Stochastic Structural Dynamics

Lecture-19

Failure of randomly vibrating systems-3

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Recall

Level crossing

$$N(0,\alpha,T) = N(T) = \int_0^T \left|\dot{X}(t)\right| \delta\left[X(t)-\alpha\right] dt = \int_0^T n(\alpha,t)\, dt$$

$$n(\alpha,t) = \left|\dot{X}(t)\right| \delta\left[X(t)-\alpha\right]$$

Number of peaks

$$M(\alpha,0,T) = \int_0^T \left|\ddot{X}(t)\right| \delta\left[\dot{X}(t)-0\right] U\left[X(t)-\alpha\right] dt = \int_0^T m(\alpha,t)\, dt$$

$$m(\alpha,t) = \left|\ddot{X}(t)\right| \delta\left[\dot{X}(t)-0\right] U\left[X(t)-\alpha\right]$$

pdf of peaks

$$p_p(\alpha) = \frac{\left(1-\varepsilon^2\right)^{\frac{1}{2}}}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{\alpha^2}{2\sigma_1^2\sqrt{2\left(1-\varepsilon^2\right)}}\right] + \frac{\varepsilon\alpha}{2\sigma_1^2}\left\{1+erf\left(\frac{\varepsilon\alpha}{\sigma_1\sqrt{2\left(1-\varepsilon^2\right)}}\right)\right\}$$

Fractional occupation time

$$\left\langle y(\alpha,t)\right\rangle = \frac{1}{2T}\int_0^T \left[1-erf\left(\frac{\alpha}{\sigma_x(t)}\right)\right] dt$$

Envelope and phase processes

Recall

$$\ddot{x}+\omega^2 x=0$$

$$x(0)=x_0;\dot{x}(0)=\dot{x}_0$$

$$x(t)=R\cos(\omega t-\theta)$$

$$R=\sqrt{x_0^2+\left(\frac{\dot{x}_0}{\omega}\right)^2};\theta=\tan^{-1}\left(\frac{\dot{x}_0}{\omega x_0}\right)$$

- $R \geq |x(t)| \forall t$

$$\ddot{x}+2\eta\omega\dot{x}+\omega^2 x=0;x(0)=x_0;\dot{x}(0)=\dot{x}_0$$

$$x(t)=\exp(-\eta\omega t)\left(x_0\cos\omega_d t+\frac{\dot{x}_0+\eta\omega x_0}{\omega_d}\sin\omega_d t\right)$$

$$x_0=R\cos\theta;\frac{\dot{x}_0+\eta\omega x_0}{\omega_d}=R\sin\theta$$

$$x(t)=\exp(-\eta\omega t)R\cos(\omega_d t-\theta)$$

$$\ddot{x}+2\eta\omega\dot{x}+\omega^2 x=\frac{P}{m}\cos\lambda t$$

$$x(0)=x_0;\dot{x}(0)=\dot{x}_0$$

$$\lim_{t\to\infty}x(t)=X_{st}(DMF)\cos(\omega_d t-\theta)$$

$$X_{st}=\frac{P}{k};DMF=\frac{1}{\sqrt{(1-r^2)^2+(2\eta r)^2}}$$

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = f(t)$$

$$x(0) = 0; \dot{x}(0) = 0; \eta < 1$$

$$x(t) = \int_0^t \frac{1}{\omega_d} \exp\left[-\eta\omega(t-\tau)\right] \sin\omega_d(t-\tau)\, f(\tau) d\tau$$

$$= \int_0^t \frac{1}{\omega_d} \exp\left[-\eta\omega(t-\tau)\right] \left\{\sin\omega_d t \cos\omega_d \tau - \cos\omega_d t \sin\omega_d \tau\right\} f(\tau) d\tau$$

$$= A(t) \sin\omega_d t + B(t) \cos\omega_d t$$

$$A(t) = \int_0^t \frac{1}{\omega_d} \exp\left[-\eta\omega(t-\tau)\right] \cos\omega_d \tau\, f(\tau) d\tau$$

$$B(t) = -\int_0^t \frac{1}{\omega_d} \exp\left[-\eta\omega(t-\tau)\right] \sin\omega_d \tau\, f(\tau) d\tau$$

$$\Rightarrow x(t) = R(t)\cos\left[\omega_d t - \theta(t)\right]$$

$$A(t) = R(t)\cos\theta(t)$$

$$B(t) = R(t)\sin\theta(t)$$

$$R(t) = \sqrt{A^2(t) + B^2(t)}$$

$$\theta(t) = \tan^{-1}\left(\frac{B(t)}{A(t)}\right)$$

Energy interpretation

$$
\begin{aligned}
R^2(t) &= x^2(t) + \frac{\dot{x}^2(t)}{\omega_d^2} \\
&\approx x^2(t) + \frac{\dot{x}^2(t)}{\omega_n^2} \\
&= x^2(t) + \frac{m\dot{x}^2(t)}{k} \\
&= \frac{2}{k}\left[\frac{kx^2}{2} + \frac{m\dot{x}^2}{2}\right] \\
&\propto \mathrm{KE+PE}
\end{aligned}
$$

$R(t) = x(t)$ whenever $\dot{x}(t) = 0$

$\Rightarrow$

$R(t)$ passes through extrema of $x(t)$.

If $x(t)$ is a sample of a narrow band process,

$R(t)$ passes through all the peaks.

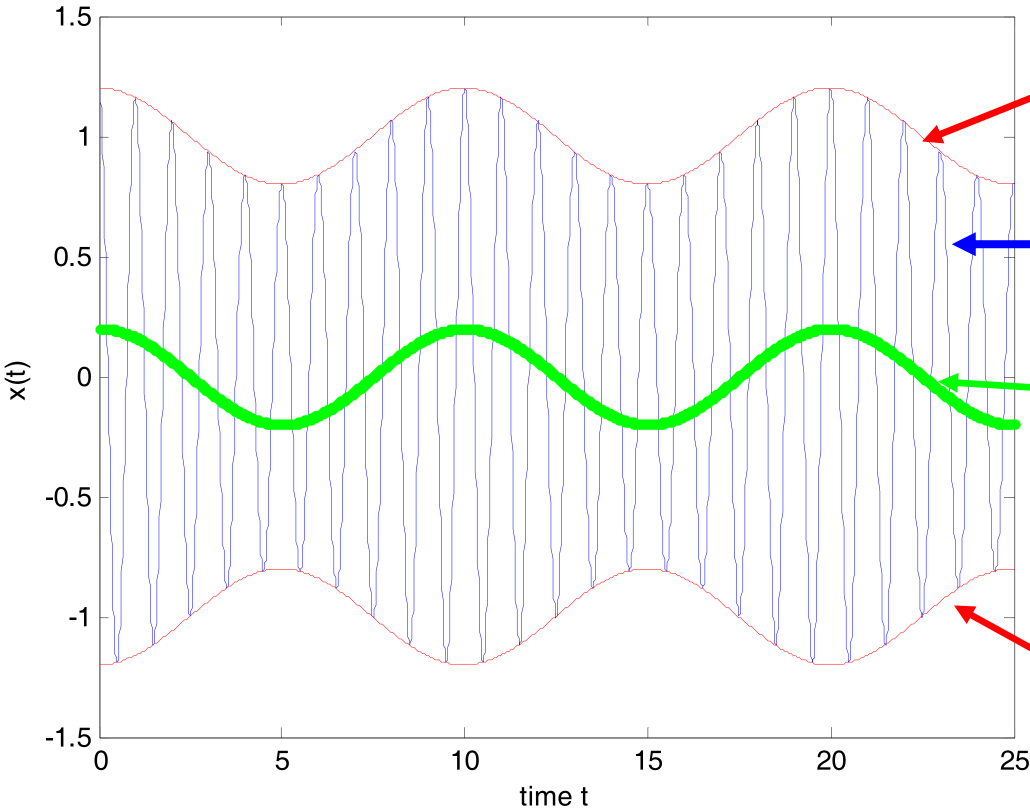
Therefore one can expect similarities in the properties of envelope and peaks of $x(t)$.

