Stochastic Structural Dynamics

Lecture-20

Failure of randomly vibrating systems-4

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Recall

Rice's definition of envelope and phase processes

$$X(t) = \sum_{n=1}^{\infty} a_n \left[\cos(\omega_n - \omega_r) t \cos \omega_r t - \sin(\omega_n - \omega_r) t \sin \omega_r t \right]$$

$$= I_c(t) \cos \omega_r t + I_s(t) \sin \omega_r t$$

$$X(t) = R(t) \cos\left[\omega_r t + \Phi(t) \right]$$

- $p_{R\Phi}(r, \phi; t)$
- $p_{RR\Phi\Phi}(r_1, r_2, \phi_1, \phi_2; t_1, t_2)$
- $p_{R\dot{R}\Phi\dot{\Phi}}(r, \dot{r}, \phi, \dot{\phi}; t)$
- Level crossing problem for $R(t)$
- Clumping for narrow banded processes

$T_f(\alpha)$ is exponentially distributed

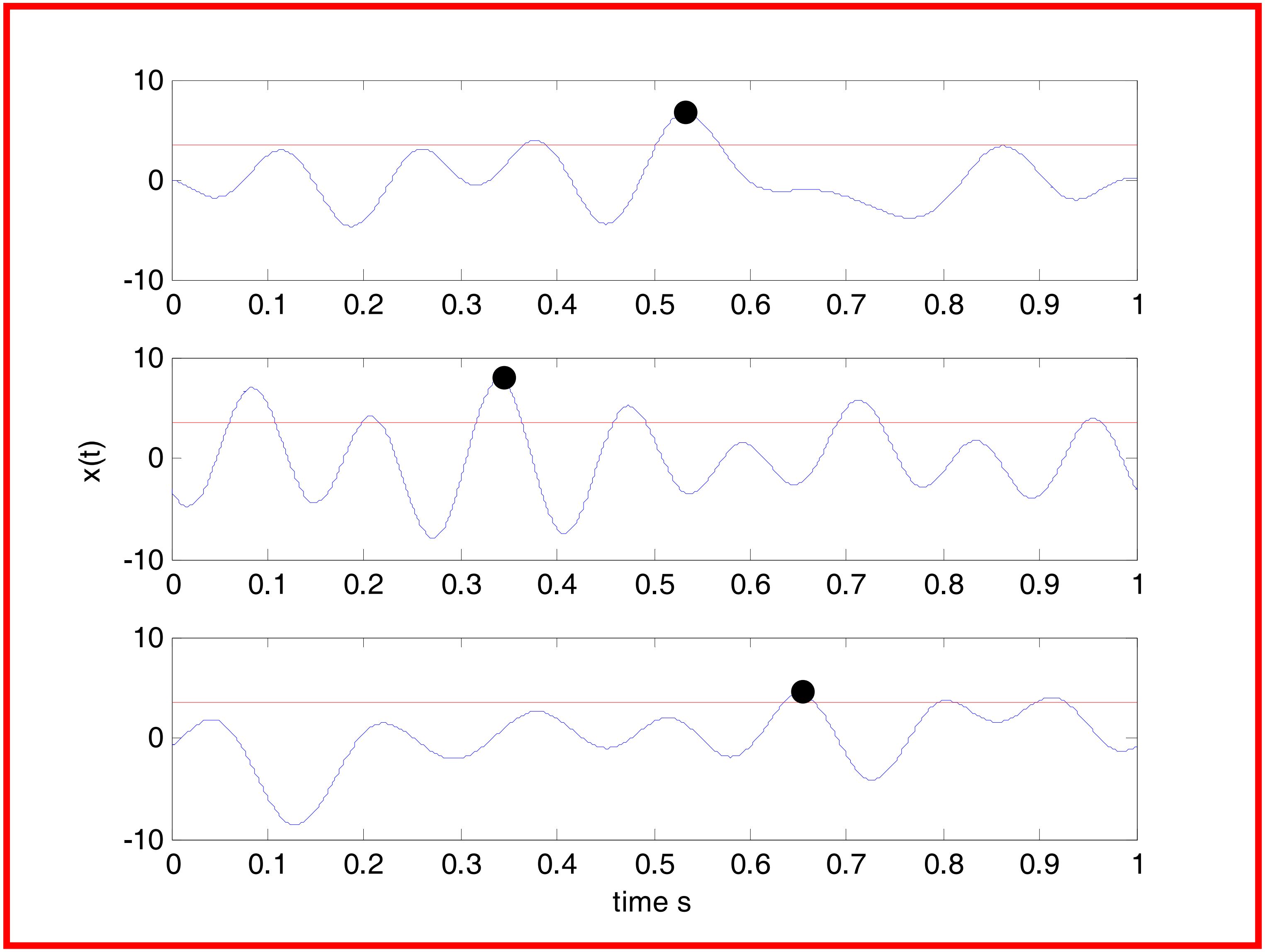
$$P_{T_f}(t) = 1 - \exp[-\lambda t]$$

$$p_{T_f}(t) = \lambda \exp[-\lambda t] \quad 0 < t < \infty$$

$$\lambda = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\langle T_f \rangle = \int_0^\infty t\lambda \exp[-\lambda t] dt = \frac{1}{\lambda}$$

The maximum value of $X(t)$ in interval 0 to T



Recall : results for iid sequence of RV - s

Let $\left\{X_i\right\}_{i=1}^{n}$ be an iid sequence of random variables.

Define $Y_n = \max_{i=1,2,\cdots,n} X_i$

$$P\left(Y_n \le y\right) = P\left(\max_{i=1,2,\cdots,n} X_i \le y\right)$$

$$= P\left(\bigcap_{i=1}^{n}\left(X_i \le y\right)\right) = \left[P_X\left(y\right)\right]^n$$

$$P_{Y_n}\left(y\right) = \left[P_X\left(y\right)\right]^n$$

$$p_{Y_n}\left(y\right) = n\left[P_X\left(y\right)\right]^{n-1} p_X\left(y\right)$$

Similarly

Let $\left\{X_i\right\}_{i=1}^{n}$ be an iid sequence of random variables.

Define $Y_1 = \min_{i=1,2,\cdots,n} X_i$

$$P\left(Y_1 > z\right) = P\left(\min_{i=1,2,\cdots,n} X_i > z\right)$$

$$= P\left(\bigcap_{i=1}^{n}\left(X_i > z\right)\right) = \left[1 - P_X\left(z\right)\right]^n$$

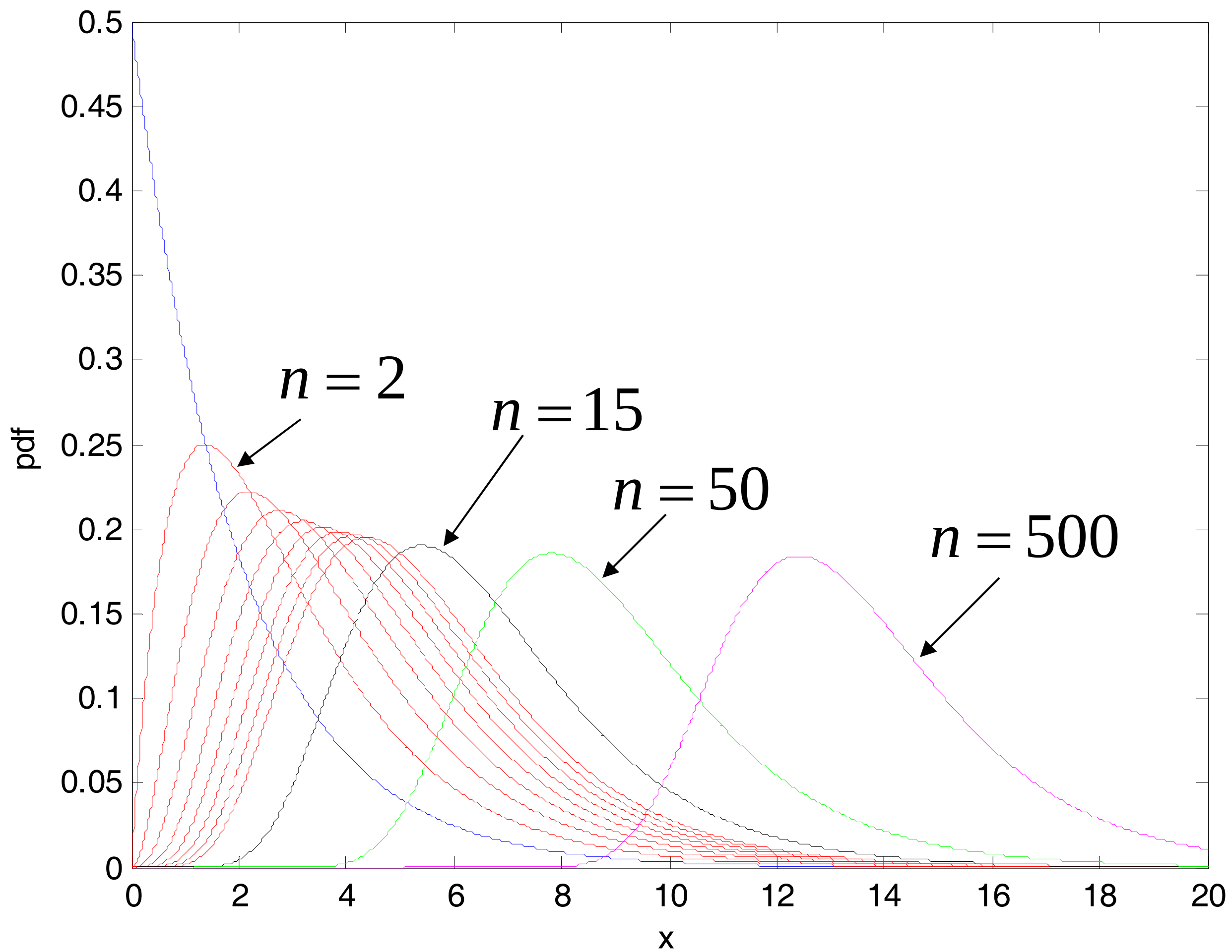
$$P_{Y_1}\left(z\right) = 1 - \left[1 - P_X\left(z\right)\right]^n$$

$$p_{Y_1}\left(z\right) = n\left[1 - P_X\left(z\right)\right]^{n-1} p_X\left(z\right)$$

