Stochastic Structural Dynamics

Lecture-18

Failure of randomly vibrating systems-2

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$N\left(\alpha,0,T\right)$

- Number of times $X(t)$ crosses α in 0 to T
- An integer valued random variable
- Given the complete description of $X(t)$,

can we characterize $N\left(\alpha,0,T\right)$?

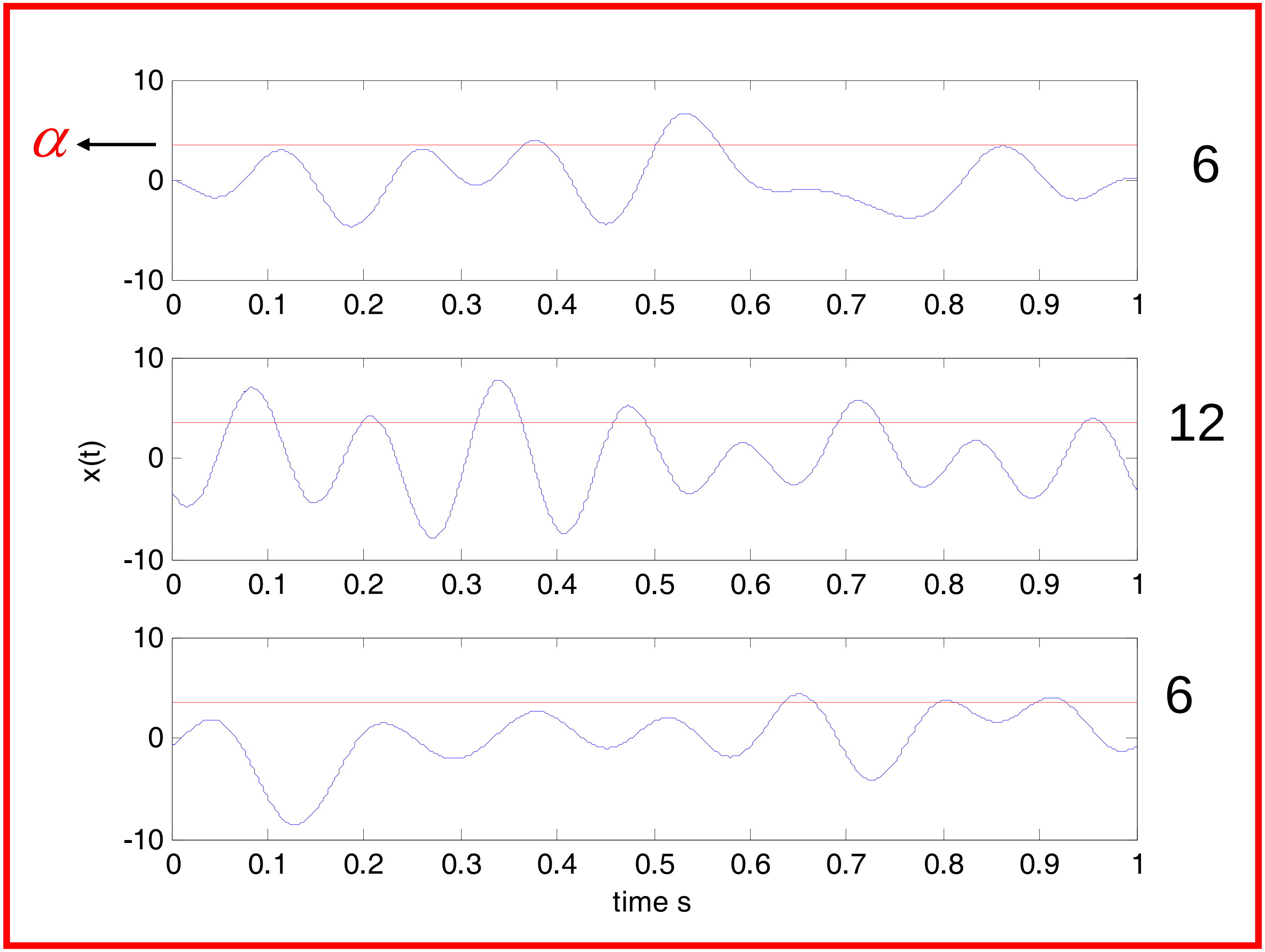
- This is known as the level crossing problem.