$$x(t) = A(1+\varepsilon \cos \omega_m t)\cos \omega t$$

$$\omega >> \omega_m$$

$$E(t) = A\varepsilon \cos \omega_m t$$

$$E_0(t) = A(1+\varepsilon \cos \omega_m t)$$



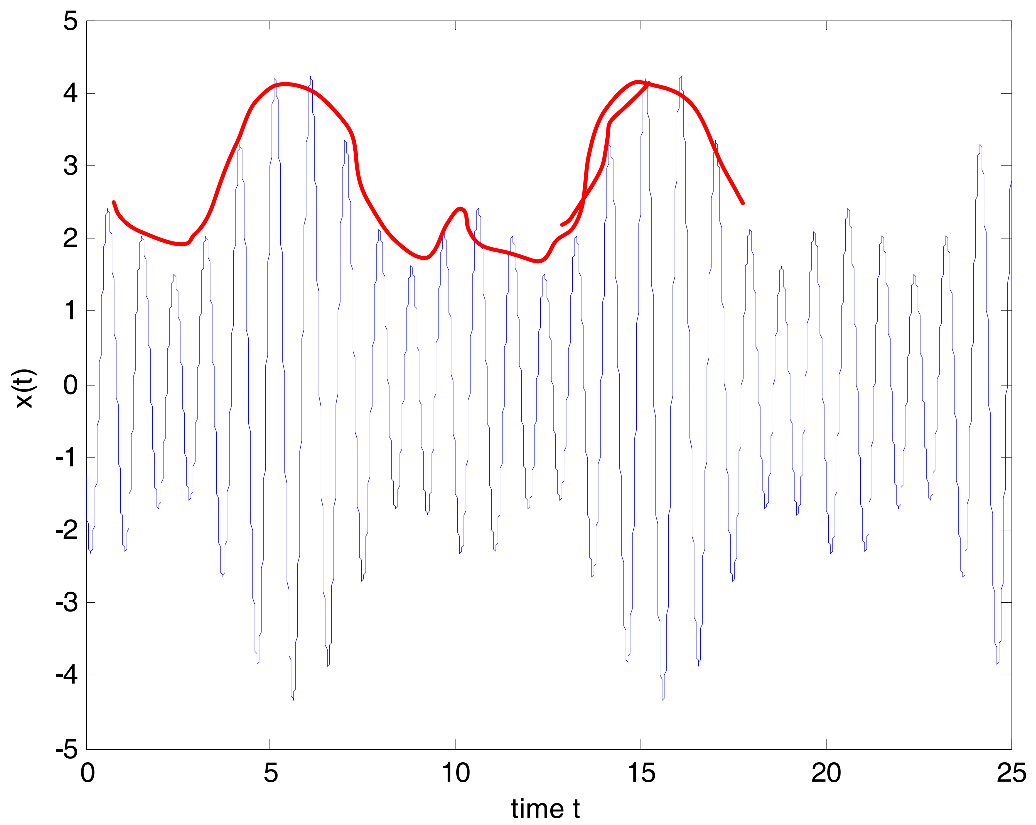
$$E_0(t) = A(1+\varepsilon \cos \omega_m t)$$

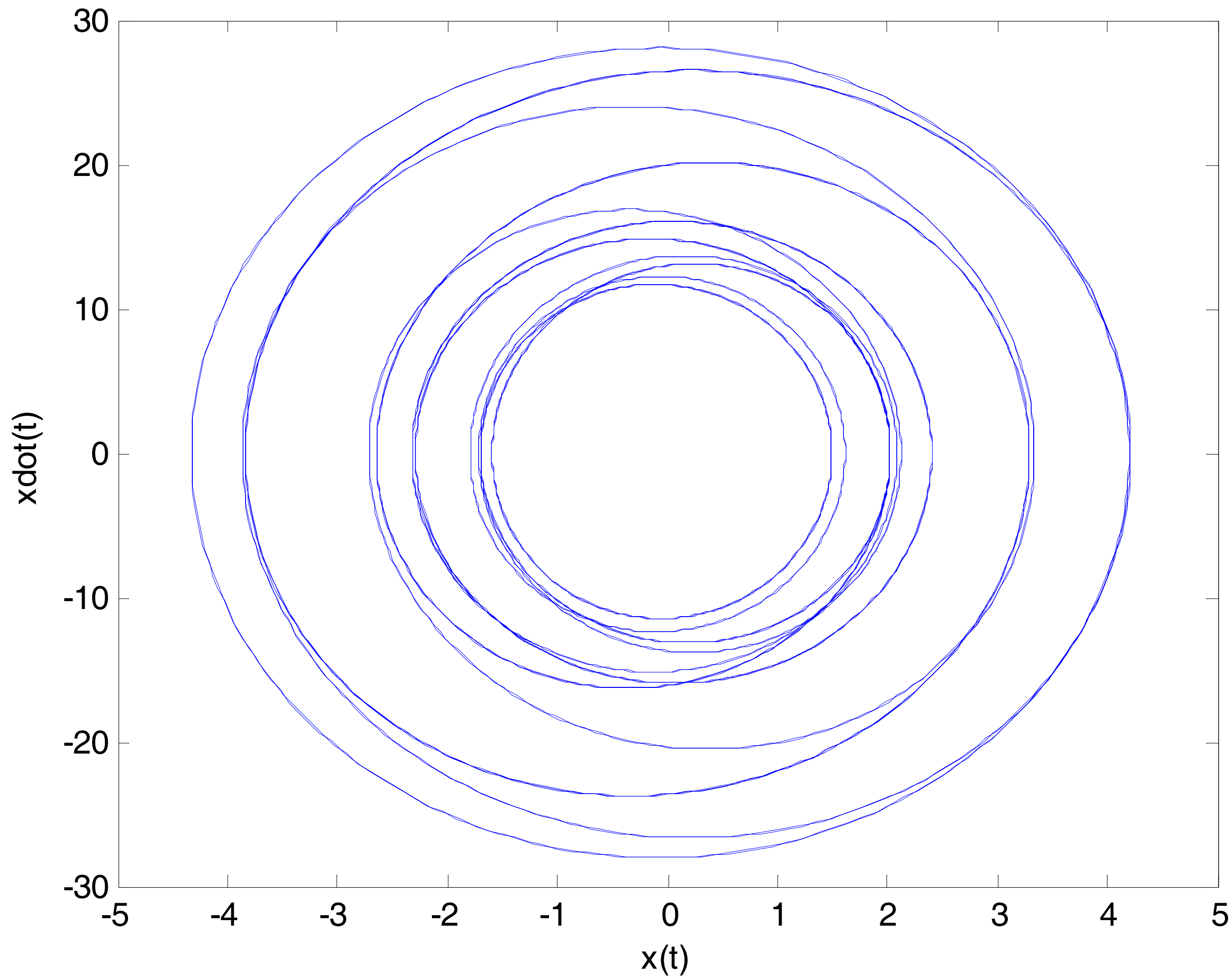
$$x(t) = A(1+\varepsilon \cos \omega_m t)\cos \omega t$$

$$E(t) = A\varepsilon \cos \omega_m t$$

$$E_0(t) = -A(1+\varepsilon \cos \omega_m t)$$

How do we generalize the notion of envelope and phase to describe random processes?





