Stochastic Structural Dynamics

Lecture-21

Markov Vector Approach-1

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Recall

Markov Property

- Referes to a property displayed by conditional PDF-s of random processes.

A scalar random process $X(t)$ is said to possess Markov property if

$$P\left[X\left(t_n\right) \le x_n \mid X\left(t_{n-1}\right) \le x_{n-1}, X\left(t_{n-2}\right) \le x_{n-2}, \cdots, X\left(t_1\right) \le x_1\right]$$

$$= P\left[X\left(t_n\right) \le x_n \mid X\left(t_{n-1}\right) \le x_{n-1}\right]$$

for any n and any choice of $0 < t_1 < t_2 < \cdots < t_n$.

Diffusion process :

A Markov process with continuous state and continuous parameter

Markov chain :

A markov process with discrete parameter

- Mathematically, it is easier to deal with Markov chains.
- In vibration problems, however, we are interested in diffusion processes.

Example

Let $\{\Theta_i\}_{i=1}^{n}$ be an iid sequence of random variables.

Define

$$S_n = \sum_{i=1}^{n} \Theta_i$$

$$S_n = \Theta_n + \sum_{i=1}^{n-1} \Theta_i = \Theta_n + S_{n-1}$$

S_n depends only upon S_{n-1} and not upon $\{S_i\}_{i=1}^{n-2}$

$p\left(s_n \mid s_{n-1}\right)$ is not a funciton of $\{S_i = s_i\}_{i=1}^{n-2}$

$\Rightarrow S_n$ is Markov.

Also, $p_{S_n}\left(s \mid S_{n-1} = u\right) = p_{\Theta_n}\left(s - u\right)$

Simple random walk, absorbing and reflecting boundaries

The model $S_n = S_{n-1} + \Theta_n; n = 1, 2, \cdots$

represents the model for a simple random walk.

Example 1

Consider an insurance company which starts business with an initial capital of X_0 at time $t = 0$. Let

$Y_1, Y_2, \cdots, Y_n =$ premiums received

$W_1, W_2, \cdots, W_n =$ claims paid

Capital at time n is given by

$$X_n = X_0 + (Y_1 - W_1) + (Y_2 - W_2) + \cdots + (Y_n - W_n)$$

At any n, if $X_n < 0$, the company is ruined and it can no longer perform. $X_n = 0$ is called an absorbing barrier.

Example 2 Gambler's ruin

Two gamblers A and B with capitals a and b play a game n number of times.

Game ends when

A wins all the capital b from B

or A loses all his capital a to B

X_n = capital gain of A at the end of n^{th} game.

$X = -a$ and $X = b$ are absorbing barriers.

This is an example of random walk with

two absorbing barriers.

Example 3 Daily water level in a dam

$b =$ capacity of dam

$X_n =$ volume of water in the dam on the n^{th} day.

$Y_n =$ inflow on n^{th} day.

X_n executes a random walk in the interval 0 to b

$X_n = 0$ & $X_n = b$ are the two reflecting barriers.

Types of questions

- **Insurance company**

What is the probability of ruin of the insurance company for a given X_0 ?

- **Two gamblers**

What is the probability of ruin of A and B?

- **Dam filling**

◦ What is the long-term equillibrium probability of water level in the dam?

◦ What is the PDF of empty periods and non-empty periods?

Example

Verify if an iid sequence of random variables form a Markov process.

- $p\left(x_n \mid x_{n-1}, x_{n-2}, \cdots, x_1\right) = p\left(x_n \mid x_{n-1}\right) = p\left(x_n\right)$ [Yes]

Example

Let $X(t) = A + Bt$ where A and B are iid random variables.

Is $X(t)$ Markov? [No]

Example

Let A, B, C and D be random variables. Consider

$$\begin{Bmatrix} X(t) \\ Y(t) \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} 1 \\ t \end{Bmatrix}$$

Investigate the conditions under which $\begin{Bmatrix} X(t) \\ Y(t) \end{Bmatrix}$ is Markov.

Complete specification of a Markov process

- $P(x_1;t_1) = P[X(t_1) \le x_1]$
- $P(x_1,x_2;t_1,t_2) = P[X(t_2) \le x_2 \mid X(t_1) \le x_1] P[X(t_1) \le x_1]$
- $P(x_1,x_2,x_3;t_1,t_2,t_3)$

$$= P[X(t_3) \le x_3 \mid X(t_2) \le x_2, X(t_1) \le x_1]$$

$$P[X(t_2) \le x_2 \mid X(t_1) \le x_1] P[X(t_1) \le x_1]$$

$$= P[X(t_3) \le x_3 \mid X(t_2) \le x_2]$$

$$P[X(t_2) \le x_2 \mid X(t_1) \le x_1] P[X(t_1) \le x_1]$$

$$= \prod_{\nu=2}^{3} P[X(t_\nu) \le x_\nu \mid X(t_{\nu-1}) \le x_{\nu-1}] P[X(t_1) \le x_1]$$

