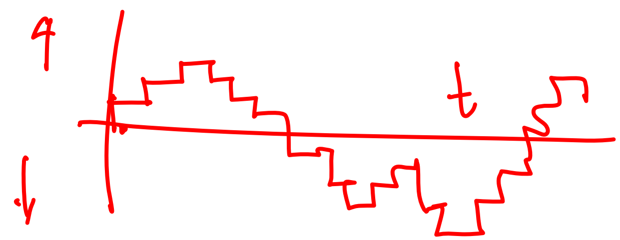
$$= \Delta x^2\left[\left(p + q\right)^2 - \left(p - q\right)^2\right] = 4pq\Delta x^2$$

Let t be the time axis and let us divide the interval $(0, t)$ into n subintervals each of width Δt such that $n\Delta t = t$.

Define $S(t) = \sum_{i=1}^{n} X_i$



$$\Rightarrow \langle S(t) \rangle = \sum_{i=1}^{n} \langle X_i \rangle = \sum_{i=1}^{n} (p-q)\Delta x$$



$$= n(p-q)\Delta x$$

$$= t(p-q)\frac{\Delta x}{\Delta t}$$

$$\mathrm{Var}\left[S(t)\right] = t4pq\Delta x^2$$

$$= t4pq\frac{\Delta x^2}{\Delta t}$$

Remarks

- $S(t)$ is known as a simple random walk.
- $S(t)$ is a discrete state, discrete parameter random process.
- Consider the limit of $\Delta x \to 0$ as $\Delta t \to 0$

$\Rightarrow$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \langle S \rangle = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} t(p-q)\frac{\Delta x}{\Delta t}$$

and

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \mathrm{Var}\left[S(t)\right] = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} t4pq\frac{\Delta x^2}{\Delta t} \to 0$$

$\Rightarrow$

In the limit of $\Delta x \to 0$ as $\Delta t \to 0$, $S(t)$ becomes a deterministic function.

This is not an interesting limit from probabilistic point of view.

Wiener and Brownian motion Processes

Consider the following limit of the simple random walk

$\Delta x^2 \to 0$ as $\Delta t \to 0$

with

$$\Delta x = \sigma \Delta t; \quad p = \frac{1}{2}\left[1 + \frac{\mu\sqrt{\Delta t}}{\sigma}\right]; \quad q = \frac{1}{2}\left[1 - \frac{\mu\sqrt{\Delta t}}{\sigma}\right]$$

$\Rightarrow$

$$\langle S(t) \rangle \to \mu t$$

$$\mathrm{Var}\left[S(t)\right] \to \sigma^2 t$$

This is an interesting limit!

Remarks

- The resulting process is known as the Wiener process.
- This is a process with continuous state and continuous parameter.
- The process is a Gaussian process (central limit theorem).
- The process is nonstationary
- If $\mu = 0$, the process is known as a Brownian motion process.
- Without loss of generality we take $B(0) = 0$.

$$S(t) = \sum_{i=1}^{n} X_i$$

$$\Rightarrow S_n = X_n + \sum_{i=1}^{n-1} X_i$$

$$\Rightarrow S_n = S_{n-1} + X_n$$

S_n is a process with independent increments.

S_n is Markov

$\Rightarrow$ Wiener and Brownian motion processes are also Markov.

Autocovariance of $B(t)$

We have $\langle B(t)\rangle = 0$ & $\langle B^2(t)\rangle = \sigma^2 t$

Let $t > s$ & consider $\left[B(t)-B(s)\right]$ & $\left[B(s)-B(0)\right]$

$$\left\langle \left[B(t)-B(s)\right]\left[B(s)-B(0)\right]\right\rangle = 0 \quad //$$

$$\Rightarrow \left\langle \left[B(t)-B(s)\right]B(s)\right\rangle = 0$$

$$\langle B(t)B(s)\rangle = \langle B^2(s)\rangle = \sigma^2 s$$

Similarly, if $s > t$ we get

$$\langle B(t)B(s)\rangle = \langle B^2(s)\rangle = \sigma^2 t$$

$$\Rightarrow \langle B(t)B(s)\rangle = \sigma^2 \min(t,s)$$

$$\langle B(t)B(s)\rangle = \sigma^2 \min(t,s)$$

$$R_{BB}(t,s) = \sigma^2 \min(t,s)$$

Consider $s > t$

$$R_{BB}(t,s) = \sigma^2 t \text{ with } s > t$$

$$\frac{\partial R_{BB}(t,s)}{\partial t} = \sigma^2 U(s-t)$$

$$\frac{\partial^2 R_{BB}(t,s)}{\partial t \partial s} = \sigma^2 \delta(s-t)$$

Consider $t > s$

$$R_{BB}(t,s) = \sigma^2 \text{s with } t > s$$

$$\frac{\partial R_{BB}(t,s)}{\partial s} = \sigma^2 U(t-s)$$

$$\frac{\partial^2 R_{BB}(t,s)}{\partial t \partial s} = \sigma^2 \delta(t-s)$$

Recall $\delta(ax) = \frac{1}{|a|}\delta(x)$.

$$\Rightarrow \frac{\partial^2 R_{BB}(t,s)}{\partial t \partial s} = \sigma^2 \delta(t-s)$$

BMP and Gaussian white noise process

$$R_{BB}(t,s)=\sigma^2\min(t,s)\ \&\ \frac{\partial^2 R_{BB}(t,s)}{\partial t\partial s}=\sigma^2\delta(t-s)$$

Notice: $\sigma^2\delta(t-s)$ is the autocovariance function of a white noise process.