30
20
10
0
-10
-20
-30
xdot(t)
-5
-4
-3
-2
-1
0
1
2
3
4
5
x(t)

Recall

Fourier representation of a Gaussian random process

Let $X(t)$ be a zero mean, stationary, Gaussian random process defined as

$$X(t) = \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t; \;\; \omega_n = n\omega_0$$

Assumptions

Here $a_n \sim N(0,\sigma_n), b_n \sim N(0,\sigma_n)$,

$\langle a_n a_k \rangle = 0 \forall n \neq k, \langle b_n b_k \rangle = 0 \forall n \neq k,$

$\langle a_n b_k \rangle = 0 \forall n, k = 1, 2, \cdots, \infty$

$$\Rightarrow \langle X(t) \rangle = \sum_{n=1}^{\infty} \{ \langle a_n \rangle \cos\omega_n t + \langle b_n \rangle \sin\omega_n t \} = 0$$

$$\left\langle X(t)X(t+\tau)\right\rangle = \left\langle \sum_{n=1}^{\infty}\left\{a_n \cos\omega_n t + b_n \sin\omega_n t\right\}\sum_{n=1}^{\infty}\left\{a_n \cos\omega_n (t+\tau) + b_n \sin\omega_n (t+\tau)\right\}\right\rangle$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\langle \left(a_n \cos\omega_n t + b_n \sin\omega_n t\right)\left(a_n \cos\omega_n (t+\tau) + b_n \sin\omega_n (t+\tau)\right)\right\rangle$$

$$\Rightarrow R_{XX}(\tau) = \sum_{n=1}^{\infty}\sigma_n^2 \cos\omega_n \tau \quad //$$

$X(t)$ is a WSS random process.

$X(t)$ is Gaussian.

$\Rightarrow X(t)$ is a SSS process.

Fourier representation of a Gaussian random process (continued)

Consider the psd function

$$S_{XX}(\omega)=\sum_{n=1}^{\infty}S(\omega_n)\Delta\omega_n\delta(\omega-\omega_n)$$

$$\Rightarrow \tilde{R}_{XX}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}S(\omega_n)\Delta\omega_n\delta(\omega-\omega_n)\cos(\omega\tau)$$

$$\Rightarrow \tilde{R}_{XX}(\tau)=\frac{1}{2\pi}\sum_{n=1}^{\infty}S(\omega_n)\Delta\omega_n\cos(\omega_n\tau)$$

Compare this with

$$R_{XX}(\tau)=\sum_{n=1}^{\infty}\sigma_n^2\cos\omega_n\tau$$

By choosing $\sigma_n^2=\dfrac{S(\omega_n)\Delta\omega_n}{2\pi}$, we see that the two ACF-s coincide.

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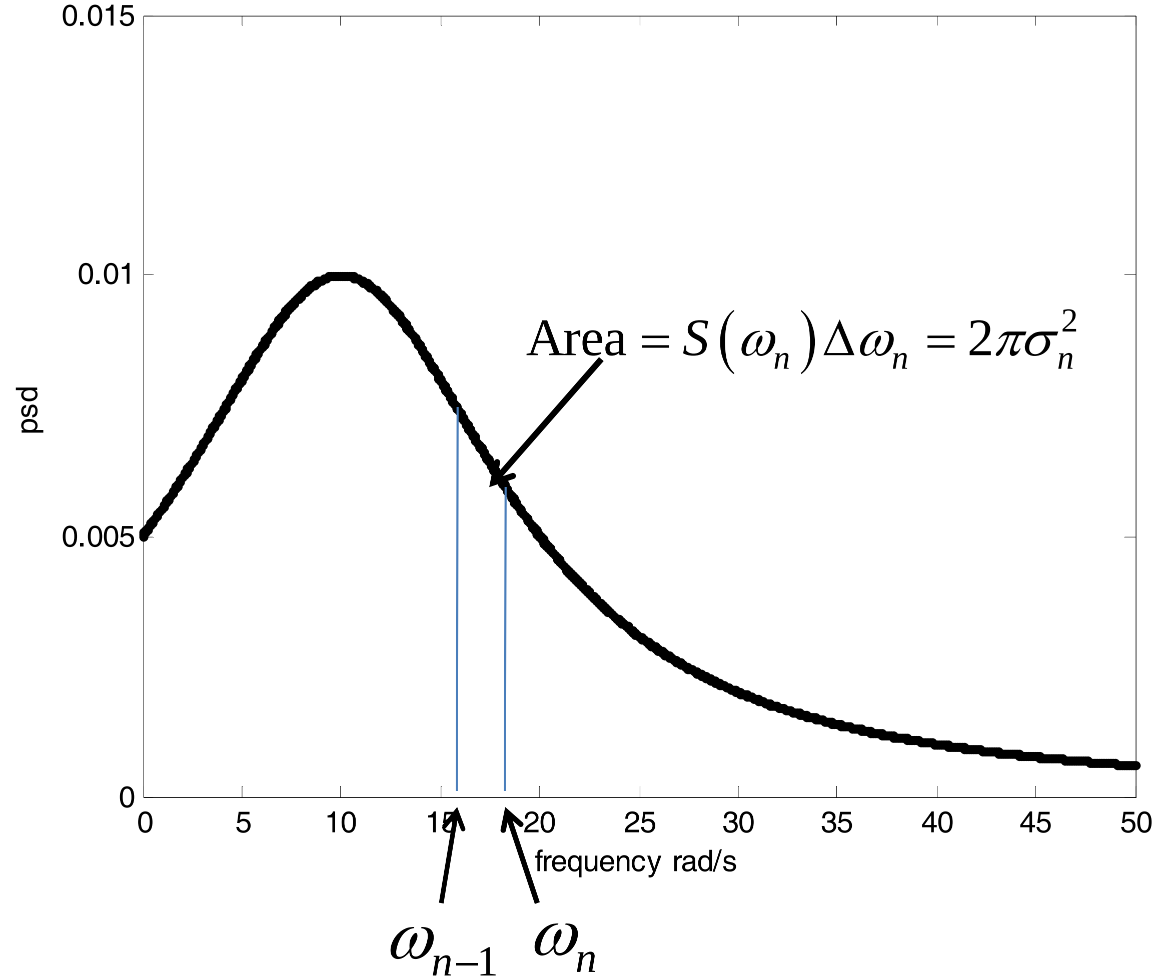
By discretizing the psd function as shown we can simulate samples of $X(t)$ using the Fourier representation

$$X\left(t\right)=\sum_{n=1}^{\infty}\left\{a_n \cos \omega_n t + b_n \sin \omega_n t\right\};\ \omega_n = n\omega_0$$