- $P(x_1,x_2,\cdots,x_n;t_1,t_2,\cdots,t_n) =$

$$\prod_{\nu=2}^{n} P[X(t_\nu) \le x_\nu \mid X(t_{\nu-1}) \le x_{\nu-1}] P[X(t_1) \le x_1]$$

Complete specification

$$\bullet P\left[X\left(t_\nu\right)\le x_\nu \mid X\left(t_{\nu-1}\right)\le x_{\nu-1}\right]$$

$$\& P\left[X\left(t_1\right)\le x_1\right]\forall n \,\&\left\{t_\nu\right\}_{\nu=1}^{n}$$

$$\bullet P\left[X\left(t_\nu\right)\le x_\nu, X\left(t_{\nu-1}\right)\le x_{\nu-1}\right]\forall n \,\&\left\{t_\nu\right\}_{\nu=1}^{n}$$

- $P\left(x_1, x_2, \cdots, x_n; t_1, t_2, \cdots, t_n\right)$

$$= \prod_{\nu=2}^{n} P\left[X\left(t_\nu\right) \le x_\nu \mid X\left(t_{\nu-1}\right) \le x_{\nu-1}\right] P\left[X\left(t_1\right) \le x_1\right]$$

- Transistional PDF of $X(t)$ [TPDF]

$$P\left[X\left(t_\nu\right) \le x_\nu \mid X\left(t_{\nu-1}\right) \le x_{\nu-1}\right]$$

- TPDF represents the evolution mechanism.
- $P\left[X\left(t_{n+1}\right) \le x_{n+1} \mid X\left(t_n\right) \le x_n, X\left(t_{n-1}\right) \le x_{n-1}\right]$

$$= P\left[X\left(t_{n+1}\right) \le x_{n+1} \mid X\left(t_n\right) \le x_n\right].$$

$$\bullet P\left[X\left(t_{n+1}\right) \le x_{n+1} \mid X\left(t_n\right) \le x_n, X\left(t_{n-1}\right) \le x_{n-1}\right]$$

$$= P\left[X\left(t_{n+1}\right) \le x_{n+1} \mid X\left(t_n\right) \le x_n\right]$$

$n+1 =$ Tomorrow

$n =$ Today

$n-1 =$ Yesterday

Tomorrow's happenings depend on the what is happening today and and not on what happened yesterday!

Recall : Model for dynamical systems using ODE - s

$$\frac{dy}{dt} = h\left[y(t), t\right]; y(0) = y_0$$

Let $t_2 > t_1$.

$$y(t_2) = q\left[t_2, y(t_1), t_1\right]$$

That is, solution at t_2 depends upon $y(t_1)$ and not on $y(\tau)$ $(0 < \tau < t_1)$.

Markov property can be viewed as the stochastic analog of the above property of ODE-s in deterministic systems

• Transistional prbability density function $\left[\text{tpdf}\right]$

$$p\left(x_\nu;t_\nu \mid x_{\nu\text{-}1};t_{\nu\text{-}1}\right)=\frac{\partial}{\partial x_\nu}P\left[X\left(t_\nu\right)\le x_\nu \mid X\left(t_{\nu-1}\right)=x_{\nu-1}\right]$$

$$p\left(x_1,x_2,\cdots,x_n;t_1,t_2,\cdots,t_n\right)=$$

$$p\left(x_n;t_n \mid x_{n-1};t_{n-1},x_{n-2};t_{n-2},\cdots,x_1;t_1\right)$$

$$p\left(x_{n-1};t_{n-1} \mid x_{n-2};t_{n-2},\cdots,x_1;t_1\right)$$

$$\cdots p\left(x_2;t_2 \mid x_1;t_1\right)p(x_1;t_1)$$

$$=p\left(x_n;t_n \mid x_{n-1};t_{n-1}\right)p\left(x_{n-1};t_{n-1} \mid x_{n-2};t_{n-2}\right)$$

$$\cdots p\left(x_2;t_2 \mid x_1;t_1\right)p(x_1;t_1)$$

$$=p(x_1;t_1)\prod_{\nu=2}^{n}p\left(x_\nu;t_\nu \mid x_{\nu-1};t_{\nu-1}\right)$$

Remark

Markov process is also completely specified in terms of

- $p\left(x_\nu;t_\nu \mid x_{\nu-1};t_{\nu-1}\right) \,\&\, p(x_1;t_1)\,\forall n \,\&\, \left\{t_\nu\right\}_{\nu=1}^{n}$
- $p\left(x_\nu;t_\nu, x_{\nu-1};t_{\nu-1}\right)\,\forall n \,\&\, \left\{t_\nu\right\}_{\nu=1}^{n}$

$$\underbrace{p\left(x_1,x_2,\cdots,x_n;t_1,t_2,\cdots,t_n\right)}_{\text{Multi-dimensional jpdf}} = \underbrace{p(x_1;t_1)}_{\text{Initial pdf}} \underbrace{\prod_{\nu=2}^{n} p\left(x_\nu;t_\nu \mid x_{\nu-1};t_{\nu-1}\right)}_{\text{Product of transistional pdfs}}$$