$\Rightarrow$ Gaussian white noise can be viewed as the formal derivative of a Brownian motion process.

• **Note**: BMP is not pathwise differentiable in the meansquare sense because $\lim\limits_{t\to s}\dfrac{\partial^2 R_{BB}(t,s)}{\partial t\partial s}\to\infty$.

$\frac{dB}{dt}=W(t)$

• $dB(t)=W(t)\,dt$

Increments of BMP

$$\Delta B(t) = B(t+\Delta t) - B(t)$$

$$\langle \Delta B(t) \rangle = \langle B(t+\Delta t) - B(t) \rangle = 0$$

$$\langle \Delta B^2(t) \rangle = \left\langle \left\{ B(t+\Delta t) - B(t) \right\}^2 \right\rangle$$

$$= \langle B^2(t+\Delta t) + B^2(t) - 2B(t+\Delta t)B(t) \rangle$$

$$= \sigma^2 \left[t + \Delta t + t - 2t \right]$$

$$= \sigma^2 \Delta t$$

Fokker Planck equation

Example

$$\frac{dx}{dt} = w(t); x(0) = x_0$$

$3\sigma^4$ Δt^2

$$\langle w(t) \rangle = 0; \langle w(t_1) w(t_2) \rangle = 2D\delta(t_1 - t_2)$$

$$dx(t) = dB(t); x(0) = x_0$$

Recall

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\alpha_1(x,t)\, p(x;t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\alpha_2(x,t)\, p(x;t)\right]$$

$$\alpha_n(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[X(t+\Delta t) - X(t)\right]^n \mid X(t) = x \right\rangle$$

$$\alpha_1(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle dB(t) \mid X(t) = x \right\rangle = 0$$

$$\alpha_2(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[dB(t)\right]^2 \mid X(t) = x \right\rangle = \lim_{\Delta t \to 0} \frac{1}{\Delta t} 2D\Delta t = 2D$$

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}; p(x;0) = \delta(x); p(\pm\infty;t) = 0$$

FPK equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2};\ p(x;0) = \delta(x - x_0);\ p(\pm\infty;t) = 0$$

Solution

Consider the characteristic function

$$M(\theta,t) = \int_{-\infty}^{\infty} p(x;t)\exp(i\theta x)dx$$

$$p(x;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(\theta,t)\exp(-i\theta x)d\theta$$

$$\frac{\partial p}{\partial t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial M(\theta,t)}{\partial t}\exp(-i\theta x)d\theta$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\theta^2 M(\theta,t)\exp(i\theta x)d\theta$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial M(\theta,t)}{\partial t}\exp(i\theta x)d\theta = D\frac{1}{2\pi}\int_{-\infty}^{\infty} -\theta^2 M(\theta,t)\exp(i\theta x)d\theta$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial M(\theta,t)}{\partial t}\exp(i\theta x)d\theta = D\frac{1}{2\pi}\int_{-\infty}^{\infty}-\theta^2 M(\theta,t)\exp(i\theta x)d\theta$$

$$\Rightarrow \frac{\partial M(\theta,t)}{\partial t} + D\theta^2 M(\theta,t) = 0$$

$$M(\theta,t) = M_0\exp\left(-D\theta^2 t\right)$$

$$M(\theta,0) = \int_{-\infty}^{\infty}\underline{\delta(x-x_0)}\exp(i\theta x)dx = \underline{\exp(i\theta x_0)}$$

$$\Rightarrow M(\theta,t) = \cancel{M_0}\exp\left(i\theta x_0 - D\theta^2 t\right)$$

$$\Rightarrow p(x;t) \sim N(x_0, Dt)$$

$$p(x;t) = \frac{1}{\sqrt{2\pi Dt}}\exp\left[-\frac{1}{2}\left(\frac{x-x_0}{\sqrt{Dt}}\right)^2\right]; -\infty < x < \infty$$

$X(t)$ is a nonstationary, Gaussian, Markov random process

Alternative derivation of the FPK equation

Let $X(t)$ be a scalar Markov random process.

By virtue of CKS equation, the following is true.

$$p(x_2,t_2\,|\,x_1,t_1)=\int p(x_2,t_2\,|\,x,\tau)\,p(x,\tau\,|\,x_1,t_1)\,dx$$

for all $t_1<\tau<t_2$.

This is an integral equation. The FPK equation is the associated PDE and can be derived as follows.

Consider

$$I=\int_{-\infty}^{\infty}R(y)\frac{\partial}{\partial t}p(y;t\,|\,x_0,t_0)\,dy$$

Here $R(y)$ is an arbitrary fuinction that admits Taylor's expansion and $R(\pm\infty)\to 0$ sufficiently fast.

$$I=\int_{-\infty}^{\infty}R(y)\frac{\partial}{\partial t}p(y;t\,|\,x_0;t_0)\,dy$$

$$=\int_{-\infty}^{\infty}R(y)\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[p(y;t+\Delta t\,|\,x_0,t_0)-p(y;t\,|\,x_0;t_0)\right]dy$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}R(y)\left[p(y;t+\Delta t\,|\,x_0;t_0)-p(y;t\,|\,x_0;t_0)\right]dy$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\{\int_{-\infty}^{\infty}R(y)\left[\int_{-\infty}^{\infty}p(y;t+\Delta t\,|\,x;t)\,p(x;t\,|\,x_0,t_0)\,dx\right]dy$$

$$-\int_{-\infty}^{\infty}R(y)p(y;t\,|\,x_0,t_0)\,dy\}\cdots(1)$$

$$R(y)=R(x+y-x)=R(x)+(y-x)R'(x)+\frac{(y-x)^2}{2!}R''(x)+\cdots$$

The first of the integral reads

$$\int_{-\infty}^{\infty} R(y)\left[\int_{-\infty}^{\infty} p(y;t+\Delta t\,|\,x;t)\,p(x;t\,|\,x_0,t_0)\,dx\right]dy$$

$$=\int_{-\infty}^{\infty}\left[R(x)+(y-x)R'(x)+\frac{(y-x)^2}{2!}R''(x)+\cdots\right]$$

$$\left[\int_{-\infty}^{\infty} p(y;t+\Delta t\,|\,x;t)\,p(x;t\,|\,x_0,t_0)\,dx\right]dy$$