Note

$$Y_1 = \max_{i=1,2,\cdots,n}\left[-X_i\right]$$

$$X = \max_{i=1,n} X_i; \quad X_i : \text{iid sequence with } p_X(x) = \lambda \exp(-\lambda x); 0 < x < \infty$$



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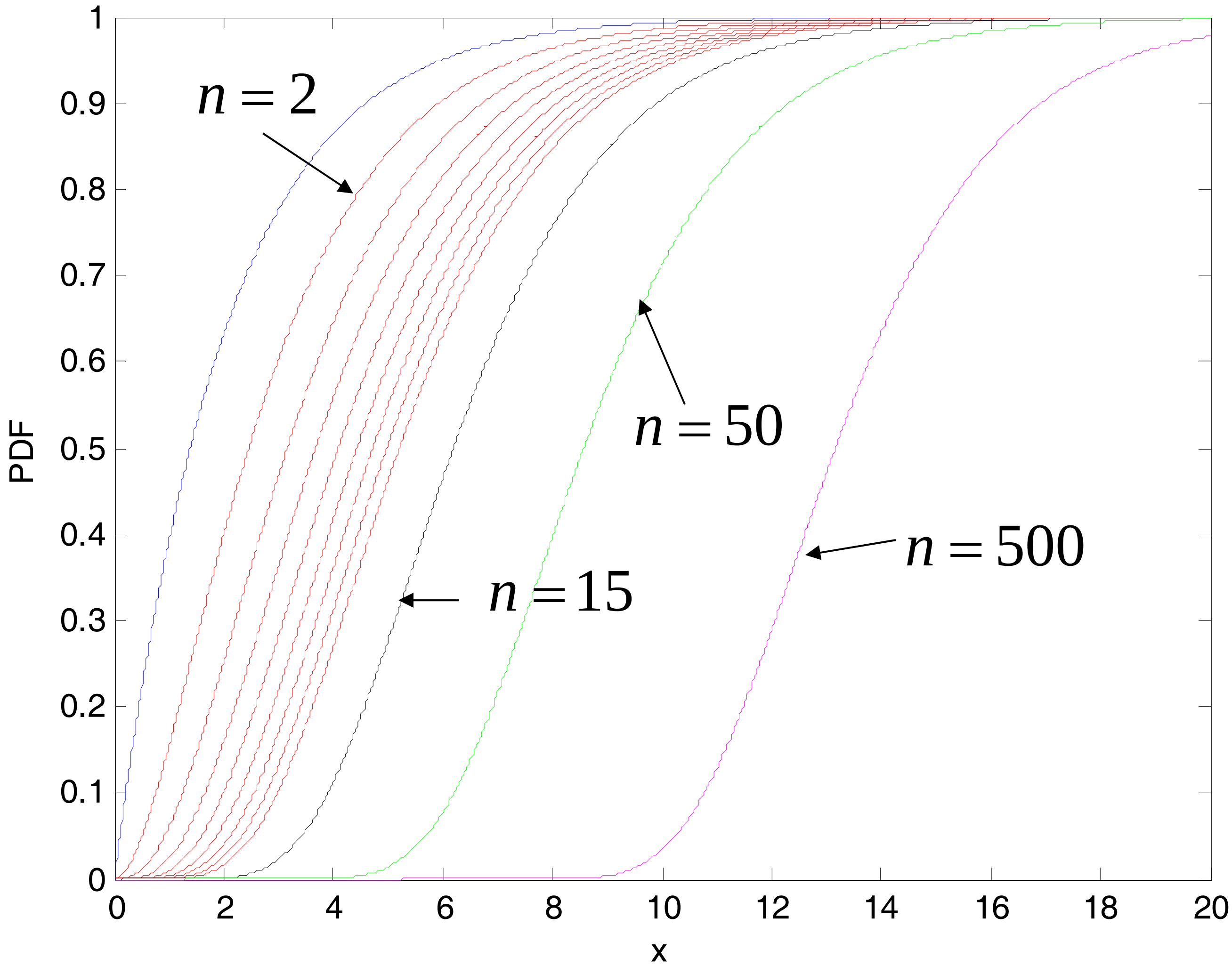
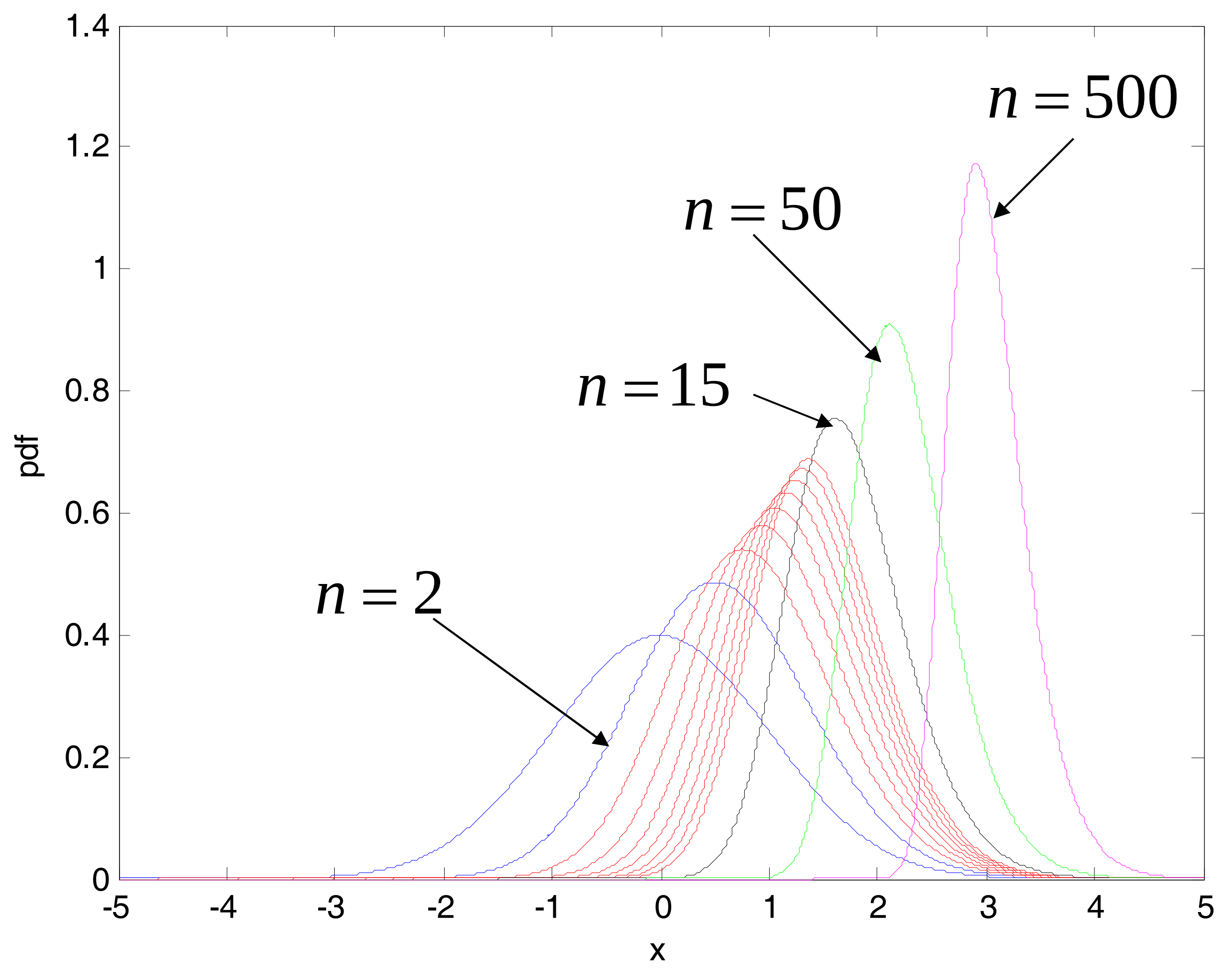
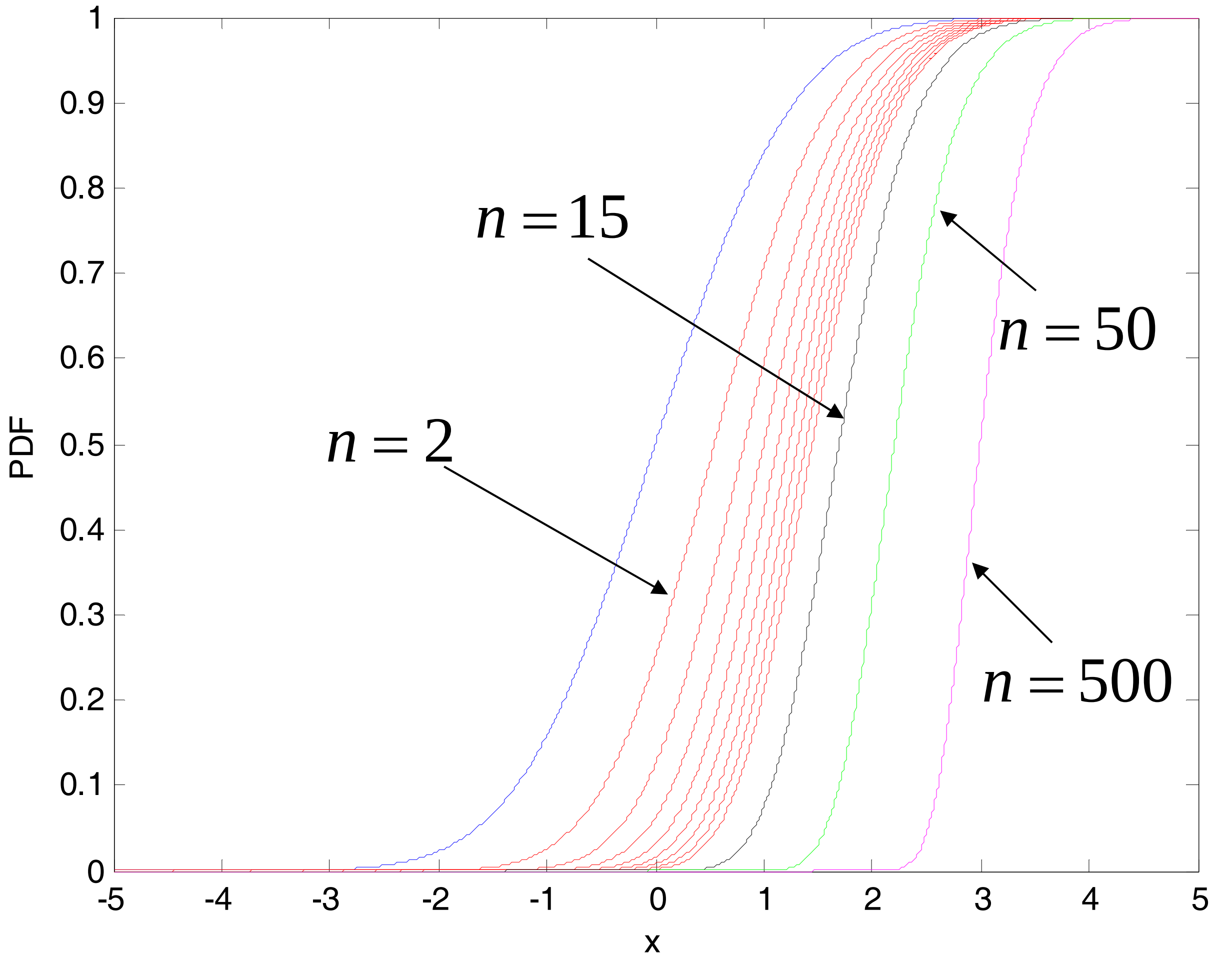
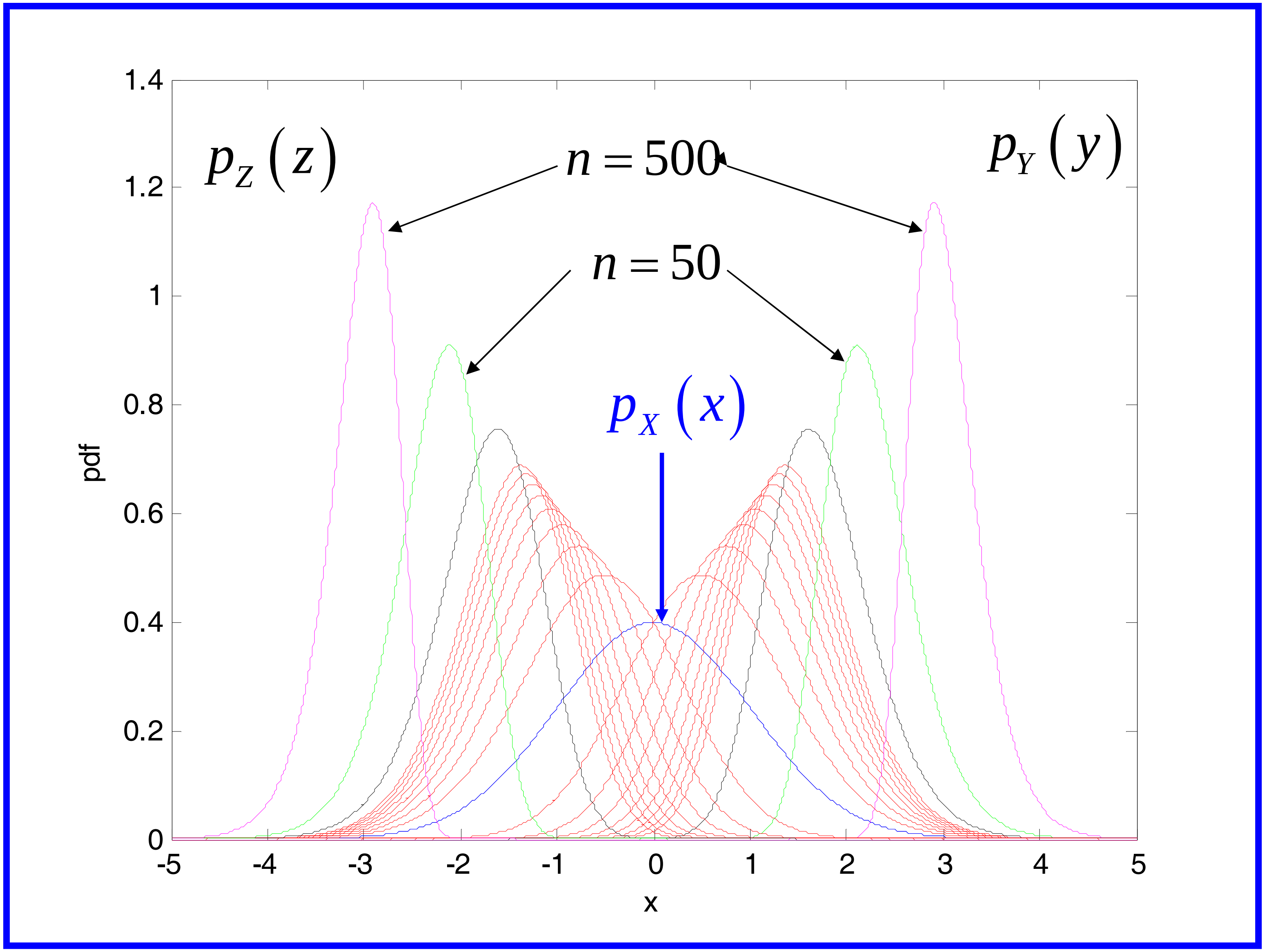