Chapman - Kolmogorov - Smoluchowski Equation

$t = t_1$ $\quad$ $t = \tau$ $\quad$ $t = t_2$

$$p(x_2,t_2;x_1,t_1) = p(x_2,t_2 \mid x_1,t_1)\,p(x_1,t_1)$$
$$= \int p(x_2,t_2;x,\tau;x_1,t_1)\,dx$$
$$= \int p(x_2,t_2 \mid x,\tau;x_1,t_1)\,p(x,\tau \mid x_1,t_1)\,p(x_1,t_1)\,dx$$
$$\Rightarrow$$
$$p(x_2,t_2 \mid x_1,t_1) = \int p(x_2,t_2 \mid x,\tau;x_1,t_1)\,p(x,\tau \mid x_1,t_1)\,dx$$
$$= \int p(x_2,t_2 \mid x,\tau)\,p(x,\tau \mid x_1,t_1)\,dx$$

Consistency condition for the process to be Markov

$$p\left(x_2,t_2 \mid x_1,t_1\right)=\int p\left(x_2,t_2 \mid x,\tau\right) p\left(x,\tau \mid x_1,t_1\right) dx$$

for all $t_1 < \tau < t_2$

Question

How to utilize this result in characterizing response of randomly driven dynamical systems?

Independent increment processes

Let $X(t)$ be a random process with continuous state and continuous parameter (time t).

Let $t_1 < t_2 < \cdots < t_n$ be n time instants.

Define $X(t_1, t_2) = X(t_2) - X(t_1)$

$X(t_1, t_2)$ is called the increment of $X(t)$ on $[t_1, t_2)$.

If, for all $t_1 < t_2 < \cdots < t_n$, the increments $X(t_1, t_2), X(t_2, t_3), \cdots, X(t_{n-1}, t_n)$ form a sequence of independent random variables, then the process $X(t)$ is said to be a process with independent increments.

Remark

Let $X(t)$ $(t \geq 0)$ be a random process with independent increments.

Define $Y(t) = X(t) - X(0)$ with $(t \geq 0)$.

$Y(t)$ is a process with independent increments and also has the property $P\left[Y(0) = 0\right] = 1$.

Also, $Y(t_{n-1}, t_n) = X(t_{n-1}, t_n)$ and has the properties of incremets of $X(t)$.

Without loss of generality, therefore, it may be taken that $P\left[X(0) = 0\right] = 1$.

Processes with independent increments possess Markovian property

Define $Y(t) = X(t)$ if $\mathrm{P}\left[X(0)=0\right]=1$

$$= X(t) - X(0) \text{ if } \mathrm{P}\left[X(0)=0\right] \neq 1$$

$\Rightarrow Y(t_0) = 0$

Consider $t_n > t_{n-1} > t_{n-2} > \cdots > t_1 > t_0 = 0$

and the associated random variables $Y(t_n), Y(t_{n-1}), \cdots, Y(t_1)$

$$Y(t_n) = \sum_{j=1}^{n}\left[Y(t_j) - Y(t_{j-1})\right]$$

$$= Y(t_1) - Y(t_0) + Y(t_2) - Y(t_1) \cdots + Y(t_n) - Y(t_{n-1}) = Y(t_n)$$

Processes with independent increments possess

Denote $Z_j = Y(t_j) - Y(t_{j-1})$

$\Rightarrow Z_j \left(j = 1, 2, \cdots, n \right)$ are sequence of independent random variables.

$$Y(t_n) = \sum_{j=1}^{n} Z_j = Z_n + Y(t_{n-1})$$

$\Rightarrow Y(t)$ is Markov.

Generalization : Markov property for vector random processes

Let $X(t)$ be a vector random process with continuous state and continuous parameter (time t).

Let $t_1 < t_2 < \cdots < t_n$ be the n time instants.

This defines n vector random variables

$X(t_1), X(t_2), \cdots, X(t_n)$.

The vector process $X(t)$ is said to possess Markov property if the n^{th} order conditional joint PDF

$$P\left[X(t_n) \le x_n \mid X(t_{n-1}) \le x_{n-1}, X(t_{n-2}) \le x_{n-2}, \cdots, X(t_1) \le x_1\right]$$

$$= P\left[X(t_n) \le x_n \mid X(t_{n-1}) \le x_{n-1}\right]$$

for any n and any choice of $t_1 < t_2 < \cdots < t_n$.