Consider $\int_{-\infty}^{\infty} R(x)\left[\int_{-\infty}^{\infty} p(y;t+\Delta t\,|\,x;t)\,p(x;t\,|\,x_0,t_0)\,dx\right]dy$

$$=\int_{-\infty}^{\infty} R(x)\,p(x;t\,|\,x_0,t_0)\underbrace{\left[\int_{-\infty}^{\infty} p(y;t+\Delta t\,|\,x;t)\,dy\right]}_{1}dx$$

$=\int_{-\infty}^{\infty} R(x)\,p(x;t\,|\,x_0,t_0)\,dx$ (This cancels with the last term in equation 1)

$$
I = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} \left[\underline{(y-x) R'(x) + \frac{(y-x)^2}{2!} R''(x)} + \cdots \right]
$$

$$
\left[\int_{-\infty}^{\infty} p(y; t+\Delta t \mid x; t)\, p(x; t \mid x_0, t_0)\, dx \right] dy
$$

$$
= \int_{-\infty}^{\infty} R'(x)\, p(x; t \mid x_0, t_0) \left[\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y-x)\, p(y; t+\Delta t \mid x; t)\, dy \right] dx
$$

$$
+ \int_{-\infty}^{\infty} \frac{R''(x)}{2}\, p(x; t \mid x_0, t_0) \left[\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (y-x)^2\, p(y; t+\Delta t \mid x; t)\, dy \right] dx + \ldots
$$

$$
I = \int_{-\infty}^{\infty} \left[R'(x) A(x,t) + R''(x) B(x,t) + R'''(x) C(x,t) + \cdots \right] p(x; t \mid x_0, t_0)\, dx
$$

$$I=\int_{-\infty}^{\infty}\left[R'(x)A(x,t)+R''(x)\underset{B}{A}(x,t)+R'''(x)C(x,t)+\cdots\right]p(x;t\mid x_0,t_0)\,dx$$

$$A(x,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(y-x)\,p(y;t+\Delta t\mid x;t)\,dy$$

$$B(x,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(y-x)^2\,p(y;t+\Delta t\mid x;t)\,dy$$

$$C(x,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(y-x)^3\,p(y;t+\Delta t\mid x;t)\,dy\cdots$$

Consider the first term

$$I=\int_{-\infty}^{\infty}R'(x)A(x,t)p(x;t\mid x_0,t_0)\,dx$$

$$=\underbrace{\left[A(x,t)\,p(x;t\mid x_0,t_0)\,R(x)\right]_{-\infty}^{\infty}}_{0}-\int_{-\infty}^{\infty}R(x)\frac{\partial}{\partial x}\left[A(x,t)\,p(x;t\mid x_0,t_0)\right]dx$$

$$=-\int_{-\infty}^{\infty}R(x)\frac{\partial}{\partial x}\left[A(x,t)\,p(x;t\mid x_0,t_0)\right]dx$$

Consider the second term

$$I = \int_{-\infty}^{\infty} R''(x) B(x,t) p(x;t \mid x_0, t_0)\, dx$$

$$= \underbrace{\left[B(x,t)\, p(x;t \mid x_0, t_0)\, R'(x) \right]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} R'(x) \frac{\partial}{\partial x} \left[B(x,t)\, p(x;t \mid x_0, t_0) \right] dx$$

$$= -\underbrace{\left\{ \frac{\partial}{\partial x} \left[B(x,t)\, p(x;t \mid x_0, t_0) \right] R(x) \right\}_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} R(x) \frac{\partial^2}{\partial x^2} \left[B(x,t)\, p(x;t \mid x_0, t_0) \right] dx$$

$$= \int_{-\infty}^{\infty} R(x) \frac{\partial^2}{\partial x^2} \left[B(x,t)\, p(x;t \mid x_0, t_0) \right] dx$$

$$I=\int_{-\infty}^{\infty} R(y)\frac{\partial}{\partial t}p(y;t\mid x_0,t_0)\,dy$$

$$=\int_{-\infty}^{\infty} R(x)\left\{-\frac{\partial}{\partial x}\left[A(x,t)\,p(x;t\mid x_0,t_0)\right]+\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B(x,t)\,p(x;t\mid x_0,t_0)\right]\right\}dx$$

$$\int_{-\infty}^{\infty} R(x)[\frac{\partial}{\partial t}p(y;t\mid x_0,t_0)+\frac{\partial}{\partial x}\left[A(x,t)\,p(x;t\mid x_0,t_0)\right]$$

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B(x,t)\,p(x;t\mid x_0,t_0)\right]+\cdots]dx=0$$

Since $R(x)$ is arbitrary, it follows that

$$\frac{\partial}{\partial t}p(y;t\mid x_0,t_0)+\frac{\partial}{\partial x}\left[A(x,t)\,p(x;t\mid x_0,t_0)\right]$$

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B(x,t)\,p(x;t\mid x_0,t_0)\right]+\cdots=0$$

This is the FPK equation.

$$\frac{\partial}{\partial t}p\left(y;t\,|\,x_0,t_0\right)+\frac{\partial}{\partial x}\left[A\left(x,t\right)p\left(x;t\,|\,x_0,t_0\right)\right]-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B\left(x,t\right)p\left(x;t\,|\,x_0,t_0\right)\right]+\cdots=0$$

Remarks

- This equation is also known as the Kolmogorov forward equation
- "Forward" because $\frac{\partial}{\partial t}p\left(y;t\,|\,x_0,t_0\right)$ refers to time derivative

with respect to $t>t_0$.