$$N(0,\alpha,T) = N(T) = \int_0^T \left|\dot{X}\left(t\right)\right| \delta\left[X\left(t\right)-\alpha\right] dt$$

$$= \int_0^T n\left(\alpha,t\right) dt$$

$$n\left(\alpha,t\right) = \left|\dot{X}\left(t\right)\right| \delta\left[X\left(t\right)-\alpha\right]$$

The number of times the level α is crossed by $X(t)$ in interval 0 to T



$$\langle n(\alpha,t)\rangle = \left\langle \left|\dot{X}(t)\right| \delta\left[X(t)-\alpha\right]\right\rangle$$
$$= \int_{-\infty}^{\infty} |\dot{x}| p_{X\dot{X}}(\alpha,\dot{x};t)\, d\dot{x}$$

Stationary Gaussian random process

$$\langle n(\alpha,t)\rangle = \frac{\sigma_{\dot{x}}}{\pi\sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\langle n(\alpha,t)\rangle = \frac{\sigma_{\dot{x}}}{\pi\sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\sigma_x^2 = \int_0^{\infty} S_{XX}(\omega)\, d\omega$$

$$\sigma_{\dot{x}}^2 = \int_0^{\infty} \omega^2 S_{XX}(\omega)\, d\omega$$

Spectral moments

$$\lambda_n = \int_0^{\infty} \omega^n S_{XX}(\omega)\, d\omega$$

$$\langle n(\alpha,t)\rangle = \frac{1}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\lambda_2}\right\}$$

Poisson model for $N(\alpha, 0, T)$

Assumptions

•The threshold level α is high (so that crossing is a rare event)

•Crossing times are mutually independent

•$N(\alpha\ , 0, T)$is a Poisson random variable

$$P\left[N(\alpha, 0, T) = k\right] = \frac{(\lambda T)^k}{k!} \exp(-\lambda T)$$

$\lambda =$ rate of crossing of level $\alpha = \langle n(\alpha, t) \rangle$

If $X(t)$ is a stationary gaussian random process with zero mean

$$\langle n(\alpha, t) \rangle = \frac{\sigma_{\dot{x}}}{\pi \sigma_x} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$P\left[N(\alpha,0,T)=k\right]=\frac{(\lambda T)^k}{k!}\exp(-\lambda T)$$

$$\lambda=\langle n(\alpha,t)\rangle=\frac{\sigma_{\dot{x}}}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$P\left[N(\alpha,0,T)=k\right]=\frac{\left(\frac{\sigma_{\dot{x}}T}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right)^k}{k!}\exp\left[-\frac{\sigma_{\dot{x}}T}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\right];$$

$$k=0,1,2,\cdots,\infty$$

Narrow band and broad band processes

Example 1 **Ideal narrow band process**

$x(t) = P\cos(\lambda t + \theta); P \sim$ Rayleigh and $\theta \sim U[0, 2\pi];\ P \perp \theta$

$$\langle x(t) \rangle = \langle P\cos(\lambda t + \theta) \rangle = \langle P \rangle \langle \cos(\lambda t + \theta) \rangle = 0$$

$$\langle x(t)x(t+\tau) \rangle = \langle P\cos(\lambda t + \theta) P\cos(\lambda t + \lambda\tau + \theta) \rangle$$

$$= \langle P^2 \rangle \cos\lambda\tau$$

$\Rightarrow x(t)$ is a stationary random process

$$\Rightarrow S_{xx}(\omega) = \langle P^2 \rangle 2\pi\delta(\lambda - \omega)$$

Check

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\cos(\omega\tau)\,d\omega = \langle P^2 \rangle \cos\lambda\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle P^2 \rangle 2\pi\delta(\lambda - \omega)\cos(\omega\tau)\,d\omega = \langle P^2 \rangle \cos\lambda\tau \text{ (ok)}$$

Example 2 Realistic narrow band processes

$$m\ddot{x} + c\dot{x} + kx = w(t)$$

$$\langle w(t) \rangle = 0; \langle w(t) w(t+\tau) \rangle = I\delta(\tau)$$

$$\lim_{\substack{t_1 \to \infty \\ t_2 \to \infty \\ t_2 - t_1 = \tau}} R_{xx}(t_1, t_2) \to \frac{I}{4\eta\omega^3 m^2} \exp\left[-\eta\omega|\tau|\right] \left[\cos\omega_d \tau + \frac{\eta}{\sqrt{1-\eta^2}} \sin\omega_d |\tau|\right]$$

$$S_{XX}(\omega) = \left|H(\omega)\right|^2 I$$

$$H(\omega) = \frac{1/m}{\left(\omega_n^2 - \omega^2\right) + i2\eta\omega\omega_n}$$