$$X = \max_{i=1,n} X_i; \quad X_i : \text{iid sequence with } p_X(x) \sim N(0,1); -\infty < x < \infty$$



$$X = \max_{i=1,n} X_i; \quad X_i : \text{iid sequence with } p_X(x) \sim N(0,1); -\infty < x < \infty$$





1.4
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pdf
-5
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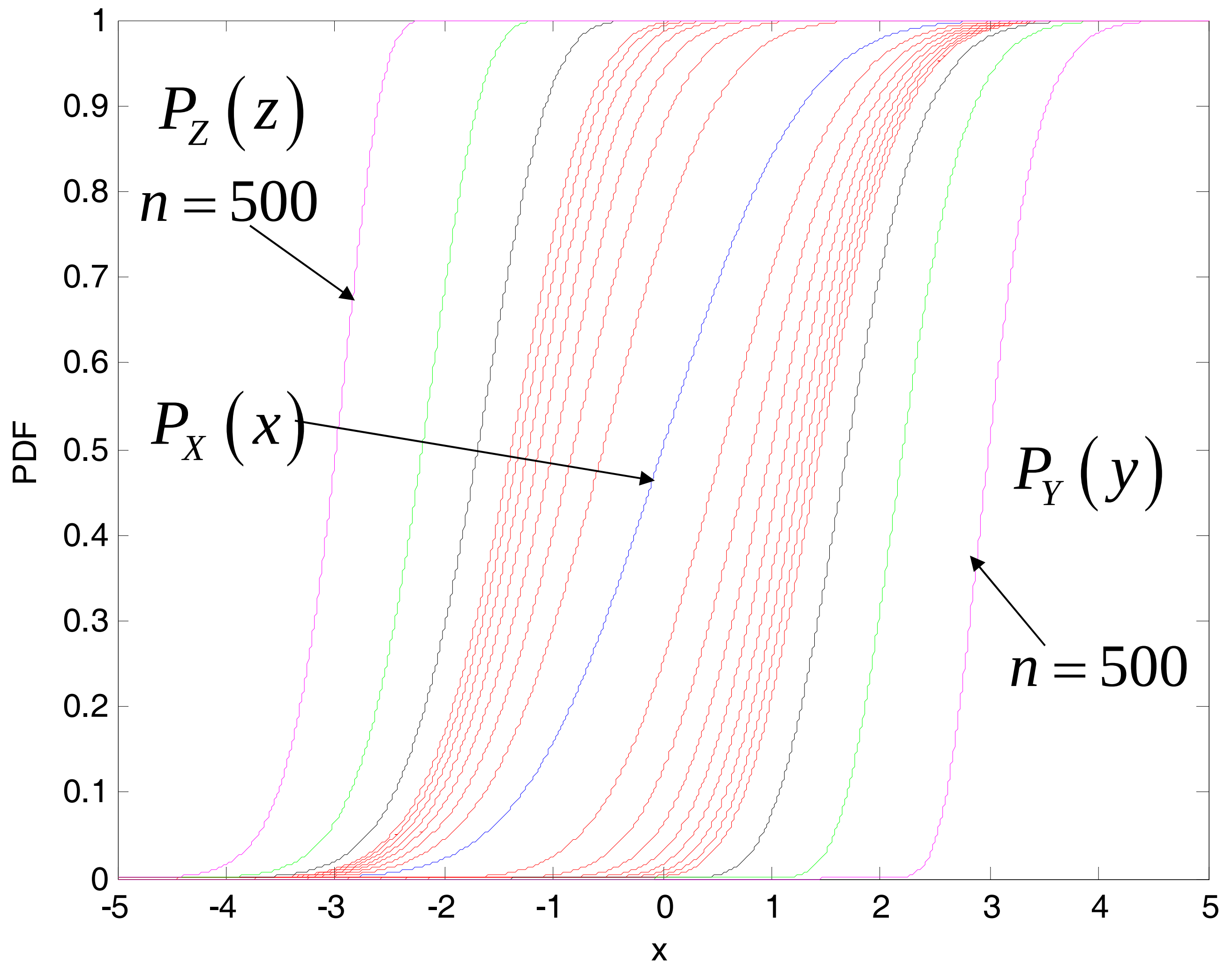
x
$p_Z(z)$
$n = 500$
$p_Y(y)$
$n = 50$
$p_X(x)$