The m-dimensional vector random process

$$\boldsymbol{X}(t) = \begin{bmatrix} X_1(t) & X_2(t) & \cdots & X_m(t) \end{bmatrix}^t$$

is said to be Markov if

$$P\left[\bigcap_{i=1}^{m}\left\{X_j(t_n) \le x_j\right\} \mid \boldsymbol{X}(t_{n-1}) = y_{n-1}, \boldsymbol{X}(t_{n-2}) = y_{n-2}, \cdots, \boldsymbol{X}(t_1) = y_1\right]$$

$$= P\left[\bigcap_{i=1}^{m}\left\{X_j(t_n) \le x_j\right\} \mid \boldsymbol{X}(t_{n-1}) = y_{n-1}\right] \forall t_n > t_{n-1} > \cdots > t_1$$

$$\boldsymbol{TPDF} = P\left[\bigcap_{i=1}^{m}\left\{X_j(t_n) \le x_j\right\} \mid \boldsymbol{X}(t_{n-1}) = y_{n-1}\right]$$

tpdf

$$p(\mathbf{x};t \mid \mathbf{x}_0, t_0) = \frac{\partial^m}{\partial x_1 \partial x_2 \cdots \partial x_m} P\left[\bigcap_{i=1}^{m}\left\{X_j(t_n) \le x_j\right\} \mid \boldsymbol{X}(t_{n-1}) = y_{n-1}\right]$$

Remarks

- A component of a vector random process (which itself is a scalar random process) need not have Markov property
- Different components of a Markov vector random process could be differentiable (in the mean square sense) to different levels.
- If a vector random process has independent random increments (vectorially), the components of the random process need not have independent increments.

Consistency condition for a vector Markov process (CKS equation)

$$p\left(\tilde{x}_3;t_3\,|\,\tilde{x}_1;t_1\right)=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p\left(\tilde{x}_3;t_3\,|\,\tilde{y};t_2\right)p\left(\tilde{y};t_2\,|\,\tilde{x}_1;t_1\right)dy_1dy_2\cdots dy_n$$

Generalization : higher order Markov property

Let $X(t)$ be a scalar random process with continuous state and continuous parameter (time t).

Let $t_1 < t_2 < \cdots < t_n$ be n time instants.

This defines n random variables

$X(t_1), X(t_2), \cdots, X(t_n)$.

$X(t)$ is said to possess 2nd order Markov property if

$$P\left[X(t_n) \leq t_n \mid X(t_{n-1}) \leq t_{n-1}, X(t_{n-2}) \leq t_{n-2}, \cdots, X(t_1) \leq t_1\right]$$

$$= P\left[X(t_n) \leq t_n \mid X(t_{n-1}) \leq t_{n-1}, X(t_{n-2}) \leq t_{n-2}\right]$$

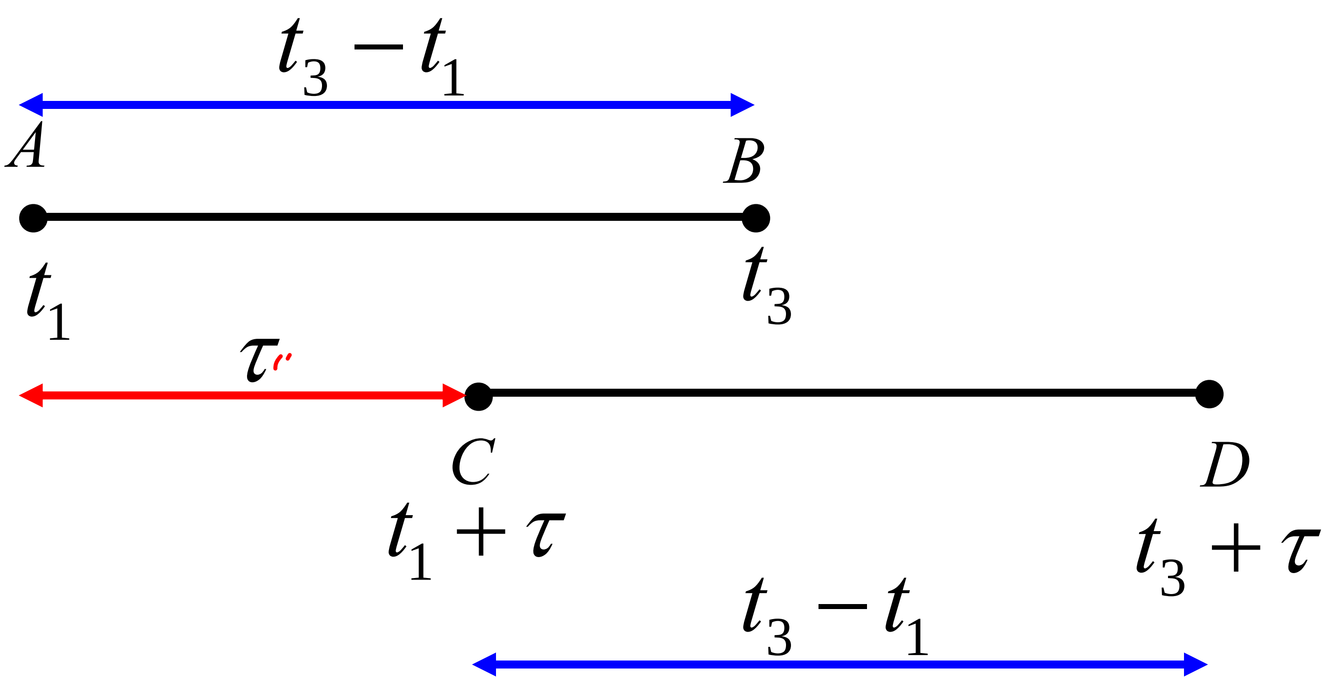
for any n and any choice of $t_1 < t_2 < \cdots < t_n$.