- $A,B,C,\cdots$ are known as the derivative moments

$$A\left(x,t\right)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{\infty}\left(y-x\right)p\left(y;t+\Delta t\,|\,x;t\right)dy$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle\left[X\left(t+\Delta t\right)-X\left(t\right)\right]\,|\,X\left(t\right)=x\right\rangle$$

$$B\left(x,t\right)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle\left[X\left(t+\Delta t\right)-X\left(t\right)\right]^2\,|\,X\left(t\right)=x\right\rangle$$

$$C\left(x,t\right)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle\left[X\left(t+\Delta t\right)-X\left(t\right)\right]^3\,|\,X\left(t\right)=x\right\rangle\cdots$$

Models for systems driven by white noise excitations : Ito's stochastic differential equations

Consider the differential equation governing the $n \times 1$ vector $x(t)$

$$\frac{dx}{dt} = F\left[t, x(t), f(t)\right]; t \ge t_0 \;\&\; x(t_0) \text{ specified}$$

Here $f(t)$ is a $m \times 1$ vector random process and F is a $n \times 1$ nonlinear function. If $f(t)$ and F are such that the integral

$\int_{t_0}^{t} F\left[\tau, x(\tau), f(\tau)\right] d\tau$ exists in a mean square sense, then

$$x(t) - x(t_0) = \int_{t_0}^{t} F\left[\tau, x(\tau), f(\tau)\right] d\tau \cdots (\mathrm{A})$$

is the solution.

If elements of $f(t)$ are Gaussian white noises, then

$\int_{t_0}^{t} F\left[\tau, x(\tau), f(\tau)\right] d\tau$ does not exist in mean square sense and

equation (A) loses its meaning.

For the sake of illustration, let us consider the scalar equation

$$\frac{dx}{dt} = f\left[t, x(t)\right] + G\left[t, x(t)\right] w(t); t \geq 0 \,\&\, x(0) \text{ is specified.}$$

Here $w(t)$ is a zero mean Gaussian white noise.

Recall $w(t)$ is a formal derivative of Brownian motion process.

That is, $\mathrm{dw}(t) = dB(t)$.

$$dx(t) = f\left[t, x(t)\right] dt + G\left[t, x(t)\right] dB(t); t \geq 0 \,\&\, x(0) \text{ is specified.}$$

$$dx(t) = f\left[t, x(t)\right]dt + G\left[t, x(t)\right]dB(t); t \geq 0 \,\&\, x(0) \text{ is specified} \cdots (B)$$

$$\Rightarrow x(t) - x(0) = \int_0^t f\left[\tau, x(\tau)\right]d\tau + \int_0^t G\left[\tau, x(\tau)\right]dB(\tau)d\tau$$

$\int_0^t f\left[\tau, x(\tau)\right]d\tau$: This integral can be interpreted as the traditional Riemann integral.

$\int_0^t G\left[\tau, x(\tau)\right]dB(\tau)d\tau$: This integral does not exist in a sample sense but can be defined in a mean square sense. (Ito's stochastic integral)

Equation (B) is called the Ito's stochastic differential equation.

References:

A H Jazwiniski, 1970, Stochastic processes and filtering theory, Academic Press, NY.

T T Soong, 1973, Random differential equations in science and engineering, Academic Press, NY.

$dx(t) = f[t, x(t)]dt + G[t, x(t)]dB(t); t \geq 0 \,\&\, x(0)$ is specified

• $B(t)$ has independent increments from this it follows that

$x(t)$ is Markov.

Intutive explanation

Consider the time instants $t,\ t+\Delta t$ and $t+2\Delta t$.

The change $x(t)$ to $x(t+\Delta t)$ is due to $[B(t+\Delta t) - B(t)]$.

The change $x(t+\Delta t)$ to $x(t+2\Delta t)$ is due to $[B(t+2\Delta t) - B(t+\Delta t)]$.

$[B(t+\Delta t) - B(t)] \,\&\, [B(t+2\Delta t) - B(t+\Delta t)]$ are independent.

$\Rightarrow$ The changes $[x(t) - x(t+\Delta t)] \,\&\, [x(t+\Delta t) - x(t+2\Delta t)]$

are independent.

$\Rightarrow x(t)$ is Markov.

Question : Can we derive the governing FPK equation and solve it?