$P_Z(z)$
$n = 500$
$P_X(x)$
$P_Y(y)$
$n = 500$
PDF
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5

Asymptotic behavior as $n \to \infty$

Consider

$$P_{Y_n}(y) = \left[P_X(y)\right]^n$$

$$p_{Y_n}(y) = n\left[P_X(y)\right]^{n-1} p_X(y)$$

Question: $\lim_{n\to\infty} P_{Y_n}(y) \to ?$

Introduce the random variable

$$\Xi_n = n\left[1 - P_X(Y_n)\right]$$

$$P_{\Xi_n}(\xi) = P\left[\Xi_n \le \xi\right] = P\left\{n\left[1 - P_X(Y_n)\right] \le \xi\right\} = P\left\{P_X(Y_n) \ge 1 - \frac{\xi}{n}\right\}$$

$$= 1 - P\left\{Y_n \le P_X^{-1}\left(1 - \frac{\xi}{n}\right)\right\} = 1 - P_{Y_n}\left[P_X^{-1}\left(1 - \frac{\xi}{n}\right)\right]$$

$$P_{\Xi_n}(\xi) = 1 - P_{Y_n}\left[P_X^{-1}\left(1-\frac{\xi}{n}\right)\right] = 1 - \left\{P_X\left[P_X^{-1}\left(1-\frac{\xi}{n}\right)\right]\right\}^n = 1 - \left(1-\frac{\xi}{n}\right)^n$$

Consider

$$\theta = \left(1-\frac{\xi}{n}\right)^n$$

$$\Rightarrow \log\theta = n\log\left(1-\frac{\xi}{n}\right) = \frac{\log\left(1-\frac{\xi}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$\lim_{n\to\infty}\frac{\log\left(1-\frac{\xi}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{\frac{1}{\left(1-\frac{\xi}{n}\right)}\left(-\frac{\xi}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} = -\xi$$

$$\Rightarrow \theta = \exp(-\xi)$$

$$P_{\Xi_n}(\xi) = 1 - \exp(-\xi)$$

$$p_{\Xi_n}(\xi) = \exp(-\xi)$$

Now consider the transformation

$$\Xi_n = n\left[1 - P_X(Y_n)\right]$$

$$Y_n = P_X^{-1}\left[1 - \frac{\Xi_n}{n}\right]$$

For large n, pdf of Ξ_n is given by $p_{\Xi_n}(\xi) = \exp(-\xi)$.

Also, Ξ_n decreases as Y_n inreases.

$\Rightarrow$ For large n

$P_{Y_n}(y) = P\left[\Xi_n > g(y)\right]$ where $g(y) = n\left[1 - P_X(y)\right]$.

$$\Rightarrow P_{Y_n}(y) = 1 - P_{\Xi_n}\left[g(y)\right] = 1 - \left\{1 - \exp\left[-g(y)\right]\right\} = \exp\left[-g(y)\right]$$

$$p_{Y_n}(y) = -\frac{dg}{dy}\exp\left[-g(y)\right]$$

Example

$$p_X(x) = \lambda \exp(-\lambda x); 0 < x < \infty$$

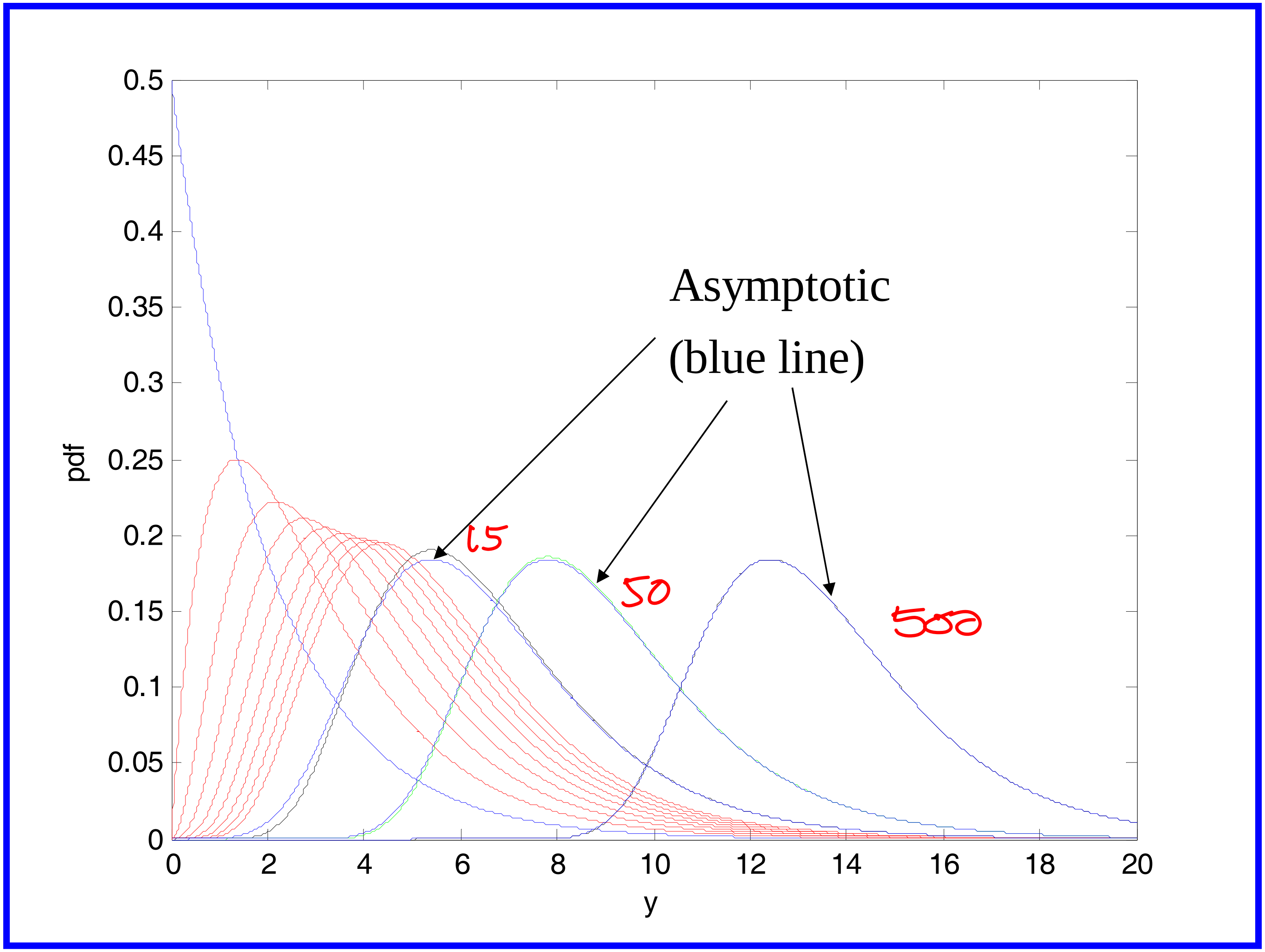
$$P_X(x) = 1 - \exp(-\lambda x); 0 < x < \infty$$

$$\Xi_n = n\left[1 - \left\{1 - \exp(-\lambda Y_n)\right\}\right] = n \exp(-\lambda Y_n)$$

$$\Rightarrow g(y) = n \exp(-\lambda y)$$

$$P_{Y_n}(y) = \exp\left[-n \exp(-\lambda y)\right]$$

$$p_{Y_n}(y) = n\lambda \exp(-\lambda y) \exp\left[-n \exp(-\lambda y)\right]$$


0.5
0.45
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0.2
0.15
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pdf
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2
4
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8
10
12
14
16
18
20
y
Asymptotic
(blue line)
15
50
500

Example

$$p_X(x) = \frac{x}{\sigma^2} \exp\left[-\frac{x^2}{2\sigma^2}\right]; 0 < x < \infty$$

$$P_X(x) = 1 - \exp\left[-\frac{x^2}{2\sigma^2}\right]; 0 < x < \infty$$

$$\Xi_n = n\left\{1 - 1 + \exp\left[-\frac{Y_n^2}{2\sigma^2}\right]\right\} = n\exp\left[-\frac{Y_n^2}{2\sigma^2}\right]$$