The idea can be generalized to 3rd and higher orders.

Markov process with stationary increments

$X(t)$ is said to be a Markov process with stationary increments if

$$p(x_3;t_3+\tau \mid x_1;t_1+\tau) = p(x_3;t_3 \mid x_1;t_1)\ \forall t_3 > t_1\ \&\ \tau$$



tpdf from A to B	=	tpdf from C to D

Remarks

• $p(x_3;t_3 \mid x_1;t_1) = p(x_3;t_3+\tau \mid x_1;t_1+\tau)$

$\Rightarrow$ tpdf can be written as $p(x_3;\tau \mid x_1)$;

$\tau =$ transistion time

• $\lim_{\tau\to\infty} p(x_3;\tau \mid x_1) \to p(x_3;\tau)$

•Stationary Markov random process is completely specified in terms of the tpdf.

• $X(t)$ is stationary $\Rightarrow X(t)$ has stationary increments.

• $X(t)$ has stationary inrements need not mean that $X(t)$ is stationary.

Kinetic equation

Let $X(t)$ be a scalar random process.

Let $p_X(x;t)$ be the first order pdf of $X(t)$.

Can we derive a differential equation satisfied by $p_X(x;t)$?

Here x and t would be the independent variables and $p_X(x;t)$ would be the dependent variable. Thus the differential equation that we are looking for would be a partial differential equation.

Furthermore, can we also derive the PDE governing the n-dimensional joint pdf of $X(t)$?

Notation: $p_X(x;t) = p(x;t)$

Reference: T T Soong, 1973, Random differential equations in science and engineering, Academic Press, NY.

Consider the random variables $X(t)$ & $X(t+\Delta t)$

$$p(x,t+\Delta t)=\int_{-\infty}^{\infty} p(x,x';t,t+\Delta t)\,dx'$$

$$=\int_{-\infty}^{\infty} p(x;t+\Delta t\,|\,x';t)\,p(x';t)\,dx'\cdots(1)$$

Define $\Delta X(t)=X(t+\Delta t)-X(t)$.

$\langle \exp[iu\Delta X(t)\,|\,X(t)=x']\rangle=$ Conditional characteristic

function of $\Delta X(t)$ given $X(t)=x'$.

Denote $\Phi(u,t+\Delta t\,|\,X(t)=x')=\langle \exp[iu\Delta X(t)\,|\,X(t)=x']\rangle$.

$$\Phi(u,t+\Delta t\,|\,X(t)=x')=\int_{-\infty}^{\infty}\exp(iu\Delta x)\,p(x;t+\Delta t\,|\,X(t)=x')\,dx\cdots(2)$$

with $\Delta X(t)=X(t+\Delta t)-X(t)$ such that $\Delta x=x-x'\cdots(3)$.

$$\Phi\left(u,t+\Delta t \mid X(t)=x'\right)=\int_{-\infty}^{\infty}\exp(iu\Delta x)\,p\left(x;t+\Delta t \mid X(t)=x'\right)dx$$

$$\Rightarrow p\left(x;t+\Delta t \mid X(t)=x'\right)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iu\Delta x)\,\Phi\left(u,t+\Delta t \mid X(t)=x'\right)du\cdots(4)$$

Recall : $f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\cdots=\sum_{n=0}^{\infty}\frac{h^n}{n!}f^n(x)$

$$\Rightarrow \Phi\left(u,t+\Delta t \mid X(t)=x'\right)=\Phi\left(0+u,t+\Delta t \mid X(t)=x'\right)$$

$$=\sum_{n=0}^{\infty}\frac{u^n}{n!}\frac{d^n}{du^n}\Phi\left(u,t+\Delta t \mid X(t)=x'\right)\Big|_{u=0}$$

$$=\sum_{n=0}^{\infty}\frac{(iu)^n}{n!}\left\langle\left[\Delta X^n(t) \mid X(t)=x'\right]\right\rangle$$

Recall : Characteristic function of a random variable

Definition : $\varphi_X(\omega) = \langle \exp(i\omega X) \rangle = \int_{-\infty}^{\infty} \exp(i\omega x)\, p_X(x)\, dx$

Here ω is real valued. Thus, $\varphi_X(\omega)$ is the Fourier transform of the pdf.

It can be shown that $\frac{1}{i^n} \frac{d^n \varphi_X}{d\omega^n}\bigg|_{\omega=0} = \langle X^n \rangle$.