Example

$$\dot{x} + \beta(x) = w(t); t \geq 0 \;\&\; x(0) = x_0$$

$$\langle w(t) \rangle = 0; \langle w(t_1) w(t_2) \rangle = 2D\delta(t_1 - t_2)$$

$$dx = -\beta\left[x(t)\right]dt + dB(t)$$

Quantity of interest: $p(x; t \mid x_0; 0)$

Initial condition: $p(x; 0 \mid x_0; 0) = \delta(x - x_0)$

Boundary conditions: $\lim_{x \to \pm\infty} p(x; t \mid x_0; 0) \to 0$

$$\alpha_n(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[X(t + \Delta t) - X(t) \right]^n \mid X(t) = x \right\rangle; n = 1, 2, \cdots$$

$$\alpha_1(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[X(t+\Delta t) - X(t) \right] \mid X(t) = x \right\rangle$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle -\beta\left[x(t)\right]dt + dB(t) \right\rangle = -\beta(x)$$

$$\alpha_2(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[X(t+\Delta t) - X(t) \right]^2 \mid X(t) = x \right\rangle$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[-\beta\left[x(t)\right]dt + \overset{dB(t)}{\cancel{\beta\left[x(t)\right]}}dt \right]^2 \right\rangle$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \underline{\beta^2(x)\Delta t^2} + \left[dB(t)\right]^2 \underline{\underline{- 2\beta(x)\,dt\,\overset{dB(t)}{\cancel{\beta x(t)}}\,dt}} \right\rangle$$

$$= \underline{\underline{2D}}$$

$$\Rightarrow \frac{\partial p}{\partial t} = \frac{\partial\left[\beta(x)p\right]}{\partial x} + D\frac{\partial^2 p}{\partial x^2};\ p \equiv p(x;t \mid x_0;0)$$

$$\frac{\partial p}{\partial t}=\frac{\partial\left[\beta(x)p\right]}{\partial x}+D\frac{\partial^2 p}{\partial x^2};\ p\equiv p\left(x;t\,|\,x_0;0\right)$$

$$p\left(x;t\,|\,x_0;0\right)=\Psi(x)T(t)$$

$$\Psi(x)\dot{T}(t)=\left[\beta(x)\Psi(x)\right]'T(t)+D\Psi''(x)T(t)$$

$$\Rightarrow\frac{\Psi(x)\dot{T}(t)}{\Psi(x)T(t)}=\frac{\left[\beta(x)\Psi(x)\right]'T(t)}{\Psi(x)T(t)}+\frac{D\Psi''(x)T(t)}{\Psi(x)T(t)}$$

$$\frac{\dot{T}(t)}{T(t)}=\frac{\left[\beta(x)\Psi(x)\right]'}{\Psi(x)}+\frac{D\Psi''(x)}{\Psi(x)}=\lambda$$

$$\dot{T}-\lambda T=0$$

$$D\Psi''(x)+\left[\beta(x)\Psi(x)\right]'-\lambda\Psi(x)=0;\ \Psi(\pm\infty)=0$$

$$D\Psi''(x)+\left[\beta(x)\Psi(x)\right]'-\lambda\Psi(x)=0;\Psi(\pm\infty)=0$$

This is an eigenvalue problem. Depending upon nature of $\beta(x)$, the solution can be obtained in the form

$$p(x;t\,|\,x_0;0)=\sum_{i=1}^{\infty}a_i\exp(\lambda_i t)\Psi_i(x)$$

The constants a_i-s can be obtained using the condition

$$p(x;0\,|\,x_0;0)=\delta(x-x_0)$$

Remark

As $t\to\infty,\dfrac{\partial p}{\partial t}\to 0\Rightarrow$

Stationary solution: $\dfrac{d\left[\beta(x)p\right]}{dx}+D\dfrac{d^2p}{dx^2}=0$

Example

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = w(t); t \geq 0; x(0) = x_0; \dot{x}(0) = \dot{x}_0$$

$$\langle w(t) \rangle = 0; \langle w(t) w(t+\tau) \rangle = 2D\delta(\tau)$$

$$X(t) = \begin{Bmatrix} X_1(t) \\ X_2(t) \end{Bmatrix} = \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix}$$

$\partial B(t) = dt\, w(t)$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2 - \omega^2 X_1\right] dt + dB(t)$$

$$p \equiv p\left[\tilde{x}; t \mid \tilde{x}_0\right] = p\left[x_1, x_2; t \mid X_1(0) = x_0, \dot{X}_1(0) = \dot{x}_0\right]$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left[\alpha_j p\right] + \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \frac{\partial}{\partial x_j \partial x_k}\left[\alpha_{jk} p\right]$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left[\alpha_j p\right] + \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \frac{\partial}{\partial x_j \partial x_k}\left[\alpha_{jk} p\right]$$

$$\alpha_j = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle \left[X_j(t+\Delta t) - X_j(t)\right] \mid X(t) = \tilde{x} \right\rangle$$

$$\alpha_{ij} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle \left[X_i(t+\Delta t) - X_i(t)\right]\left[X_j(t+\Delta t) - X_j(t)\right] \mid X(t) = \tilde{x} \right\rangle$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2 - \omega^2 X_1\right] dt + dB(t)$$

$$\alpha_1 = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle X_2(t) dt \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = x_2$$

$$\alpha_2 = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle \left[-2\eta\omega X_2 - \omega^2 X_1\right] dt + dB(t) \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle$$

$$= -2\eta\omega x_2 - \omega^2 x_1$$

$$\alpha_{11} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle X_2^2 (dt)^2 \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = 0$$

Similarly, $\alpha_{12} = \alpha_{21} = 0$ & $\alpha_{22} = 2D$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left[\alpha_j p\right] + \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \frac{\partial}{\partial x_j \partial x_k}\left[\alpha_{jk} p\right]$$

$$\alpha_1 = x_2; \alpha_2 = -2\eta\omega x_2 - \omega^2 x_1; \alpha_{11} = \alpha_{12} = \alpha_{21} = 0 \ \& \ \alpha_{22} = 2D$$

$$\frac{\partial p}{\partial t} = -x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left\{2\eta\omega x_2 + \omega^2 x_1\right\} p\right] + D\frac{\partial^2 p}{\partial x_2^2}$$