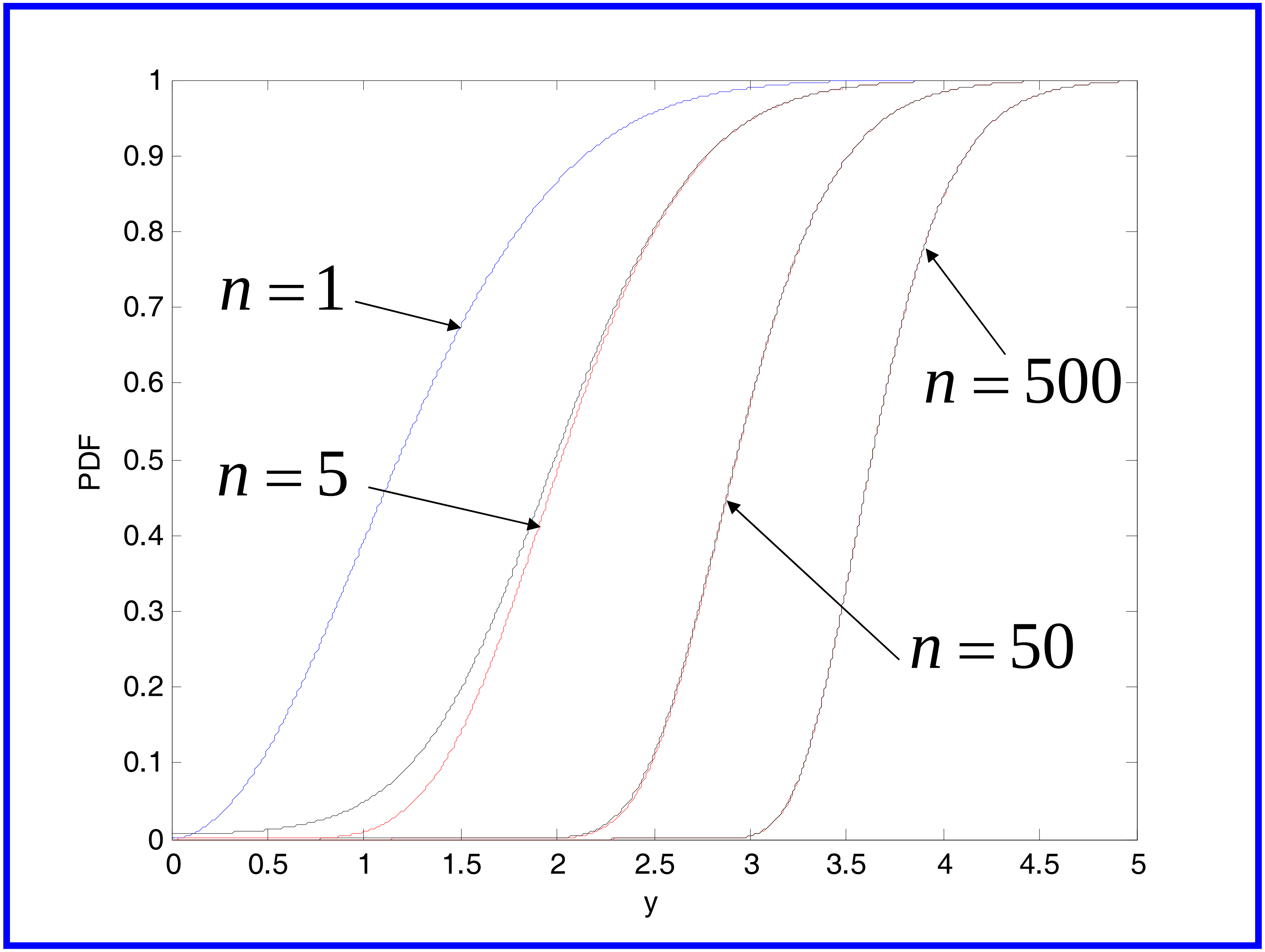
$$\Rightarrow g(y) = n\exp\left[-\frac{y^2}{2\sigma^2}\right]$$

$$P_{Y_n}(y) = \exp\left\{-n\exp\left[-\frac{y^2}{2\sigma^2}\right]\right\}$$

$$p_{Y_n}(y) = \frac{ny}{\sigma^2}\exp\left[-\frac{y^2}{2\sigma^2}\right]\exp\left\{-n\exp\left[-\frac{y^2}{2\sigma^2}\right]\right\}$$