By using inverse Fourier transform, it follows that

$p_X(x) = \int_{-\infty}^{\infty} \varphi_X(\omega) \exp(-i\omega x)\, dx.$

Characteristic function of a random process

Definition :

$$\phi_X\left(u_1;t_1,u_2;t_2,\cdots,u_n;t_n\right)=\left\langle \exp\left[i\sum_{j=1}^{n}u_jX\left(t_j\right)\right]\right\rangle$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left[i\sum_{j=1}^{n}u_jx_j\right]p\left(x_1;t_1,x_2;t_2,\cdots,x_n;t_n\right)dx_1dx_2\cdots dx_n$$

Here $\left\{u_j\right\}_{j=1}^{n}$ is real valued.

$$\alpha_{j_1,j_2,\cdots,j_n}\left(t_1,t_2,\cdots,t_n\right)=\left\langle X^{j_1}\left(t_1\right)X^{j_2}\left(t_2\right)\cdots X^{j_n}\left(t_n\right)\right\rangle$$

$$\frac{\partial^m}{\partial u_1^{j_1}\partial u_2^{j_2}\cdots\partial u_n^{j_n}}\phi_X\left(u_1;t_1,u_2;t_2,\cdots,u_n;t_n\right)\Big|_{\left\{u_j\right\}_{j=1}^{n}=0}=i^m\alpha_{j_1,j_2,\cdots,j_n}\left(t_1,t_2,\cdots,t_n\right)$$

such that $m=j_1+j_2+\cdots+j_n$.

$$\Rightarrow p\left(x;t+\Delta t \mid X(t)=x'\right)=$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iu\Delta x)\left\{\sum_{n=0}^{\infty}\frac{(iu)^n}{n!}\left\langle\left[\Delta X^n(t) \mid X(t)=x'\right]\right\rangle\right\}du$$

$$p\left(x;t+\Delta t \mid X(t)=x'\right)=\sum_{n=0}^{\infty}\frac{i^n}{2\pi}\frac{1}{n!}a_n(x',t)\int_{-\infty}^{\infty}u^n\exp(-iu\Delta x)\,du$$

with

$$a_n(x',t)=\left\langle\left[\Delta X^n(t) \mid X(t)=x'\right]\right\rangle$$

$$=\left\langle\left[\left\{X(t+\Delta t)-x(t)\right\}^n \mid X(t)=x'\right]\right\rangle\cdots(5)$$

= **Incremental moments**

$$p\left(x;t+\Delta t \mid X(t)=x'\right)=\sum_{n=0}^{\infty}\frac{a_n(x',t)}{n!}\frac{1}{2\pi}\int_{-\infty}^{\infty}(iu)^n\exp(-iu\Delta x)\,du$$

$$p\left(x;t+\Delta t\,|\,X(t)=x'\right)=\sum_{n=0}^{\infty}\frac{a_n(x',t)}{n!}\frac{1}{2\pi}\int_{-\infty}^{\infty}(iu)^n\exp(-iu\Delta x)\,du\cdots(6)$$

Recall

$$\int_{-\infty}^{\infty}\delta(x)\exp(iux)\,dx=1$$

$$\Rightarrow\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iux)\,du$$

$$\Rightarrow\frac{d^n}{dx^n}\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(-1)^n(iu)^n\exp(-iux)\,du$$

$$p\left(x;t+\Delta t\,|\,X(t)=x'\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a_n(x',t)\frac{d^n}{dx^n}\delta(\Delta x)\cdots(7)$$

Substitute equation 7 in 1

$$p\left(x;t+\Delta t \mid X(t)=x'\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a_n(x',t)\frac{d^n}{dx^n}\delta(\Delta x)\cdots(7)$$

$$p(x,t+\Delta t)=\int_{-\infty}^{\infty}p(x;t+\Delta t \mid x';t)\,p(x';t)\,dx'\cdots(1)$$

$$\Rightarrow p(x,t+\Delta t)=\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}a_n(x',t)\frac{d^n}{dx^n}\delta(\Delta x)\,p(x';t)\,dx'$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\int_{-\infty}^{\infty}\delta(\Delta x)\,a_n(x',t)\,p(x';t)\,dx'$$

Recall $\frac{d^n}{dx^n}\int_{-\infty}^{\infty}\delta(x-a)f(x)dx=\frac{d^n f(a)}{dx^n}$

$$p(x,t+\Delta t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[a_n(x,t)\,p(x;t)\right]$$

$$p(x,t+\Delta t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[a_n(x,t)\,p(x;t)\right]$$

$$=a_0(x,t)\,p(x;t)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[a_n(x,t)\,p(x;t)\right]\ldots(8)$$

$$a_0(x,t)=\left\langle \Delta X^0(t)\,|\,X(t)=x'\right\rangle=1$$

$$\Rightarrow p(x,t+\Delta t)-p(x;t)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[a_n(x,t)\,p(x;t)\right]$$

$$\frac{\partial p}{\partial t}=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[a_n(x,t)\,p(x;t)\right]$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\frac{d^n}{dx^n}\left[a_n(x,t)\,p(x;t)\right]$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[\alpha_n(x,t)\,p(x;t)\right]$$

$$\frac{\partial p}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left[\alpha_n(x,t)\, p(x;t) \right]$$

$$\alpha_n(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\langle \left[X(t+\Delta t) - X(t) \right]^n \mid X(t) = x \right\rangle; n = 1, 2, \cdots$$

= Derivative moments

$$\lambda(x,t) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \left[\alpha_n(x,t)\, p(x;t) \right]$$

$$\frac{\partial p}{\partial t} + \frac{\partial \lambda}{\partial x} = 0$$

- Equation of conservation of probability

Similar to equation of continuity in fluid mechanics.