Alternative representation

Let $X(t)$ be a zero mean, stationary, random process defined as

$$X(t)=\sum_{n=1}^{\infty} A_n \cos(\omega_n t-\theta_n)$$

Here $\{A_n\}_{n=1}^{\infty}$ are deterministic constants and $\{\theta_n\}_{n=1}^{\infty}$ form an iid sequence of random variables with a common PDF that is uniformly distributed in 0 to 2π.

$$\langle X(t)\rangle=\left\langle \sum_{n=1}^{\infty} A_n \cos(\omega_n t-\theta_n)\right\rangle$$

$$=\sum_{n=1}^{\infty} A_n \langle \cos\omega_n t\cos\theta_n-\sin\omega_n t\sin\theta_n\rangle$$

$$=\sum_{n=1}^{\infty} A_n \cos\omega_n t\int_0^{2\pi}\frac{1}{2\pi}\cos\theta_n d\theta_n-\sin\omega_n t\int_0^{2\pi}\frac{1}{2\pi}\sin\theta_n d\theta_n$$

$$=0$$

$$X(t)=\sum_{n=1}^{\infty}A_n\left[\cos\omega_n t\cos\theta_n-\sin\omega_n t\sin\theta_n\right]$$

$$\left\langle X(t)X(t+\tau)\right\rangle=<\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_nA_m\left[\cos\omega_n t\cos\theta_n-\sin\omega_n t\sin\theta_n\right]$$

$$\left[\cos\omega_n(t+\tau)\cos\theta_n-\sin\omega_n(t+\tau)\sin\theta_n\right]>$$

$$=\left\langle\sum_{n=1}^{\infty}A_n^2\left[\cos^2\theta_n\cos\omega_n t\cos\omega_n(t+\tau)+\sin^2\theta_n\sin\omega_n t\sin\omega_n(t+\tau)\right]\right\rangle$$

$$=\sum_{n=1}^{\infty}A_n^2\cos\omega_n\tau$$

$X(t)$ is a WSS random process.

$X(t)$ is Gaussian (apply central limit theorem).

$\Rightarrow X(t)$ is a SSS process.

Rice's definition of envelope and phase processes

$$X(t)=\sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t;\ \ \omega_n = n\omega_0$$

$$a_n \sim N(0,\sigma_n), b_n \sim N(0,\sigma_n),$$

$$\langle a_n a_k \rangle = 0 \forall n \neq k, \langle b_n b_k \rangle = 0 \forall n \neq k,$$

$$\langle a_n b_k \rangle = 0 \forall n,k = 1,2,\cdots,\infty$$

ω_r = Central frequency

$$X(t)=\sum_{n=1}^{\infty} a_n \cos\left[(\omega_n-\omega_r)t+\omega_r t\right]+b_n \sin\left[(\omega_n-\omega_r)t+\omega_r t\right]$$

$$=\sum_{n=1}^{\infty} a_n \left[\cos(\omega_n-\omega_r)t\cos\omega_r t-\sin(\omega_n-\omega_r)t\sin\omega_r t\right]$$

$$+\sum_{n=1}^{\infty} b_n \left[\sin(\omega_n-\omega_r)t\cos\omega_r t+\cos(\omega_n-\omega_r)t\sin\omega_r t\right]$$

$$X(t)=\sum_{n=1}^{\infty}a_n\left[\cos(\omega_n-\omega_r)t\cos\omega_r t-\sin(\omega_n-\omega_r)t\sin\omega_r t\right]$$

$$+\sum_{n=1}^{\infty}b_n\left[\sin(\omega_n-\omega_r)t\cos\omega_r t+\cos(\omega_n-\omega_r)t\sin\omega_r t\right]$$

$$=\left\{\sum_{n=1}^{\infty}a_n\cos(\omega_n-\omega_r)t+b_n\sin(\omega_n-\omega_r)t\right\}\cos\omega_r t$$

$$-\left\{\sum_{n=1}^{\infty}a_n\sin(\omega_n-\omega_r)t-b_n\cos(\omega_n-\omega_r)t\right\}\sin\omega_r t$$

$$=I_c(t)\cos\omega_r t+I_s(t)\sin\omega_r t$$

$$I_c(t)=\sum_{n=1}^{\infty}a_n\cos(\omega_n-\omega_r)t+b_n\sin(\omega_n-\omega_r)t$$

$$I_s(t)=-\sum_{n=1}^{\infty}a_n\sin(\omega_n-\omega_r)t-b_n\cos(\omega_n-\omega_r)t$$

$$I_c(t)=\sum_{n=1}^{\infty} a_n \cos(\omega_n-\omega_r)t+b_n \sin(\omega_n-\omega_r)t$$

$$\Rightarrow$$

$$\left\langle I_c^2(t)\right\rangle=\sum_{n=1}^{\infty}\sigma_n^2=\left\langle X^2(t)\right\rangle$$

$$I_s(t)=-\sum_{n=1}^{\infty} a_n \sin(\omega_n-\omega_r)t-b_n \cos(\omega_n-\omega_r)t$$

$$\Rightarrow$$

$$\left\langle I_s^2(t)\right\rangle=\sum_{n=1}^{\infty}\sigma_n^2=\left\langle X^2(t)\right\rangle$$

$$X(t) = I_c(t)\cos\omega_r t + I_s(t)\sin\omega_r t$$

$$I_c(t) = a(t)\cos\theta(t); \quad I_s(t) = a(t)\sin\theta(t)$$

$$a^2(t) = I_c^2(t) + I_s^2(t)$$

$$\theta(t) = \tan^{-1}\left[\frac{I_s(t)}{I_c(t)}\right]$$

$$X(t) = a(t)\cos\left[\omega_r t + \theta(t)\right]$$

$a(t) =$ Envelope process associated with $X(t)$

$\theta(t) =$ Phase process associated with $X(t)$

$\omega_r =$ Central frequency associated with $X(t)$

Alternative representation

$$X(t) = \sum_{n=1}^{\infty} A_n \cos(\omega_n t - \theta_n)$$

Here $\{A_n\}_{n=1}^{\infty}$ are deterministic constants and $\{\theta_n\}_{n=1}^{\infty}$ form an iid sequence of random variables with a common PDF that is uniformly distributed in 0 to 2π.

$$X(t) = \sum_{n=1}^{\infty} A_n \cos\left[(\omega_n - \omega_r)t - \theta_n + \omega_r t\right]$$

$$= \sum_{n=1}^{\infty} A_n \left\{\cos\left[(\omega_n - \omega_r)t - \theta_n\right]\cos\omega_r t - \sin\left[(\omega_n - \omega_r)t - \theta_n\right]\sin\omega_r t\right\}$$

$$= E_c(t)\cos\omega_r t - E_s(t)\sin\omega_r t$$

$$E_c(t) = \sum_{n=1}^{\infty} A_n \cos\left[(\omega_n - \omega_r)t - \theta_n\right]$$

$$E_s(t) = \sum_{n=1}^{\infty} A_n \sin\left[(\omega_n - \omega_r)t - \theta_n\right]$$

$$X(t) = E_c(t)\cos\omega_r t - E_s(t)\sin\omega_r t$$

$$E_c(t) = a(t)\cos\theta(t)$$

$$E_s(t) = a(t)\sin\theta(t)$$

$$\Rightarrow$$

$$X(t) = a(t)\cos\left[\omega_r t - \theta(t)\right]$$

$$a(t) = \sqrt{E_c^2(t) + E_s^2(t)}$$

$$\theta(t) = \tan^{-1}\left[\frac{E_s(t)}{E_c(t)}\right]$$

$a(t) =$ Envelope process associated with $X(t)$

$\theta(t) =$ Phase process associated with $X(t)$

$\omega_r =$ Central frequency associated with $X(t)$

Probability distributions of Envelope and phase processes

Let $A(t)$ and $B(t)$ be two random processes.