$t \to 0$

$$p\left(x_1, x_2; t \mid X_1(0) = x_{10}, X_2(0) = x_{20}\right) = \delta\left(x_1 - x_{10}\right)\delta\left(x_2 - x_{20}\right)$$

$$p\left(\pm\infty, x_2; t \mid X_1(0) = x_{10}, X_2(0) = x_{20}\right) = 0$$

$$p\left(x_2, \pm\infty; t \mid X_1(0) = x_{10}, X_2(0) = x_{20}\right) = 0$$

Remark

The FPK equation can be viewed as the equation of motion governing the evolution of pdf $p\left(\tilde{x}; t \mid \tilde{x}_0; 0\right)$

$$\frac{\partial p}{\partial t} \to 0$$

Equation in the steady state $(t \to \infty)$

$$-x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left\{2\eta\omega x_2 + \omega^2 x_1\right\} p\right] + D \frac{\partial^2 p}{\partial x_2^2} = 0$$

BCS

$$p\left(\pm\infty, x_2; t \mid X_1(0) = x_{10}, X_2(0) = x_{20}\right) = 0$$

$$p\left(x_2, \pm\infty; t \mid X_1(0) = x_{10}, X_2(0) = x_{20}\right) = 0$$

Example : nonstationary inputs

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = e(t)w(t); t \geq 0; x(0) = x_0; \dot{x}(0) = \dot{x}_0$$

$$\langle w(t) \rangle = 0; \langle w(t)w(t+\tau) \rangle = 2D\delta(\tau)$$

$e(t) =$ deterministic modulating function

$$X(t) = \begin{Bmatrix} X_1(t) \\ X_2(t) \end{Bmatrix} = \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix}$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2 - \omega^2 X_1\right] dt + e(t) dB(t)$$

$$p \equiv p\left[\tilde{x}; t \mid \tilde{x}_0\right] = p\left[x_1, x_2; t \mid X_1(0) = x_0, \dot{X}_1(0) = \dot{x}_0\right]$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\left[\alpha_j p\right] + \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} \frac{\partial}{\partial x_j \partial x_k}\left[\alpha_{jk} p\right]$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2 - \omega^2 X_1\right]dt + e(t)\,dB(t)$$

$$\alpha_1 = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle X_2(t)\,dt \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = x_2$$

$$\alpha_2 = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle \left[-2\eta\omega X_2 - \omega^2 X_1\right]dt + dB(t) \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle$$

$$= -2\eta\omega x_2 - \omega^2 x_1$$

$$\alpha_{11} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle X_2^2 (dt)^2 \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = 0$$

Similarly, $\alpha_{12} = \alpha_{21} = 0$ & $\alpha_{22} = 2De^2(t)$

$$\frac{\partial p}{\partial t} = -x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left\{2\eta\omega x_2 + \omega^2 x_1\right\} p\right] + De^2(t)\frac{\partial^2 p}{\partial x_2^2}$$

Note: no steady state solution exists.

Example : Nonlinear system

$$\ddot{x}+2\eta\omega\dot{x}+\omega^2 x+\alpha x^3=w(t);t\ge 0;x(0)=x_0;\dot{x}(0)=\dot{x}_0$$

$$\langle w(t)\rangle=0;\langle w(t)w(t+\tau)\rangle=2D\delta(\tau)$$

$$X(t)=\begin{Bmatrix} X_1(t) \\ X_2(t) \end{Bmatrix}=\begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix}$$

$$dX_1=X_2dt$$

$$dX_2=\left[-2\eta\omega X_2-\omega^2 X_1-\alpha X_1^3\right]dt+dB(t)$$

$$p\equiv p\left[\tilde{x};t\,|\,\tilde{x}_0\right]=p\left[x_1,x_2;t\,|\,X_1(0)=x_0,\dot{X}_1(0)=\dot{x}_0\right]$$

$$\frac{\partial p}{\partial t}=-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[\alpha_j p\right]+\frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{\partial}{\partial x_j\partial x_k}\left[\alpha_{jk}p\right]$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2 - \omega^2 X_1 - \alpha X_1^3\right] dt + dB(t)$$

$$\alpha_1 = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle X_2(t) dt \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = x_2$$

$$\alpha_2 = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[-2\eta\omega X_2 - \omega^2 X_1 - \alpha X_1^3\right] dt + dB(t) \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle$$

$$= -2\eta\omega x_2 - \omega^2 x_1 - \alpha x_1^3$$

$$\alpha_{11} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle X_2^2 (dt)^2 \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = 0$$

Similarly, $\alpha_{12} = \alpha_{21} = 0 \,\&\, \alpha_{22} = 2D$

$$\frac{\partial p}{\partial t} = -x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left\{2\eta\omega x_2 + \omega^2 x_1 + \alpha x_1^3\right\} p\right] + D \frac{\partial^2 p}{\partial x_2^2}$$

Steady state $(t \to \infty)$

$$-x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left\{2\eta\omega x_2 + \omega^2 x_1 + \alpha x_1^3\right\} p\right] + D \frac{\partial^2 p}{\partial x_2^2} = 0$$