- Diffusion equation
- $\lambda(x,t)$=amount of probability crossing x in unit time

Remarks

- The equation $\frac{\partial p}{\partial t}+\frac{\partial \lambda}{\partial t}=0$ has infinite order with respect to the spatial coordinate.

Therefore, its application is severely limited.

- **Theorem**

R F Pawula,1967, IEEE Trans.

Information theory, IT-13, 33-41

If the derivative moment $\alpha_n(x,t)$ exists for all n and is zero for some even n, then $\alpha_n(x,t)=0\,\forall n\geq 3$.

Implication :

There exist two cases of equation

$$\frac{\partial p}{\partial t}+\frac{\partial \lambda}{\partial t}=0.$$ The one in which

the order of the equaiton in x is infinite

and the other in which the order

is 2 or less. We would be interested in

applying the kinetic eqaution for

the case in which $\alpha_n\left(x,t\right)=0\forall n\geq 3.$

Proof of the statement :

If the derivative moment $\alpha_n(x,t)$ exists for all n and is zero for some even n, then $\alpha_n(x,t)=0\,\forall n\geq 3.$

Let $n\geq 3$ and let n be odd.

$$\alpha_n(x,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle\left[X(t+\Delta t)-X(t)\right]^n \mid X(t)=x\right\rangle$$

$$=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle\left[X(t+\Delta t)-X(t)\right]^{\frac{n-1}{2}}\left[X(t+\Delta t)-X(t)\right]^{\frac{n+1}{2}} \mid X(t)=x\right\rangle$$

Apply the Schwarz inequality

$$\alpha_n^2(x,t) \le \lim_{\Delta t \to 0} \frac{1}{\Delta t^2} \left\langle \left[X(t+\Delta t) - X(t) \right]^{n-1} \mid X(t) = x \right\rangle$$

$$\left\langle \left[X(t+\Delta t) - X(t) \right]^{n+1} \mid X(t) = x \right\rangle.$$

$$\Rightarrow \alpha_n^2(x,t) \le \alpha_{n-1}(x,t)\,\alpha_{n+1}(x,t) \; \forall n \ge 3 \,\&\, n \text{ odd} \cdots (1)$$

Similarly it can be shown that

$$\alpha_n^2(x,t) \le \alpha_{n-2}(x,t)\,\alpha_{n+2}(x,t) \; \forall n \ge 4 \,\&\, n \text{ even} \cdots (2)$$

Let $\alpha_r(x,t)=0$ where r is an even integer.

Let $n = r-1, r+1,$ and using

$\alpha_n^2(x,t) \le \alpha_{n-1}(x,t)\alpha_{n+1}(x,t)\,\forall n \ge 3\,\&\, n$ odd

we get

$\alpha_{r-1}^2(x,t) \le \alpha_{r-2}(x,t)\alpha_r(x,t)\,\forall r \ge 4$

$\alpha_{r+1}^2(x,t) \le \alpha_r(x,t)\alpha_{r+2}(x,t)\,\forall r \ge 2$

Let $n = r - 2, r + 2,$ and using

$$\alpha_n^2(x,t) \le \alpha_{n-2}(x,t)\,\alpha_{n+2}(x,t)\ \forall n \ge 4\ \&\ n \text{ even}$$

we get

$$\alpha_{r-2}^2(x,t) \le \alpha_{r-4}(x,t)\,\alpha_r(x,t)\ \forall r \ge 6$$

$$\alpha_{r+2}^2(x,t) \le \alpha_r(x,t)\,\alpha_{r+4}(x,t)\ \forall r \ge 2$$

$$\alpha_{r-1}^2 \le \alpha_{r-2}\alpha_r \forall r \ge 4$$

$$\alpha_{r+1}^2 \le \alpha_r\alpha_{r+2} \forall r \ge 2$$

$$\alpha_{r-2}^2 \le \alpha_{r-4}\alpha_r \forall r \ge 6$$

$$\alpha_{r+2}^2 \le \alpha_r\alpha_{r+4} \forall r \ge 2$$

Illustration:

Let α_3=0 and α_n exist for all *n*.

$$\alpha_{r+1}^2 \le \alpha_r\alpha_{r+2} \forall r \ge 2 \Rightarrow \alpha_4^2 = 0$$

$$\alpha_{r+2}^2 \le \alpha_r\alpha_{r+4} \forall r \ge 2 \Rightarrow \alpha_5^2 = 0$$

$$\Rightarrow \alpha_r = 0 \forall r > 3$$

$$\alpha_{r-1}^2 \le \alpha_{r-2}\alpha_r \forall r \ge 4$$

$$\alpha_{r+1}^2 \le \alpha_r \alpha_{r+2} \forall r \ge 2$$

$$\alpha_{r-2}^2 \le \alpha_{r-4}\alpha_r \forall r \ge 6$$

$$\alpha_{r+2}^2 \le \alpha_r \alpha_{r+4} \forall r \ge 2$$

Let α_r =0 and α_n exist for all n.