Black: Asymptotic

Red: Exact



Example

$$P_X(x) = x \quad \text{for } 0 < x \le 1$$

$$= 1 \quad \text{for } x > 1$$

$$\Rightarrow g(y) = n(1 - y); \ \text{for } 0 < y \le 1$$

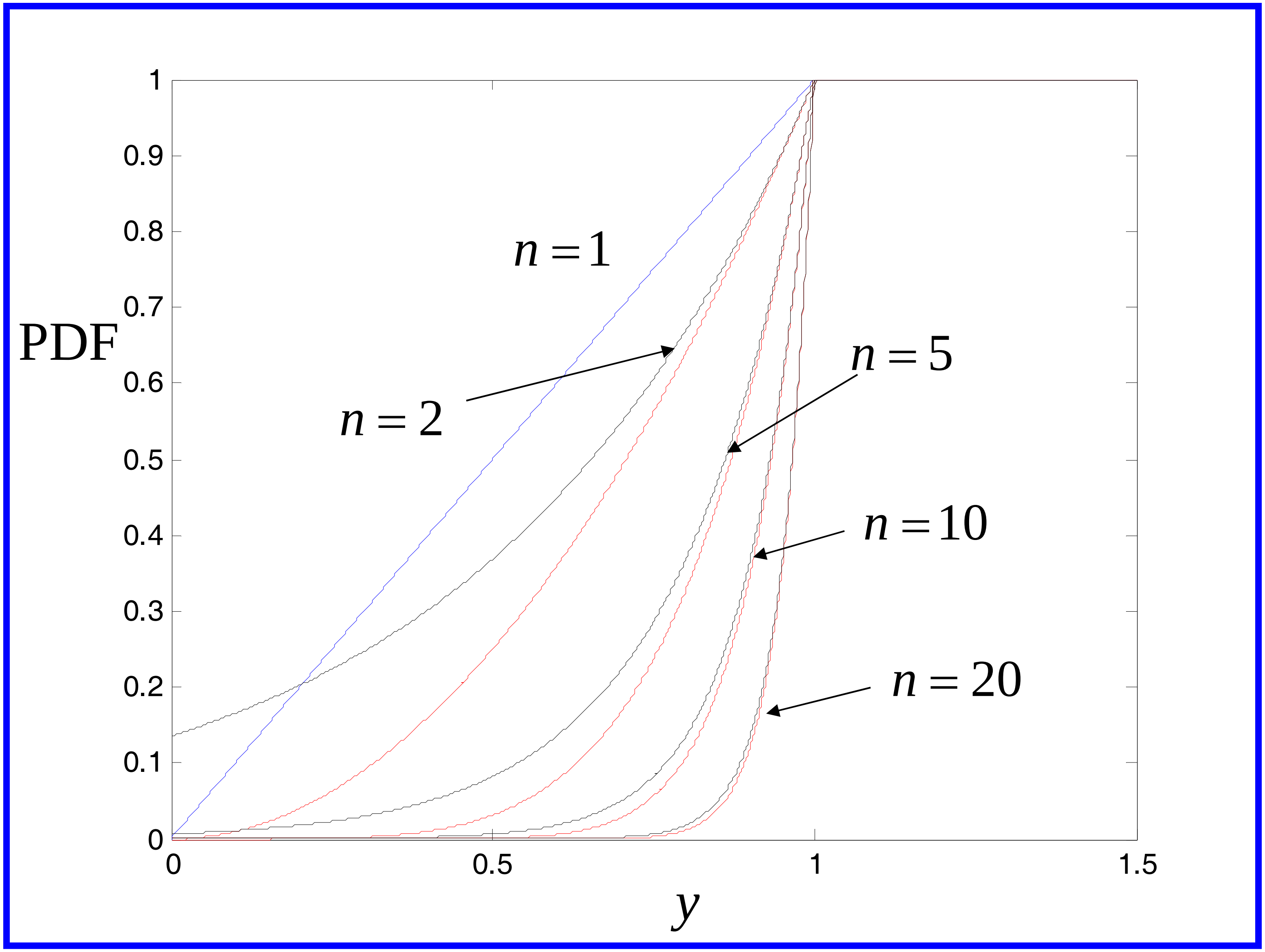
$$= 0; \ \text{for } y > 1$$

$$P_{Y_n}(y) = \exp\left[-n(1 - y)\right]; \ \text{for } 0 < y \le 1$$

$$= 1; \ \text{for } y > 1$$

Black: Asymptotic

Red: Exact



Example

$$P_X(x) = 1 - \frac{1}{x^k} \quad \text{for } x \geq 1$$

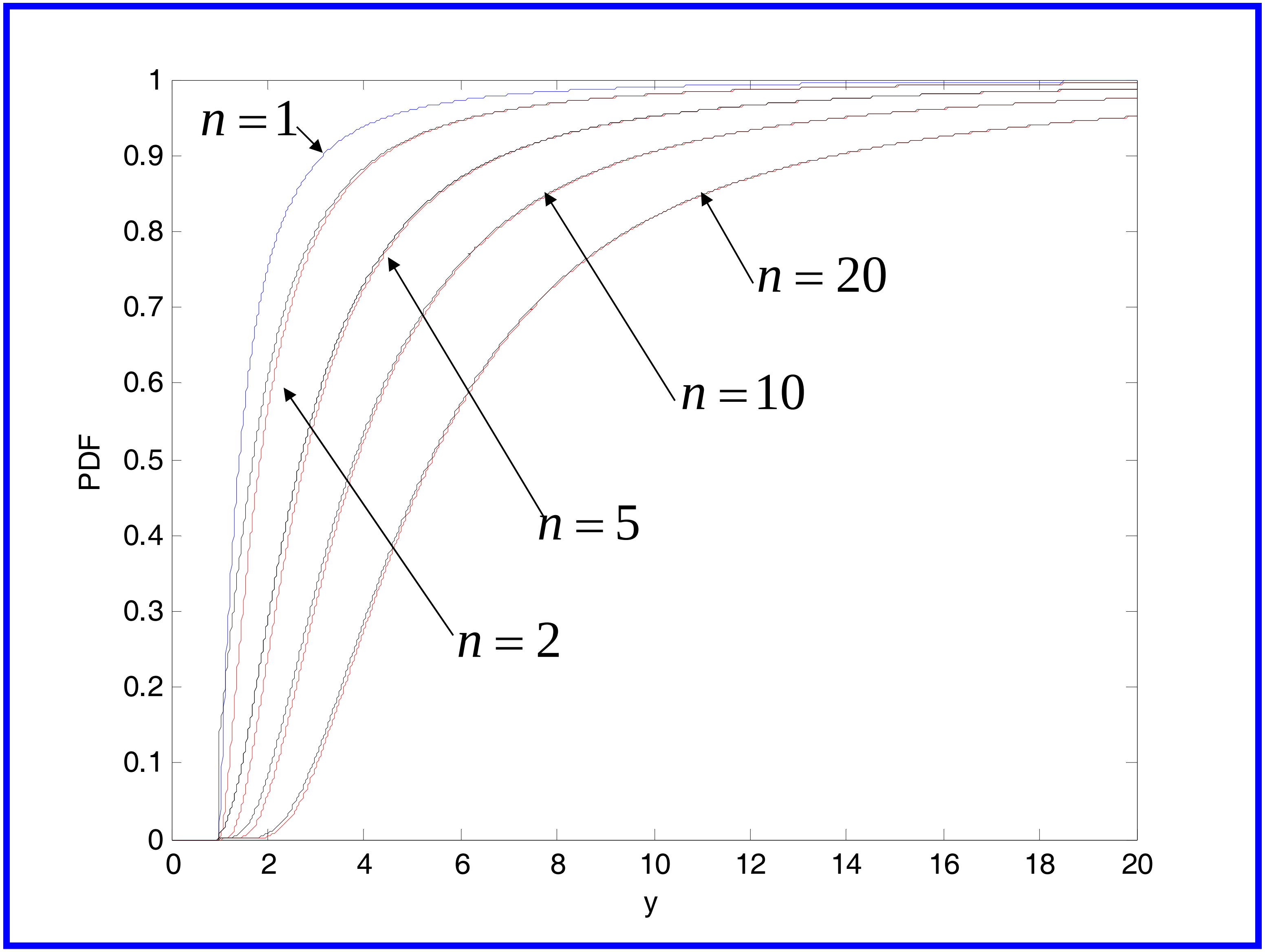
$$= 0 \quad \text{for } x < 1$$

$$\Rightarrow g(y) = n\left[1 - \left(1 - \frac{1}{y^k}\right)\right]; \ \text{for } y \geq 1$$

$$P_{Y_n}(y) = \exp\left[-ny^{-k}\right]; \ \text{for } y \geq 1$$

Black: Asymptotic

Red: Exact



Summary

Parent pdf	Asymptotic exreme value pdf
$p_X(x) = \lambda \exp(-\lambda x); 0 < x < \infty$	$P_{Y_n}(y) = \exp\left[-n \exp(-\lambda y)\right]; 0 < y < \infty$
$p_X(x) = \frac{x}{\sigma^2} \exp\left[-\frac{x^2}{2\sigma^2}\right]; 0 < x < \infty$	$P_{Y_n}(y) = \exp\left\{-n \exp\left[-\frac{y^2}{2\sigma^2}\right]\right\}; 0 < y < \infty$
$p_X(x) = x; 0 < x \le 1$	$P_{Y_n}(y) = \exp\left[-n(1-y)\right];$ for $0 < y \le 1$
$p_X(x) = kx^{-(k+1)}$ for $x \ge 1$	$P_{Y_n}(y) = \exp\left[-ny^{-k}\right];$ for $y \ge 1$

Asymptotic forms

- Double exponential forms
- Single exponential forms

Degeneracy

$$Y_n = \max_{i=1,n} X_i$$

$$P_{Y_n}(y) = \left[P_X(y) \right]^n$$

$$\lim_{n\to\infty} P_{Y_n}(y) \to 0 \text{ for } P_X(y) < 1$$

$$\to 1 \text{ for } P_X(y) = 1$$

That is, the limiting distribution takes only values 0 and 1, *i.e.*, it degnerates.

We avoid degeneracy by looking for constants

a_n and b_n such that

$$\left[P_X\left(a_n x+b_n\right)\right]^n = P\left[\frac{Y_n-b_n}{a_n}\le x\right]\to G\left(x\right) \text{ as } n\to\infty$$

where the limit distribution G is non-degenerate.

It can be shown that G needs to be one of the following three types

$$[\text{Frechet}]\, P_Y\left(y\right)=\exp\left[-\left(\frac{v_n}{y}\right)^k\right];0<y<\infty$$

$$[\text{Weibull}]\, P_Y\left(y\right)=\exp\left[-\left(\frac{\omega-y}{\omega-w_n}\right)^k\right];y\le\omega$$

$$[\text{Gumbel}]\, P_Y\left(y\right)=\exp\left[-\exp\left(-\alpha_n\left\{y-u_n\right\}\right)\right];-\infty<y<\infty$$

Remarks

- If $P_X(x)$ is such that $P_{Y_n}(y)$ is Frechet/Weibull/Gumbel then we say that $P_X(x)$ is said to belong to the basin of attraction of Frechet/Weibull/Gumbel extreme value distributions respectively.
- If $p_X(x)$ has an exponentially decaying right hand tail then $P_{Y_n}(y)$ would be Gumbel
- If $p_X(x)$ has a right hand tail that decays as a polynomial then $P_{Y_n}(y)$ would be Frechet
- If $p_X(x)$ has a finite upper bound then $P_{Y_n}(y)$ would be Weibull

Distribution	Domain of attraction for maxima	Domain of attraction for minima
Normal	Gumbel	Gumbel
Exponential	Gumbel	Weibull
Log-normal	Gumbel	Gumbel
Gamma	Gumbel	Weibull
Gumbel (maxima)	Gumbel	Gumbel
Gumbel (minima)	Gumbel	Gumbel
Rayleigh	Gumbel	Weibull
Uniform	Weibull	Weibull
Weibull (maxima)	Weibull	Gumbel
Weibull (minima)	Gumbel	Weibull
Cauchy	Frechet	Frechet
Pareto	Frechet	Frechet
Frechet (maxima)	Frechet	Gumbel
Frechet (minima)	Gumbel	Frechet

E Castillo, 1988, Extreme value theory in engineering, Academic Press, Boston

Generalized extreme value distribution

$$P_{X_m}(x) = \exp\left\{ -\left[1 + \xi \left(\frac{x-\mu}{\sigma} \right)^{\frac{1}{\xi}} \right] \right\}; 1 + \xi \left(\frac{x-\mu}{\sigma} \right) > 0$$

$\xi \in R$ Shape paramter

$\mu \in R$ Location paramter

$\sigma > 0$ Scale parameter

$\xi \to 0$ Gumbel

$\xi > 0$ Frechet

$\xi < 0$ Weibull

Occurrence of double exponential PDF models for extremes in Poisson counting models

Consider a random phenomenon E, which occurs as a Poisson process with constant arrival rate ν.

Let $t_1, t_2, \cdots, t_k$ be the times at which the event E occurs.
Let Z_i be the random variable representing the intensity measure of E occuring at the time instant t_i.

Let $Z_i, i = 1, 2, \cdots$ be an iid sequence with common PDF $P_Z(z)$.
Let $Z_{\max}(t)$ be the maximum value of Z_i observed over the time interval $(0, t)$.

Consider

$$P\left[Z_{\max} \le z \mid N(t)=k\right] = \left[P_Z(z)\right]^k$$

$$\Rightarrow P_{Z_{\max}}(z) = \sum_{k=0}^{\infty} P\left[Z_{\max} \le z \mid N(t)=k\right] P\left[N(t)=k\right]$$

$$= \sum_{k=0}^{\infty} \left[P_Z(z)\right]^k \frac{(\nu t)^k}{k!} \exp(-\nu t)$$