Define

$$X(t) = A(t)\cos\omega t + B(t)\sin\omega t$$

Let $A(t) = R(t)\cos\Phi(t)$ & $B(t) = R(t)\sin\Phi(t)$

$\Rightarrow$

$$X(t) = R(t)\cos\left[\omega t + \Phi(t)\right]$$

$$R(t) = \sqrt{A^2(t) + B^2(t)}$$

$$\Phi(t) = \tan^{-1}\left(\frac{B(t)}{A(t)}\right)$$

By definition

$R(t)$=amplitude process, envelope, or amplitude modulation of $X(t)$

$\Phi(t)$ = phase process, or phase modulation of $X(t)$

ω = carrier frequency or central frequency

Problem

Given $p_{AB}(a,b;t)$ to find $p_{R\Phi}(r,\phi;t)$.

$$A = R\cos\Phi$$

$$B = R\sin\Phi$$

$$J^{-1} = \begin{vmatrix} \cos\Phi & -R\sin\Phi \\ \sin\Phi & R\cos\Phi \end{vmatrix} = R$$

$$p_{R\Phi}(r,\phi) = r p_{AB}(a,b)\Big|_{\substack{a=r\cos\phi \\ b=r\sin\phi}}$$

Let A and B be jointly Gaussian $\Rightarrow$

$$p_{AB}(a,b) = \frac{1}{2\pi\sigma_a\sigma_b\sqrt{\left(1-r_{ab}^2\right)}} \exp\left[-\frac{1}{2\left(1-r_{ab}^2\right)}\left\{\frac{a^2}{\sigma_a^2}+\frac{b^2}{\sigma_b^2}-\frac{2r_{ab}ab}{\sigma_a\sigma_b}\right\}\right]$$

$$p_{R\Phi}(r,\phi) = r p_{AB}(a,b)\Big|_{\substack{a=r\cos\phi \\ b=r\sin\phi}}$$

$$\Rightarrow$$

$$p_{R\Phi}(r,\phi) = \frac{r}{2\pi\sigma_a\sigma_b\sqrt{\left(1-r_{ab}^2\right)}}$$

$$\exp\left[-\frac{1}{2\left(1-r_{ab}^2\right)}\left\{\frac{r^2\cos^2\phi}{\sigma_a^2}+\frac{r^2\sin^2\phi}{\sigma_b^2}-\frac{2r_{ab}r^2\sin\phi\cos\phi}{\sigma_a\sigma_b}\right\}\right]$$

$$0<r<\infty; 0<\phi<2\pi$$

$$p_R(r) = \int_0^{2\pi} p_{R\Phi}(r,\phi)\,d\phi; 0<r<\infty$$

$$p_\Phi(\phi) = \int_0^{\infty} p_{R\Phi}(r,\phi)\,dr; 0<\phi<2\pi$$

$$p_R(r) = \frac{r}{\sigma_a \sigma_b \sqrt{\left(1 - r_{ab}^2\right)}} \exp\left[-r^2 \left(\frac{\sigma_a^2 + \sigma_b^2}{4\sigma_a^2 \sigma_b^2 \left(1 - r_{ab}^2\right)}\right)\right] I_0 \left[r^2 \left(\frac{r_{ab}^2 + \left\{ \frac{\sigma_a^2 - \sigma_b^2}{2\sigma_a \sigma_b} \right\}^2}{2\sigma_a \sigma_b \left(1 - r_{ab}^2\right)} \right) \right]$$

$0 < r < \infty.$

$I_0(\bullet) =$ Bessel's function of the first kind

$$p_\Phi(\phi) = \frac{\sqrt{\left(1 - r_{ab}^2\right)}}{2\pi \sigma_a \sigma_b \left[\frac{\cos^2 \phi}{\sigma_b^2} + \frac{\sin^2 \phi}{\sigma_a^2} - \frac{r_{ab} \sin \phi \cos \phi}{\sigma_a \sigma_b} \right]}; 0 < \phi < 2\pi$$

R is generalized Rayleigh random variable

Φ is a generalized β random variable.

Note

$$\int_0^{2\pi} \exp\left(b\cos\theta\right) d\theta = 2\pi I_0\left(b\right)$$

$I_0\left(b\right) =$ modified Bessel's function of argument b and order 0.

Special case $r_{ab}=0, \sigma_a=\sigma_b=\sigma$

$$p_{R\Phi}(r,\phi)=\frac{r}{2\pi\sigma^2}\exp\left[-\frac{1}{2}\left\{\frac{r^2\cos^2\phi}{\sigma^2}+\frac{r^2\sin^2\phi}{\sigma^2}\right\}\right]$$

$$=\frac{r}{2\pi\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right];0<r<\infty;0<\phi<2\pi$$

$$p_R(r)=\int_0^{2\pi}p_{R\Phi}(r,\phi)\,d\phi=\int_0^{2\pi}\frac{r}{2\pi\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right]d\phi$$

$$p_R(r)=\frac{r}{2\pi\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right];0<r<\infty$$ [Rayleigh RV]

Similarly we get $p_\Phi(\phi)=\frac{1}{2\pi};0<\phi<2\pi$ [Uniform RV]

$\Rightarrow p_{R\Phi}(r,\phi)=p_R(r)\,p_\Phi(\phi)\Rightarrow R\perp\Phi$

Remarks

- Envelope and peaks share common properties.

This is not surprising since the envelope passes through all the peaks.

- The heuristic approach based on which we derived Rayleigh model for the peak distribution stands justified since using a more rigorous arguement we have arrived at Rayleigh model for the envelope.

- $p_R(r)$ and $p_\Phi(\phi)$ are independent of the central frequency ω_r

Joint pdf of $R(t_1), R(t_2), \Phi(t_1), \& \Phi(t_2)$

$$X(t_1) = A(t_1)\cos(\omega t_1) + B(t_1)\sin(\omega t_1)$$

$$X(t_2) = A(t_2)\cos(\omega t_2) + B(t_2)\sin(\omega t_2)$$

$$A(t_1) = R(t_1)\cos\Phi(t_1)$$

$$B(t_1) = R(t_1)\sin\Phi(t_1)$$

$$A(t_2) = R(t_2)\cos\Phi(t_2)$$

$$B(t_2) = R(t_2)\sin\Phi(t_2)$$

$$A_1 = R_1\cos\Phi_1$$

$$B_1 = R_1\sin\Phi_1$$

$$A_2 = R_2\cos\Phi_2$$

$$B_2 = R_2\sin\Phi_2$$

$$J^{-1} = \begin{vmatrix} \cos\Phi_1 & -R_1\sin\Phi_1 & 0 & 0 \\ \sin\Phi_1 & R_1\cos\Phi_1 & & \\ 0 & 0 & \cos\Phi_2 & -R_1\sin\Phi_2 \\ 0 & 0 & \sin\Phi_2 & R_2\cos\Phi_2 \end{vmatrix}$$