$$\alpha_{r+2}^2 \le \alpha_r \alpha_{r+4} \forall r \ge 2 \,\&\, \alpha_{r+1}^2 \le \alpha_r \alpha_{r+2} \forall r \ge 2 \Rightarrow \alpha_n = 0 \forall n > r$$

$$\alpha_{r-2}^2 \le \alpha_{r-4}\alpha_r \forall r \ge 6 \,\&\, \alpha_{r-1}^2 \le \alpha_{r-2}\alpha_r \forall r \ge 4 \Rightarrow \alpha_n = 0 \forall n < r \,\&\, n \ge 3$$

QED

Remarks

- For the case $\alpha_n(x,t) = 0 \forall n \geq 3,$ the kinetic equation takes the form

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[\alpha_1(x,t)\,p(x;t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\alpha_2(x,t)\,p(x;t)\right]$$

This equation is known as the Fokker-Planck equation. Here

$$\lambda(x,t) = \alpha_1(x,t)\,p(x;t) - \frac{1}{2}\frac{\partial}{\partial x}\left[\alpha_2(x,t)\,p(x;t)\right].$$

- Initial condition: $p(x;t=0) = p_0(x)$
- Boundary conditions: different possibilities exist.

Boundary conditions

If $X(t)$ takes values from -∞ to ∞, then the BCS are specified at $\pm\infty$. These boundaries are inaccessible.

$$\frac{\partial p}{\partial t}+\frac{\partial \lambda}{\partial t}=0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial p}{\partial t} dx+\int_{-\infty}^{\infty} \frac{\partial \lambda}{\partial x} dx=0$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{-\infty}^{\infty} p(x;t) dx+\int_{-\infty}^{\infty} \frac{\partial \lambda}{\partial x} dx=0$$

$$\int_{-\infty}^{\infty} p(x;t) dx=1 \Rightarrow \int_{-\infty}^{\infty} \frac{\partial \lambda}{\partial x} dx=0$$

$$\lambda(-\infty;t)=\lambda(\infty;t)$$

A stronger condition: $\lambda(-\infty;t)=\lambda(\infty;t)=0$

Still stronger condition: $p(\pm\infty;t)=0$

If $X(t)$ can take values only in the bounded region $x_1 \le X(t) \le x_2$, then the Fokker Planck equation is valid in this region with

$$\frac{\partial p}{\partial t} + \frac{\partial \lambda}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial t}\int_{x_1}^{x_2} p(x;t)\,dx + \int_{x_1}^{x_2} \frac{\partial \lambda}{\partial x}\,dx = 0$$

$$\Rightarrow \lambda(x_1;t) = \lambda(x_2;t)$$

A stronger condition: $\lambda(x_1;t) = \lambda(x_2;t) = 0$

Still stronger condition: $p(x_1;t) = p(x_2;t) = 0$

$\lambda(x_1;t) = \lambda(x_2;t) = 0$: Reflecting boundaries

$p(x_1;t) = p(x_2;t) = 0$: Absorbing boundaries

Remarks (continued)

• When $\alpha_n(x,t)$ are independent of time, stationary solutions might exist such that

$t \to \infty$

$\frac{\partial p}{\partial t} = 0$ and one gets the simplified equation

$$-\frac{\partial}{\partial x}\left[\alpha_1(x)\,p(x)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\alpha_2(x)\,p(x)\right] = 0$$

leading to the solution $\lambda(x) = 0$.

Generalization of the kinetic equaiton for the case of a vector random process

Let $X(t)=\{X_1(t),X_2(t),\cdots,X_m(t)\}^t$ be a m-dimensional vector random process.

Consider the vector random variables $X(t)$ & $X(t+\Delta t)$

$$p(\tilde{x},t+\Delta t)=\int_{-\infty}^{\infty} p(\tilde{x};t+\Delta t\,|\,\tilde{x}';t)\,p(\tilde{x}';t)\,d\tilde{x}'$$

$$\frac{\partial p(\tilde{x};t)}{\partial t}=\sum_{n_1,n_2,\cdots,n_m=1}^{\infty}\left[\prod_{j=1}^{m}\frac{(-1)^{n_j}}{(n_j)!}\frac{\partial}{\partial x_j^{n_j}}\right]\left[\alpha_{n_1,n_2,\cdots,n_m}(\tilde{x};t)\,p(\tilde{x};t)\right]$$

with

$$\alpha_{n_1,n_2,\cdots,n_m}(\tilde{x};t)=\lim_{\Delta t\to 0}\left\langle\prod_{j=1}^{m}\left[X_j(t+\Delta t)-X_j(t)\right]^{n_j}\,|X(t)=\tilde{x}\right\rangle$$

Next Lecture

- How to derive the Fokker Plank equation for response of dynamical systems driven by random excitations.
- How to solve them?