Example 3 Ideal broad band process

Gaussian white noise $w(t)$

$$\langle w(t)\rangle = 0; \langle w(t)w(t+\tau)\rangle = I\delta(\tau)$$

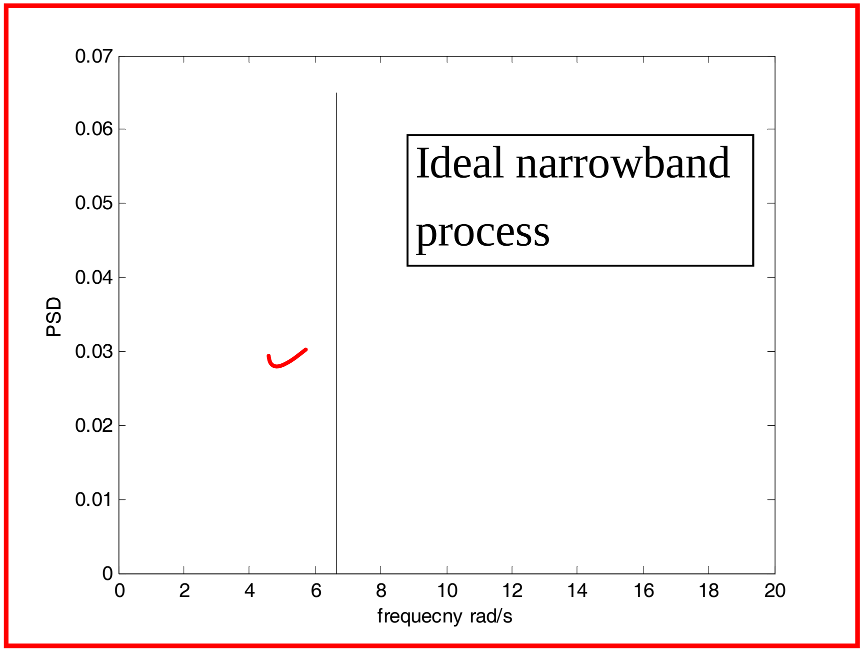
$$S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} I\delta(\tau-0)\exp(-i\omega\tau)\,d\omega = \frac{I}{2\pi}$$

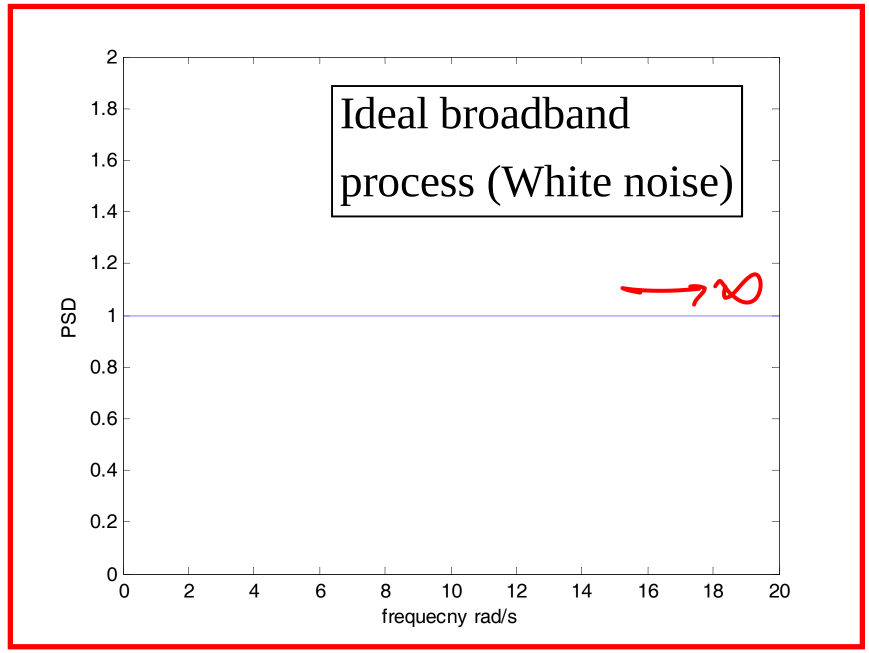
Example 4 Band limited white noise process