$$= \cos\Phi_1 \begin{vmatrix} R_1\cos\Phi_1 & 0 & 0 \\ 0 & \cos\Phi_2 & -R_1\sin\Phi_2 \\ 0 & \sin\Phi_2 & R_2\cos\Phi_2 \end{vmatrix}$$

$$+R_1\sin\Phi_1 \begin{vmatrix} \sin\Phi_1 & 0 & 0 \\ 0 & \cos\Phi_2 & -R_1\sin\Phi_2 \\ 0 & \sin\Phi_2 & R_2\cos\Phi_2 \end{vmatrix}$$

$$= R_1 R_2$$

$$p_{R_1R_2\Phi_1\Phi_2}(r_1, r_2, \phi_1, \phi_2) = r_1 r_2 \, p_{A_1B_1A_2B_2}(a_1, a_2, b_1, b_2)\Big|_{\substack{a_1 = r_1\cos\phi_1 \\ a_2 = r_2\cos\phi_2 \\ b_1 = r_1\sin\phi_1 \\ b_2 = r_2\sin\phi_2}}$$

$$p_{R_1R_2}(r_1, r_2) = \int_0^{2\pi}\int_0^{2\pi} r_1 r_2 \, p_{A_1B_1A_2B_2}(r_1\cos\phi_1, r_1\sin\phi_1, r_2\cos\phi_2, r_2\sin\phi_2)\, d\phi_1 d\phi_2$$

t_1, t_2

If $X(t)$ is a stationary Gaussian random process with zero mean it can be shown that (exercise)

$$p_{R_1R_2}(r_1, r_2) = \frac{r_1 r_2}{\Delta} I_0\left[\frac{r_1 r_2}{\Delta}\sqrt{(\mu_{13}^2 + \mu_{14}^2)}\right]\exp\left[-\frac{\sigma^2}{2\Delta}(r_1^2 + r_2^2)\right]$$

$$\sigma^2 = \langle A^2 \rangle = \langle B^2 \rangle = \langle X^2(t) \rangle$$

$$\mu_{13} = \langle A(t) A(t+\tau) \rangle = \langle B(t) B(t+\tau) \rangle$$

$$\mu_{14} = \langle A(t) B(t+\tau) \rangle = -\langle B(t+\tau) A(t) \rangle$$

$$\Delta = \sigma^4 - \mu_{13}^2 - \mu_{14}^2$$

Joint pdf of $A(t)\,\&\,\dot{A}(t)$

$p_{R\dot{R}\Phi\dot{\Phi}}(r, \dot{r}, \phi, \dot{\phi}; t)$

$$A(t) = R(t)\cos\Phi(t)$$

$$B(t) = R(t)\sin\Phi(t)$$

$$\dot{A}(t) = \dot{R}(t)\cos\Phi(t) - R(t)\dot{\Phi}(t)\sin\Phi(t)$$

$$\dot{B}(t) = \dot{R}(t)\sin\Phi(t) + R(t)\dot{\Phi}(t)\cos\Phi(t)$$

$$J^{-1} = \begin{vmatrix} \cos\Phi & -R\sin\Phi & 0 & 0 \\ \sin\Phi & R\cos\Phi & 0 & 0 \\ -R\sin\Phi & -\dot{R}\sin\Phi - R\dot{\Phi}\cos\Phi & \cos\Phi & -R\sin\Phi \\ \dot{\Phi}\cos\Phi & \dot{R}\cos\Phi - R\dot{\Phi}\sin\Phi & \sin\Phi & R\cos\Phi \end{vmatrix}$$

$$= \cos\Phi \begin{vmatrix} R\cos\Phi & 0 & 0 \\ -\dot{R}\sin\Phi - R\dot{\Phi}\cos\Phi & \cos\Phi & -R\sin\Phi \\ \dot{R}\cos\Phi - R\dot{\Phi}\sin\Phi & \sin\Phi & R\cos\Phi \end{vmatrix} + R\sin\Phi \begin{vmatrix} \sin\Phi & 0 & 0 \\ -R\sin\Phi & \cos\Phi & -R\sin\Phi \\ \dot{\Phi}\cos\Phi & \sin\Phi & R\cos\Phi \end{vmatrix}$$

$$= R^2\cos^2\Phi + R^2\sin^2\Phi = R^2$$

$$p_{R\dot{R}\Phi\dot{\Phi}}\left(r,\dot{r},\phi,\dot{\phi};t\right)=r^2\,p_{A\dot{A}B\dot{B}}\left(a,\dot{a},b,\dot{b};t\right)\Big|_{\substack{a=r\cos\phi\\ b=r\sin\phi\\ \dot{a}=\dot{r}\cos\phi-r\dot{\phi}\sin\phi\\ \dot{b}=\dot{r}\sin\phi+r\dot{\phi}\cos\phi}}$$

$$0<r<\infty$$

$$-\infty<\dot{r}<\infty$$

$$0<\phi<2\pi$$

$$-\infty<\dot{\phi}<\infty$$

Let

- $A(t)$ and $B(t)$ be zero mean, Gaussian, stationary random processes
- $\langle A^2(t)\rangle = \langle B^2(t)\rangle = \langle X^2(t)\rangle = \sigma_X^2$
- $A(t), \dot{A}(t), B(t), \dot{B}(t)$ are independent
- $\langle \dot{A}^2(t)\rangle = \langle \dot{B}^2(t)\rangle = \sigma_1^2$

$$X(t) = A(t)\cos\omega t + B(t)\sin\omega t$$

$$\dot{X}(t) = -A(t)\omega\sin\omega t + \dot{A}(t)\cos\omega t + B(t)\omega\cos\omega t + \dot{B}(t)\sin\omega t$$

$$= \left(\dot{B} - A\omega\right)\sin\omega t + \left(\dot{A} + B\omega\right)\cos\omega t$$

$$\Rightarrow$$

$$\left\langle \dot{X}^2(t)\right\rangle = \sigma_{\dot{X}}^2 = \sigma_1^2 + \omega^2\sigma_X^2$$

Exercise: verify if the following are true.

$$p_{R\dot{R}\Phi\dot{\Phi}}\left(r,\dot{r},\phi,\dot{\phi};t\right)=\frac{r^2}{\left(2\pi\right)^2\sigma_X^2\sigma_1^2}\exp\left[-\frac{r^2}{2\sigma_X^2}-\frac{\dot{r}^2+r^2\dot{\phi}^2}{\sigma_1^2}\right]$$

$$0<r<\infty;-\infty<\dot{r}<\infty;0<\phi<2\pi;-\infty<\dot{\phi}<\infty$$

$$\Rightarrow$$

$$p_{R\dot{R}\Phi\dot{\Phi}}\left(r,\dot{r};t\right)=\int_0^{2\pi}\int_{-\infty}^{\infty}p_{R\dot{R}\Phi\dot{\Phi}}\left(r,\dot{r},\phi,\dot{\phi};t\right)d\phi d\dot{\phi}$$

$$=\frac{r}{\left(2\pi\right)^{\frac{1}{2}}\sigma_X^2}\exp\left[-\frac{r^2}{2\sigma_X^2}-\frac{\dot{r}^2}{\sigma_1^2}\right];\quad 0<r<\infty;-\infty<\dot{r}<\infty$$

• Determine $p_{\Phi\dot{\Phi}}\left(\phi,\dot{\phi};t\right)$

R S Langley, 1986, On various definitions of the envelope of a random process, Journal of Sound and Vibration, 105(3), 503-512.