Example : parametric random excitations

$$\ddot{x}+\dot{x}\left[2\eta\omega+\varepsilon W_1(t)\right]+x\left[\omega^2+\alpha W_2(t)\right]=W_3(t);$$

$$t\geq 0;\, x(0)=x_0;\, \dot{x}(0)=\dot{x}_0$$

$$dB_i(t)=W_i(t)\,dt;\left\langle dB_i(t)\,dB_j(t+\tau)\right\rangle=2D_{ij}\delta(\tau)$$

$$X(t)=\begin{Bmatrix}X_1(t)\\X_2(t)\end{Bmatrix}=\begin{Bmatrix}x(t)\\\dot{x}(t)\end{Bmatrix}$$

$$dX_1=X_2dt$$

$$dX_2=\left[-2\eta\omega X_2-\omega^2X_1\right]dt-\varepsilon x_2dB_1(t)-\alpha x_1dB_2(t)+dB_3(t)$$

$$p\equiv p\left[\tilde{x};t\mid\tilde{x}_0\right]=p\left[x_1,x_2;t\mid X_1(0)=x_0,\dot{X}_1(0)=\dot{x}_0\right]$$

$$\frac{\partial p}{\partial t}=-\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left[\alpha_j p\right]+\frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\frac{\partial}{\partial x_j\partial x_k}\left[\alpha_{jk}p\right]$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2 - \omega^2 X_1\right]dt - \varepsilon x_2 dB_1(t) - \alpha x_1 dB_2(t) + dB_3(t)$$

$$\alpha_1 = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle X_2(t)dt \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = x_2$$

$$\alpha_2 = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle \left[-2\eta\omega X_2 - \omega^2 X_1\right]dt + dB(t) \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle$$

$-\varepsilon x_2 dB_1(t) - \alpha x_1 dB_2(t)$

$$= -2\eta\omega x_2 - \omega^2 x_1$$

$$\alpha_{11} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left\langle X_2^2 (dt)^2 \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = 0$$

Similarly, $\alpha_{12} = \alpha_{21} = 0$

$$\alpha_{22} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} < \left\{\left[-2\eta\omega X_2 - \omega^2 X_1\right]dt - \varepsilon x_2 dB_1(t) - \alpha x_1 dB_2(t) + dB_3(t)\right\}^2$$

$$\mid X_1(t) = x_1, X_2(t) = x_2 >$$

$$= 2\varepsilon^2 x_2^2 D_{11} + 2\alpha^2 x_1^2 D_{22} + 2D_{33} + 4\varepsilon\alpha x_1 x_2 D_{12} - 4\varepsilon x_2 D_{13} - 4\alpha x_1 D_{23}$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2 - \omega^2 X_1\right] dt - \varepsilon x_2 dB_1(t) - \alpha x_1 dB_2(t) + dB_3(t)$$

$$\alpha_1 = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle X_2(t) dt \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = x_2$$

$$\alpha_2 = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[-2\eta\omega X_2 - \omega^2 X_1\right] dt + dB(t) \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle$$

$$= -2\eta\omega x_2 - \omega^2 x_1$$

$$\alpha_{11} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle X_2^2 (dt)^2 \mid X_1(t) = x_1, X_2(t) = x_2 \right\rangle = 0$$

Similarly, $\alpha_{12} = \alpha_{21} = 0$

$$\alpha_{22} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} < \left\{ \left[-2\eta\omega X_2 - \omega^2 X_1\right] dt - \varepsilon x_2 dB_1(t) - \alpha x_1 dB_2(t) + dB_3(t) \right\}^2$$

$$= 2\varepsilon^2 x_2^2 D_{11} + 2\alpha^2 x_1^2 D_{22} + 2D_{33} + 4\varepsilon\alpha x_1 x_2 D_{12} - 4\varepsilon x_2 D_{13} - 4\alpha x_1 D_{23}$$

$$\frac{\partial p}{\partial t} = -x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left\{2\eta\omega x_2 + \omega^2 x_1\right\} p\right] +$$

$$\frac{\partial^2}{\partial x_2^2}\left[\left(2\varepsilon^2 x_2^2 D_{11} + 2\alpha^2 x_1^2 D_{22} + 2D_{33} + 4\varepsilon\alpha x_1 x_2 D_{12} - 4\varepsilon x_2 D_{13} - 4\alpha x_1 D_{23}\right) p\right]$$

$$p\left(x_1, x_2; t \mid X_1(0) = x_{10}, X_2(0) = x_{20}\right) = \delta\left(x_1 - x_{10}\right)\delta\left(x_2 - x_{20}\right)$$

$$p\left(\pm\infty, x_2; t \mid X_1(0) = x_{10}, X_2(0) = x_{20}\right) = 0$$

$$p\left(x_2, \pm\infty; t \mid X_1(0) = x_{10}, X_2(0) = x_{20}\right) = 0$$

If stationary solution exist, it is governed by

$$-x_2 \frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left\{2\eta\omega x_2 + \omega^2 x_1\right\} p\right] +$$

$$\frac{\partial^2}{\partial x_2^2}\left[\left(2\varepsilon^2 x_2^2 D_{11} + 2\alpha^2 x_1^2 D_{22} + 2D_{33} + 4\varepsilon\alpha x_1 x_2 D_{12} - 4\varepsilon x_2 D_{13} - 4\alpha x_1 D_{23}\right) p\right] = 0$$