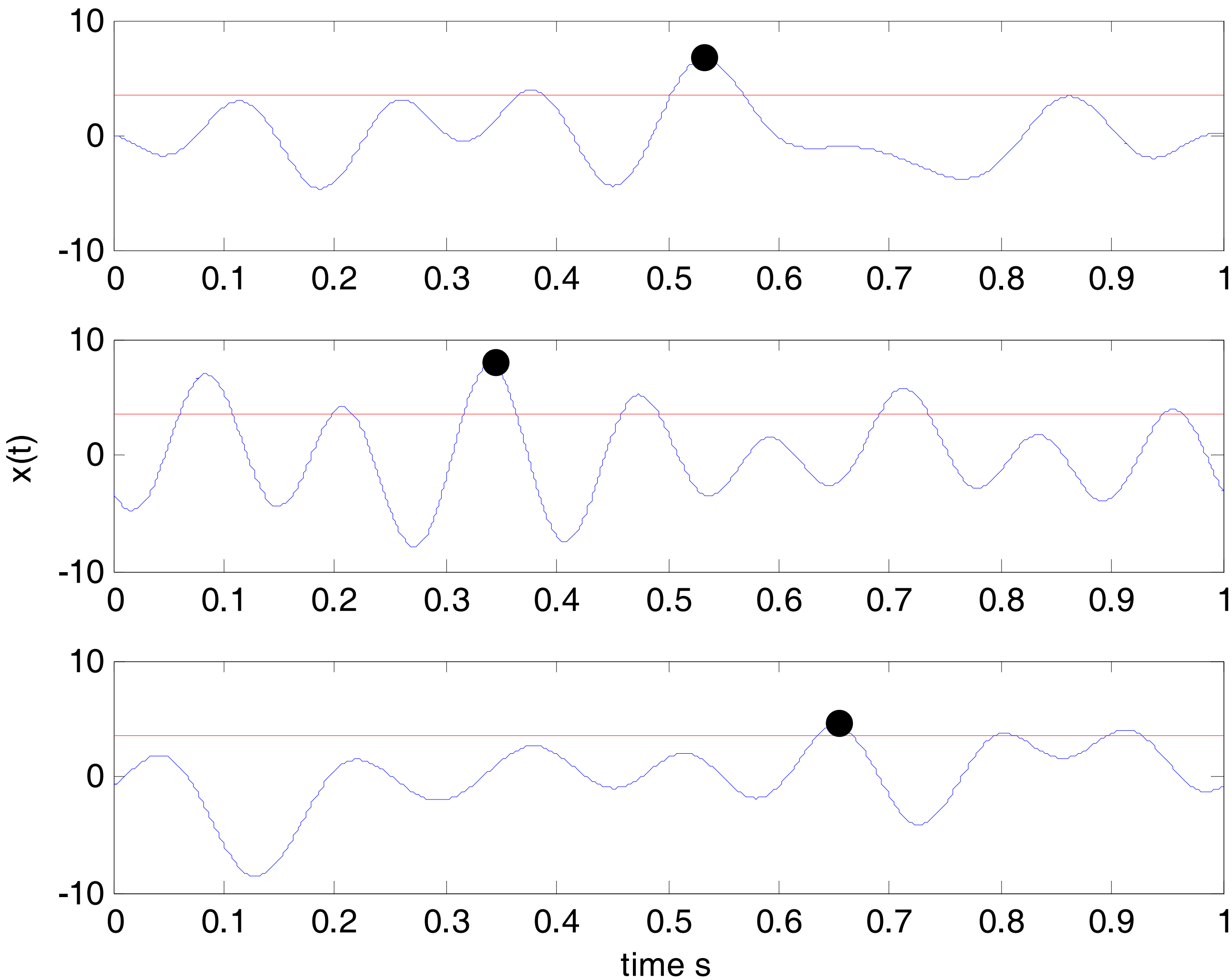
$$= \exp\left[-\nu t\left(1 - P_Z(z)\right)\right]$$

If $P_Z(z) = 1 - \exp\left[-\alpha(z - z_0)\right] \Rightarrow$

$$P_{Z_{\max}}(z) = \exp\left[-\nu t\left\{\exp\left[-\alpha(z - z_0)\right]\right\}\right]$$

This is the PDF of a Gumbel RV. The above model has been used to model the maximum earthquake ground acceleration in the time interval 0 to t.

The maximum value of $X(t)$ in interval 0 to T



Let $X(t)$ be a zero mean stationary Gaussian random process.

Define $X_m = \max_{0<t<T} X(t)$.

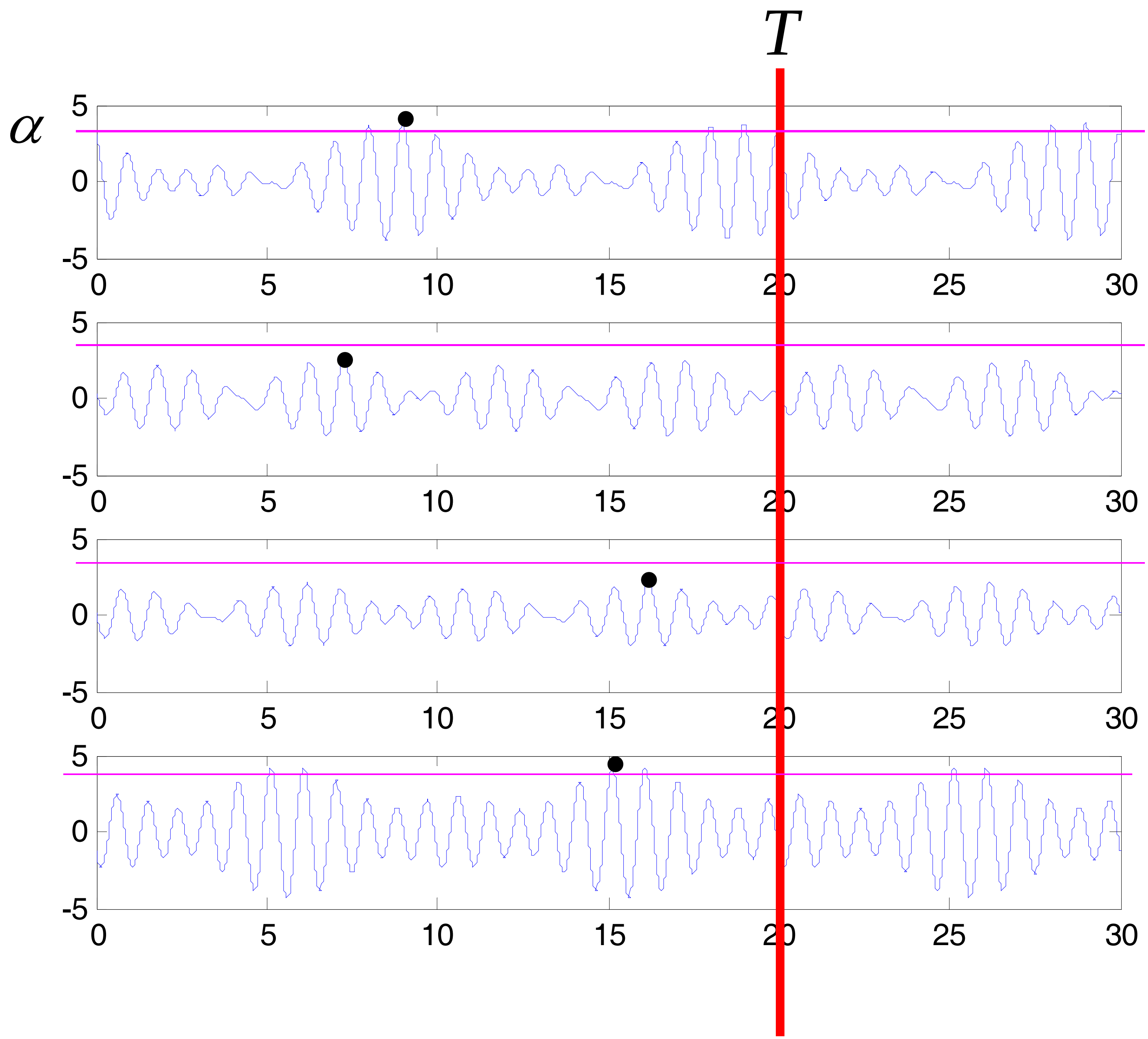
Given the complete description of $X(t)$ can we determine $P_{X_m}(x)$?

$$
\begin{aligned}
P_{X_m}(\alpha) &= P\left[X_m \le \alpha\right] \\
&= P\left[T_f(\alpha) > T\right] \\
&= 1 - P\left[T_f(\alpha) \le T\right]
\end{aligned}
$$

$$P_{T_f}(t) = 1 - \exp\left[-\lambda t\right]$$

$$p_{T_f}(t) = \lambda \exp\left[-\lambda t\right] \quad 0 < t < \infty$$

$$\lambda = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$



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$$P_{T_f}(t) = 1 - \exp[-\lambda t]$$

$$p_{T_f}(t) = \lambda \exp[-\lambda t] \quad 0 < t < \infty$$

$$\lambda = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$P_{X_m}(\alpha) = 1 - P\left[T_f(\alpha) \le T\right]$$

$$P_{X_m}(\alpha) = \exp\left[-\frac{\sigma_{\dot{x}} T}{2\pi\sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right]$$

$$p_{X_m}(\alpha) = \frac{\sigma_{\dot{x}} T \alpha}{2\pi\sigma_x^3} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\} \exp\left[-\frac{\sigma_{\dot{x}} T}{2\pi\sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right]$$

$$-\infty < \alpha < \infty$$

$$P_{X_m}(\alpha) = \exp\left[-\frac{\sigma_{\dot{x}} T}{2\pi\sigma_x} \exp\left\{ -\frac{1}{2}\frac{\alpha^2}{\sigma_x^2} \right\} \right]$$

Denote $N_X^+(0) = \dfrac{\sigma_{\dot{x}}}{2\pi\sigma_x}; \varsigma = \dfrac{\alpha}{\sigma_x}$

$$P_{X_m}(\alpha) = \exp\left[-N_X^+(0) T \exp\left\{ -\frac{\varsigma^2}{2} \right\} \right]$$

Let $\exp(-\nu) = N_X^+(0) T \exp\left\{ -\dfrac{\varsigma^2}{2} \right\}$

$$\Rightarrow -\nu = \log\left(N_X^+(0) T \right) - \frac{\varsigma^2}{2}$$

$$\Rightarrow \varsigma = \left[2\log\left(N_X^+(0) T \right) + 2\nu \right]^{\frac{1}{2}} \simeq \left[2\log\left(N_X^+(0) T \right) \right]^{\frac{1}{2}} + \frac{\nu}{\left[2\log\left(N_X^+(0) T \right) \right]^{\frac{1}{2}}}$$

provided $\nu < \left[2\log\left(N_X^+(0) T \right) \right]^{\frac{1}{2}}$. This is likley to be true for large T.