Remarks

• Knowing $p_{R\dot{R}}(r,\dot{r};t)$ the number of crossing of level α by the envelope process $R(t)$ can be characterized.

$$\langle n_R^+(\xi,t)\rangle = \int_0^\infty \dot{r}\, p_{R\dot{R}}(\xi,\dot{r};t)\, d\dot{r}$$

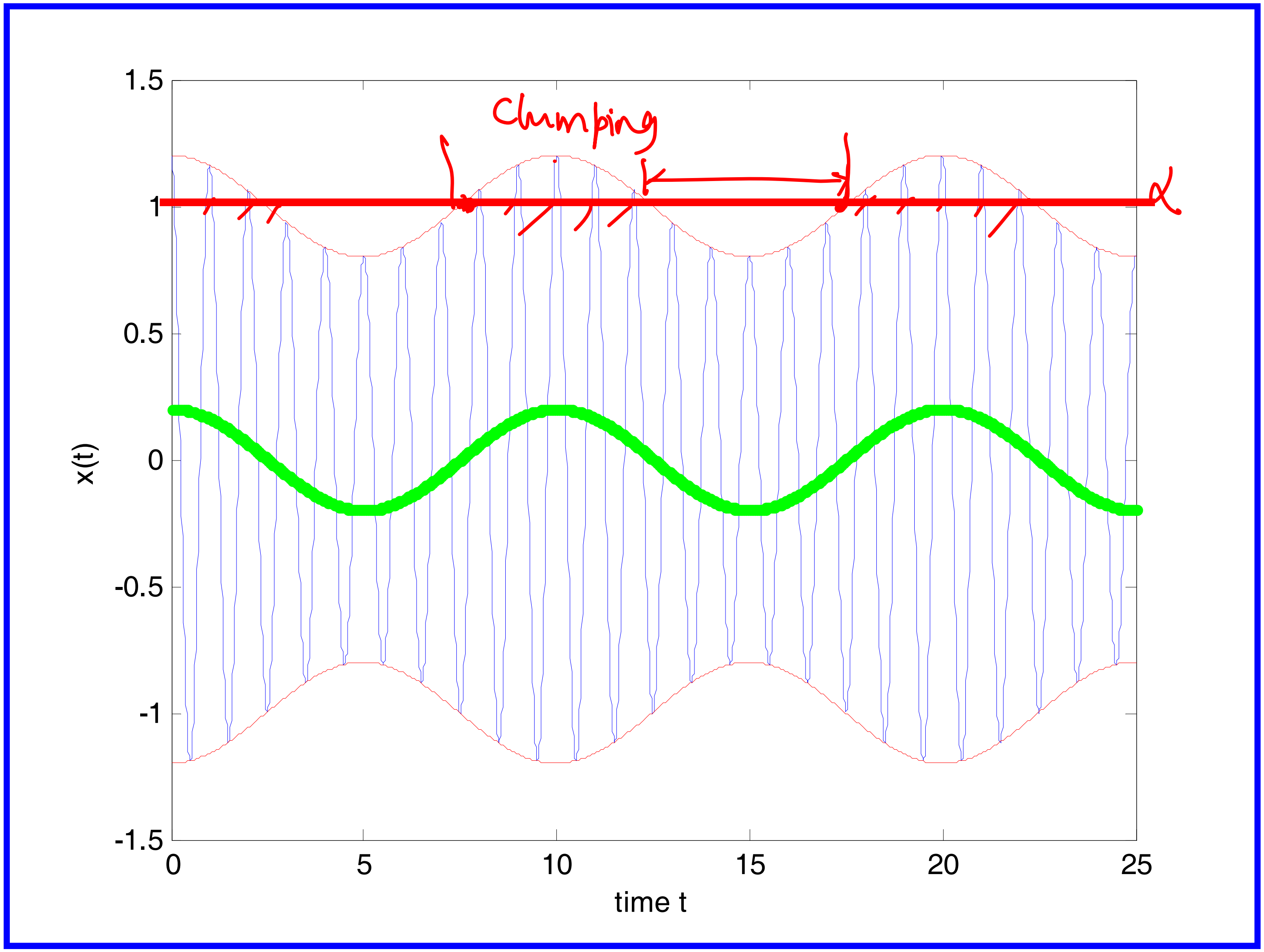
$$= \frac{\xi\sigma_1}{(2\pi)^{\frac{1}{2}}\sigma_X^2}\exp\left(-\frac{\xi^2}{2\sigma_X^2}\right)$$