Example : Filtered white noise excitations

$$\ddot{x}+2\eta\omega\dot{x}+\omega^2 x+\alpha x^3 = f(t); t\geq 0; x(0)=x_0; \dot{x}(0)=\dot{x}_0$$

$$\ddot{f}+2\xi\lambda\dot{f}+\lambda^2 f = w(t); t\geq 0; f(0)=f_0; \dot{f}(0)=\dot{f}_0$$

$$\langle w(t)\rangle = 0; \langle w(t)w(t+\tau)\rangle = 2D\delta(\tau)$$

$$X(t)=\left\{x(t) \quad \dot{x}(t) \quad f(t) \quad \dot{f}(t)\right\}^t$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2-\omega^2 X_1-\alpha X_1^3\right]dt + X_3 dt$$

$$dX_3 = X_4 dt$$

$$dX_4 = \left[-2\xi\lambda X_4-\lambda^2 X_3\right]dt + dB(t)$$

$$dX_1 = X_2 dt$$

$$dX_2 = \left[-2\eta\omega X_2 - \omega^2 X_1 - \alpha X_1^3\right] dt + X_3 dt$$

$$dX_3 = X_4 dt$$

$$dX_4 = \left[-2\xi\lambda X_4 - \lambda^2 X_3\right] dt + dB(t)$$

$$\alpha_1 = x_2$$

$$\alpha_2 = -2\eta\omega x_2 - \omega^2 x_1 - \alpha x_1^3 + x_3$$

$$\alpha_3 = x_4$$

$$\alpha_4 = -2\xi\lambda x_4 - \lambda^2 x_3$$

$$\alpha_{ij} = 0 \forall i, j = 1, 2, 3, 4 \text{ except } \alpha_{44} = 2D$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{4} \frac{\partial}{\partial x_j}\left[\alpha_j p\right] + D \frac{\partial^2 p}{\partial^2 x_4^2}$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{4} \frac{\partial}{\partial x_j}\left[\alpha_j p\right] + D\frac{\partial^2 p}{\partial^2 x_4^2}$$

$$p\left(x_1, x_2; x_3, x_4; t \mid X(0) = \tilde{x}_0\right) = \prod_{i=1}^{4} \delta\left(x_i - x_{i0}\right)$$

$$p\left(\pm\infty, x_2, x_3, x_4; t \mid X(0) = \tilde{x}_0\right) = 0$$

$$p\left(x_1, \pm\infty, x_3, x_4; t \mid X(0) = \tilde{x}_0\right) = 0$$

$$p\left(x_1, x_2, \pm\infty, x_4; t \mid X(0) = \tilde{x}_0\right) = 0$$

$$p\left(x_1, x_2, x_3, \pm\infty; t \mid X(0) = \tilde{x}_0\right) = 0$$

If stationary solution exist, it is governed by

$$-\sum_{j=1}^{4} \frac{\partial}{\partial x_j}\left[\alpha_j p\right] + D\frac{\partial^2 p}{\partial^2 x_4^2} = 0$$

Example : Linear MDOF systems

$$M\ddot{X} + C\dot{X} + KX = W(t); t \geq 0; X(0) = X_0; \dot{X}(0) = \dot{X}_0$$

$$X(t) \sim N \times 1$$

$$\langle W(t) \rangle = 0; \langle W(t) W^t(t+\tau) \rangle = \left[2D_{ij} \right] \delta(\tau)$$

$$\ddot{X} + M^{-1}C\dot{X} + M^{-1}KX = M^{-1}W(t)$$

$$Y = \begin{Bmatrix} Y_I \\ Y_{II} \end{Bmatrix} = \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix}$$

$$dY_I = Y_{II} dt$$

$$dY_{II} = -M^{-1}CY_{II} - M^{-1}KY_I + M^{-1}dB(t)$$

$$dY(t) = PYdt + QdB(t) t \geq 0; Y(0) = Y_0$$

Example : Nonlinear MDOF systems

$$M\ddot{X}+F\left[X,\dot{X}\right]=W(t);t\geq 0;X(0)=X_0;\dot{X}(0)=\dot{X}_0$$

$$X(t)\sim N\times 1$$

$$\langle W(t)\rangle=0;\langle W(t)W^t(t+\tau)\rangle=\left[2D_{ij}\right]\delta(\tau)$$

$$\ddot{X}+M^{-1}F\left[X,\dot{X}\right]=M^{-1}W(t)$$

$$Y=\begin{Bmatrix}Y_I\\Y_{II}\end{Bmatrix}=\begin{Bmatrix}X\\\dot{X}\end{Bmatrix}$$

$$dY_I=Y_{II}dt$$

$$dY_{II}=-M^{-1}F(Y)dt+M^{-1}dB(t)$$

$$dY(t)=P(Y)dt+QdB(t)t\geq 0;Y(0)=Y_0$$

General : n - dimensional Ito SDE

$$dX(t) = f[t, X(t)]dt + G[t, X(t)]dB(t); t \ge 0; X(0) = X_0$$

$$X(t), f[t, X(t)] \sim n \times 1$$

$$G[t, X(t)] \sim n \times m$$

$$dB(t) \sim m \times 1$$

$$\langle dB(t) \rangle = 0; \langle \Delta B_i(t) \Delta B_j(t+\tau) \rangle = 2D_{ij}\delta(\tau)$$

$$\alpha_j = f_j[t, x]; j = 1, 2, \cdots, n$$

$$\alpha_{ij} = 2\left[GDG^t\right]_{ij}; i, j = 1, 2, \cdots, m$$

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[\alpha_j p] + \frac{1}{2}\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{\partial}{\partial x_j \partial x_k}[\alpha_{jk} p]$$

$$p(\tilde{x}; 0 \mid \tilde{x}_0; 0) = \prod_{i=1}^{n} \delta(x_i - x_{i0}) + \text{BCS}$$

Next Lecture

- Solutions of FPK equations
  - Transient solutions
    - Linear systems with additive noises
  - Steady state solutions
    - All scalar equations
    - All linear systems with additive noises
    - A class of nonlinear and parametrically excited systems
- Moment equations