$$\varsigma \simeq \left[2\log\left(N_X^+(0)T\right)\right]^{\frac{1}{2}} + \frac{\nu}{\left[2\log\left(N_X^+(0)T\right)\right]^{\frac{1}{2}}}$$

$$\Rightarrow$$

$$\nu = C_1\left(\varsigma - C_1\right) \text{ with } C_1 = \left[2\log\left(N_X^+(0)T\right)\right]^{\frac{1}{2}}$$

$$\Rightarrow$$

$$P_{X_m}\left(\varsigma\right) = \exp\left[-\exp\left\{-C_1\left(\varsigma - C_1\right)\right\}\right]; -\infty < \varsigma < \infty$$

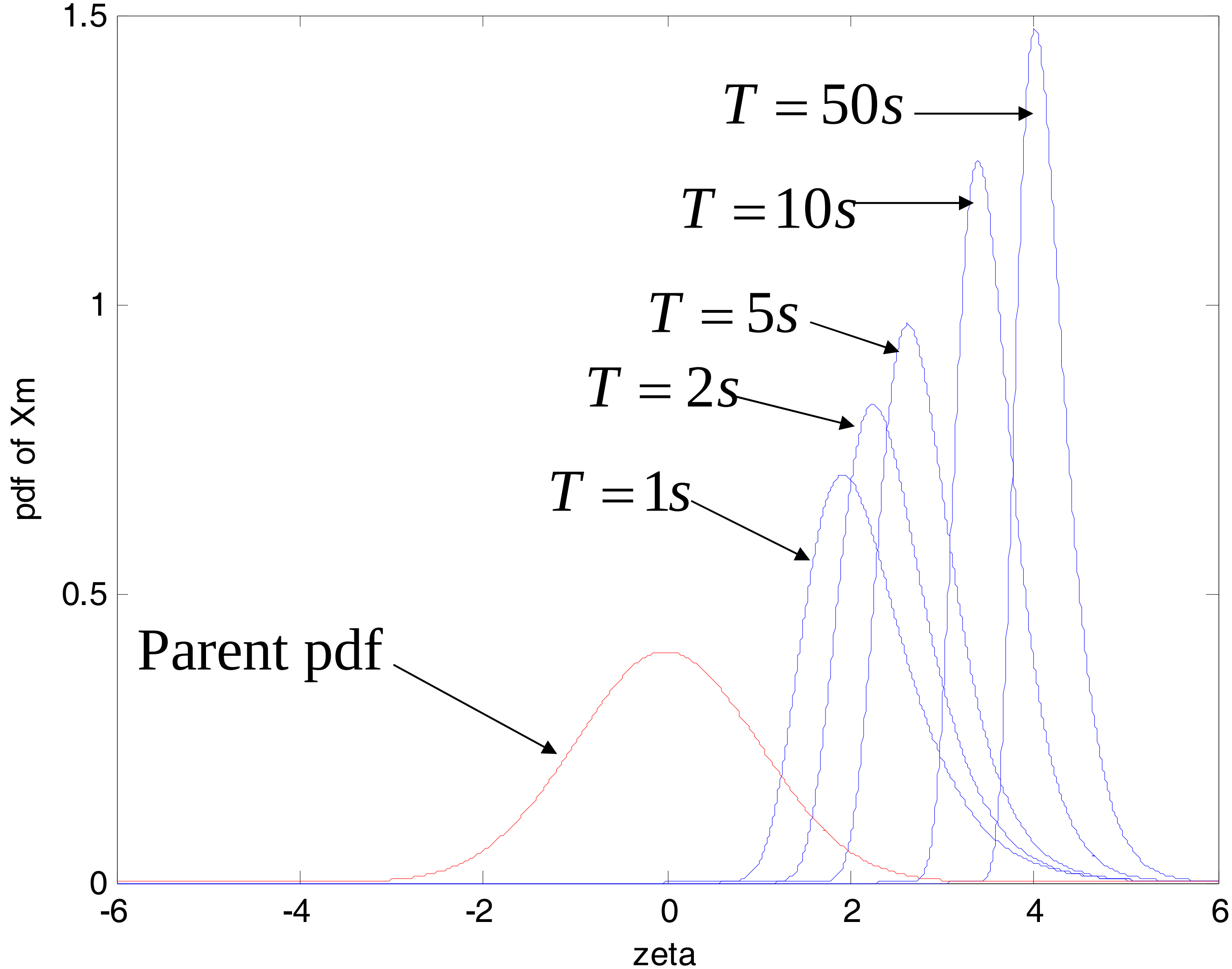
This is a Gumbel PDF.

Moments

$$\left\langle X_m \right\rangle = C\sigma_X$$

$$\text{Var}\left[X_m\right] = \sigma_{X_m}^2 = \frac{\pi^2}{6}\frac{\sigma_x^2}{C_1^2}$$

$$C = C_1 + \frac{0.5772}{C_1}$$



1.5
1
0.5
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-6
-4
-2
0
2
4
6
pdf of Xm
zeta
$T = 50s$
$T = 10s$
$T = 5s$
$T = 2s$
$T = 1s$
Parent pdf

Alternative derivation

Recall: pdf of peaks

$$p_p(\alpha) = \frac{\left(1-\varepsilon^2\right)^{\frac{1}{2}}}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{\alpha^2}{2\sigma_1^2\sqrt{2\left(1-\varepsilon^2\right)}}\right] + \frac{\varepsilon\alpha}{2\sigma_1^2}\left\{1 + erf\left(\frac{\varepsilon\alpha}{\sigma_1\sqrt{2\left(1-\varepsilon^2\right)}}\right)\right\}$$

$$1 - P_p(\alpha) = \int_\alpha^\infty p_p(s)\,ds$$

Let NT be the total number of peaks in the interval 0 to T.

$$X_m = \max_{0<t<T} X(t) = \max(\text{all peaks in 0 to T})$$

$$P_{X_m}(\alpha) = \left[P_p(\alpha)\right]^{NT}$$

For large NT asymptotic results can be used.

Exercise

What happens if $X(t)$ is non-stationary?

Recall

$$\left\langle n^{+}(\alpha,t)\right\rangle = \frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_{x}}\left(1-r^{2}\right)[\exp\left(-\frac{\alpha^{2}}{2\sigma_{x}^{2}\left(1-r^{2}\right)}\right)+$$

$$\frac{\alpha r}{\sigma_{x}}\exp\left(-\frac{\alpha^{2}}{2\sigma_{x}^{2}}\right)\left\{1-erf\left(\frac{\alpha r}{\sigma_{x}\sqrt{2\left(1-r^{2}\right)}}\right)\right\}]$$

$$P_{T_f}(t)=1-\exp\left[-\int_{0}^{t}\left\langle n_{X}^{+}(\alpha,\tau)d\tau\right\rangle\right]$$

$$P_{X_m}(\alpha)=\exp\left[-\int_{0}^{T}\left\langle n_{X}^{+}(\alpha,\tau)d\tau\right\rangle\right]$$

Applications

- Response spectrum based approaches in earthquake engineering
- Gust factor approach in wind engineering
- Accumulation of fatigue damage under random dynamic loads

Markov vector approach in random vibrations

- An alternative approach to random vibraiton analysis
- Valid when response vector satisfy Markovian property
- Source of exact solutions for nonlinear random vibration problems for a limited class of problems
- Strategies for solving wider class of problems.

Markov Property

Let $X(t)$ be a scalar random process with continuous state and continuous parameter (time t).

Let $t_1 < t_2 < \cdots < t_n$ be the n time instants.

This defines n random variables

$X(t_1), X(t_2), \cdots, X(t_n)$.

$X(t)$ is said to possess Markov property if

$$P\left[X(t_n) \le t_n \mid X(t_{n-1}) \le t_{n-1}, X(t_{n-2}) \le t_{n-2}, \cdots, X(t_1) \le t_1\right]$$

$$= P\left[X(t_n) \le t_n \mid X(t_{n-1}) \le t_{n-1}\right]$$

for any n and any choice of $t_1 < t_2 < \cdots < t_n$.

Dependence of future on past is only through the present.

$\Rightarrow$

$$P_X\left(x_n,t_n \mid x_{n-1},t_{n-1};x_{n-2},t_{n-2};\cdots,x_1,t_1\right)=P_X\left(x_n,t_n \mid x_{n-1},t_{n-1}\right)$$

$$p_X\left(x_n,t_n \mid x_{n-1},t_{n-1};x_{n-2},t_{n-2};\cdots,x_1,t_1\right)=p_X\left(x_n,t_n \mid x_{n-1},t_{n-1}\right)$$

Description of a Markov process

- $p\left(x_1,t_1\right)$
- $p\left(x_2,t_2;x_1,t_1\right)=p\left(x_2,t_2 \mid x_1,t_1\right)p\left(x_1,t_1\right)$
- $p\left(x_3,t_3;x_2,t_2;x_1,t_1\right)=p\left(x_3,t_3 \mid x_2,t_2;x_1,t_1\right)p\left(x_2,t_2 \mid x_1,t_1\right)p\left(x_1,t_1\right)$

$=p\left(x_3,t_3 \mid x_2,t_2\right)p\left(x_2,t_2 \mid x_1,t_1\right)p\left(x_1,t_1\right)$

$\vdots$

- $p\left(x_n,t_n;x_{n-1},t_{n-1};\cdots;x_1,t_1\right)=\prod_{\nu=2}^{n}p\left(x_\nu,t_\nu \mid x_{\nu-1},t_{\nu-1}\right)p\left(x_1,t_1\right)$

Transistion probability density function

tpdf : $p\left(x_{\nu}, t_{\nu} \mid x_{\nu-1}, t_{\nu-1}\right)$

• $p\left(x_1, t_1\right)$ and $p\left(x_{\nu}, t_{\nu} \mid x_{\nu-1}, t_{\nu-1}\right)$ $\forall \nu = 2, 3, \cdots$ completely specify a Markov process