= Average rate of crossing of the level ξ with positive slope by $R(t)$

Note: $R(t)$ is non-Gaussian

• Crossing of a level α for narrow band processes occur in clumps

Clumping

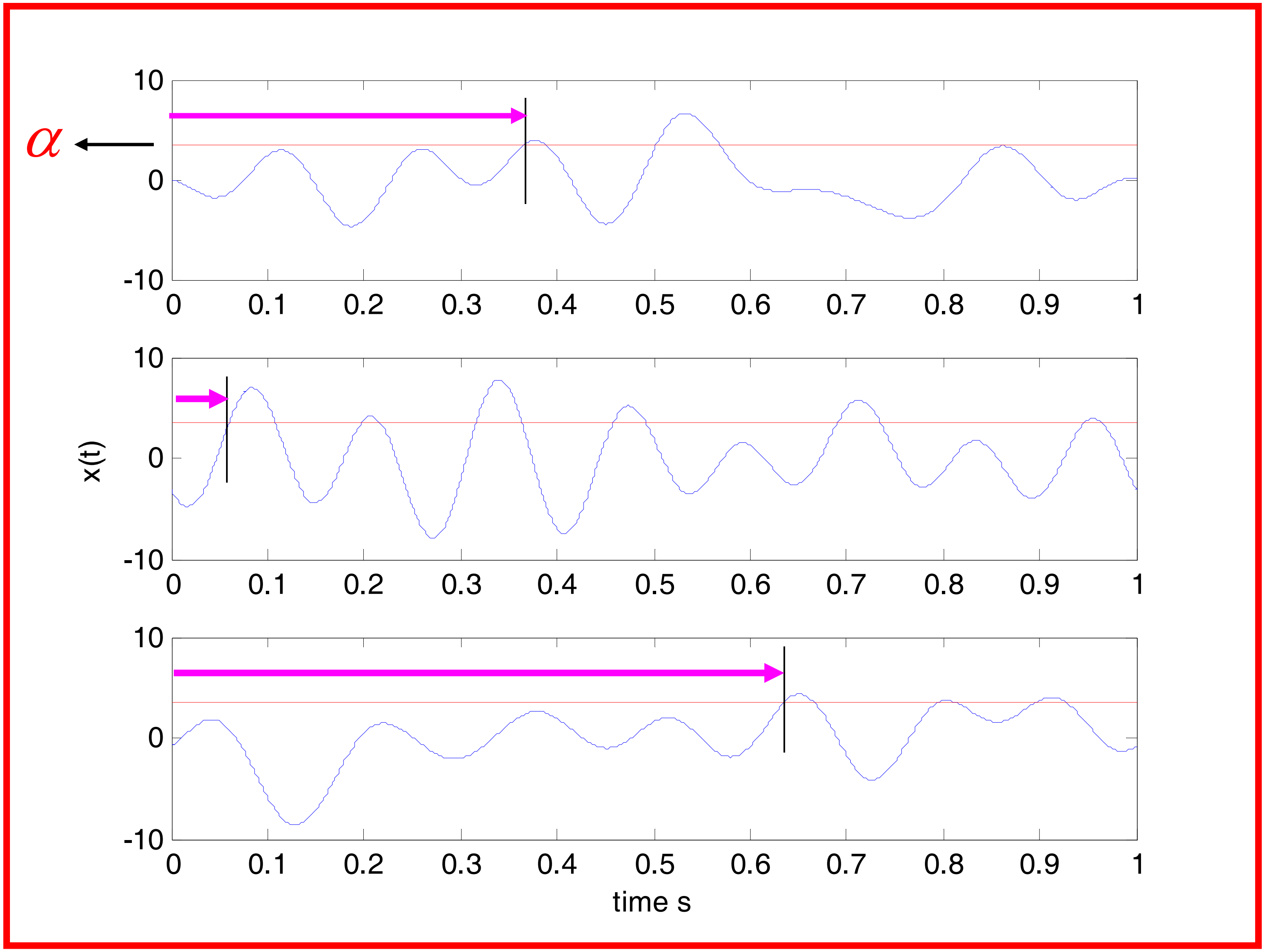


Average clump size

$$\langle cs \rangle = \frac{\langle n_X^+(\xi,t) \rangle}{\langle n_R^+(\xi,t) \rangle} = \frac{1}{\sqrt{2\pi}\xi} \frac{\sigma_X \sigma_{\dot{X}}}{\sigma_1}$$

Poisson model for number of level crossing is more appropriate for $R(t)$ than for $X(t)$

Time required by $X(t)$ to reach level α for the first time



$T_f(\alpha)$

- The time required by $X(t)$ to cross level α for the first time
- A real valued random variable taking values in 0 to ∞
- Given the complete description of $X(t)$,

can we characterize $T_f(\alpha)$?

- This is known as the
    - First passage problem
    - Barrier crossing problem
    - Outcrossing problem

Poisson model for $N(\alpha, 0, T)$

Assumptions

- The threshold level α is high (so that crossing is a rare event)
- Crossing times are mutually independent
- $N(\alpha, 0, T)$ is a Poisson random variable

$$P\left[N(\alpha,0,T)=k\right]=\frac{(\lambda T)^k}{k!}\exp(-\lambda T)$$

$$\lambda = \text{rate of crossing of level } \alpha = \langle n(\alpha,t)\rangle$$

If $X(t)$ is a stationary gaussian random process with zero mean

$$\langle n(\alpha,t)\rangle = \frac{\sigma_{\dot{x}}}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$P\left[N(\alpha,0,T)=k\right]=\frac{(\lambda T)^k}{k!}\exp(-\lambda T)$$

$$\lambda=\langle n(\alpha,t)\rangle=\frac{\sigma_{\dot{x}}}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$P\left[N(\alpha,0,T)=k\right]=\frac{\left(\frac{\sigma_{\dot{x}}T}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right)^k}{k!}\exp\left[-\frac{\sigma_{\dot{x}}T}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right];$$

$$k=0,1,2,\cdots,\infty$$

$$P\left[T_f(\alpha) > t\right] = P\left[\text{no points in 0 to } t\right]$$

$$= P\left[N^+(\alpha, 0, T) = 0\right]$$

$$= \exp\left[-\frac{\sigma_{\dot{x}} t}{2\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right]$$

$$P_{T_f}(t) = 1 - P\left[T_f(\alpha) > t\right]$$

$$= 1 - \exp\left[-\frac{\sigma_{\dot{x}} t}{2\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right]$$

$$p_{T_f}(t) = \frac{dP_{T_f}(t)}{dt}$$

$$= \frac{\sigma_{\dot{x}}}{2\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\exp\left[-\frac{\sigma_{\dot{x}} t}{2\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right]$$

$$0 < t < \infty$$

$T_f(\alpha)$ is exponentially distributed

$$P_{T_f}(t) = 1 - \exp[-\lambda t]$$

$$p_{T_f}(t) = \lambda \exp[-\lambda t] \quad 0 < t < \infty$$

$$\lambda = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\langle T_f \rangle = \int_0^\infty t\lambda \exp[-\lambda t]\,dt = \frac{1}{\lambda}$$

Example

Let $X(t)$ be a nonstationary, zero mean, Gaussian random process with autocovariance function $R_{XX}(t_1, t_2)$.

Determine the average rate of crossing of level α by the process $X(t)$.

$$\langle n(\alpha,t)\rangle = \left\langle \left|\dot{X}(t)\right| \delta\left[X(t)-\alpha\right]\right\rangle = \int_{-\infty}^{\infty} |\dot{x}| p_{X\dot{X}}(\alpha,\dot{x};t)\,d\dot{x}$$

We need the jpdf of $X(t)$ and $\dot{X}(t)$.

$$p(x,\dot{x};t) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}\sqrt{(1-r^2)}} \exp\left[-\frac{1}{2(1-r^2)}\left\{\frac{x^2}{\sigma_x^2} + \frac{\dot{x}^2}{\sigma_{\dot{x}}^2} - \frac{2rx\dot{x}}{\sigma_x\sigma_{\dot{x}}}\right\}\right]$$

$-\infty < x, \dot{x} < \infty$

$$\left\langle n^+(\alpha,t)\right\rangle = \frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_x}\left(1-r^2\right)[\exp\left(-\frac{\alpha^2}{2\sigma_x^2\left(1-r^2\right)}\right)+$$

$$\frac{\alpha r}{\sigma_x}\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right)\left\{1-erf\left(\frac{\alpha r}{\sigma_x\sqrt{2\left(1-r^2\right)}}\right)\right\}]$$

Note

- The quantities $r, \sigma_x, \& \sigma_{\dot{x}}$ are time varying.
- If $r=0$ and $\sigma_x, \& \sigma_{\dot{x}}$ are time invariant, the above expression reduces to the expression for the case when $X(t)$ is stationary.

This is what is expected.

$$1 - \exp(-\lambda t)$$

$$P_{T_f}(t) = 1 - \exp\left[-\int_0^t \left\langle n_X^+(\alpha, \tau)\, d\tau \right\rangle \right]$$

First passage time for envelope process $R(t)$

Recall

$$\left\langle n_R^+(\xi,t)\right\rangle = \int_0^\infty \dot{r}\, p_{R\dot{R}}(\xi,\dot{r};t)\, d\dot{r} = \frac{\xi\sigma_1}{(2\pi)^{\frac{1}{2}}\sigma_X^2}\exp\left(-\frac{\xi^2}{2\sigma_X^2}\right)$$

$$P_{T_f}(t) = 1 - \exp\left[-\frac{\xi\sigma_1 t}{(2\pi)^{\frac{1}{2}}\sigma_X^2}\exp\left(-\frac{\xi^2}{2\sigma_X^2}\right)\right]$$

The maximum value of $X(t)$ in interval 0 to T