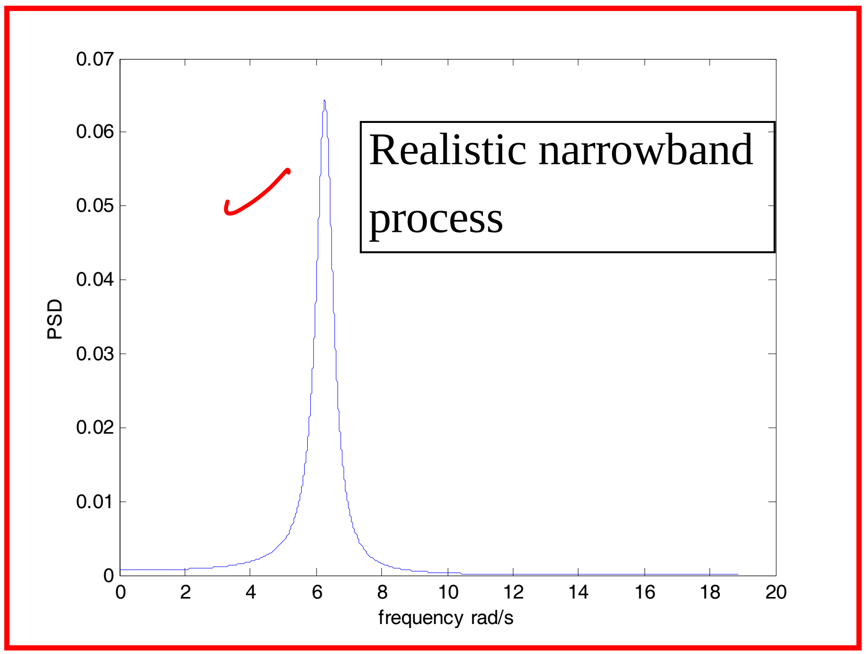
$$S_{xx}(\omega) = 1 \text{ for } |\omega| < \sigma$$

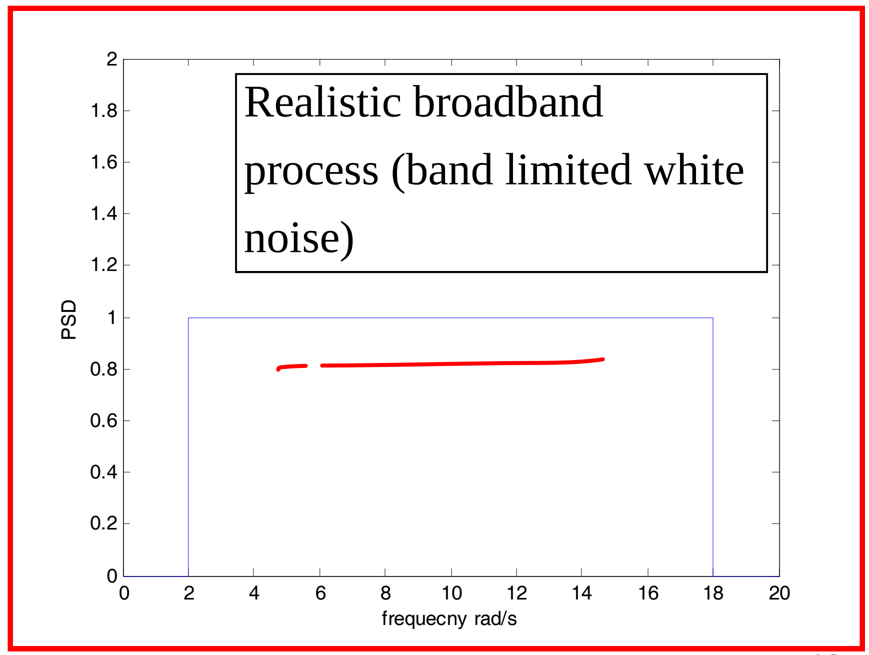
$$= 0 \text{ for } |\omega| > \sigma$$

$$R_{xx}(\tau) = \frac{\sin\sigma\tau}{\pi\tau}$$

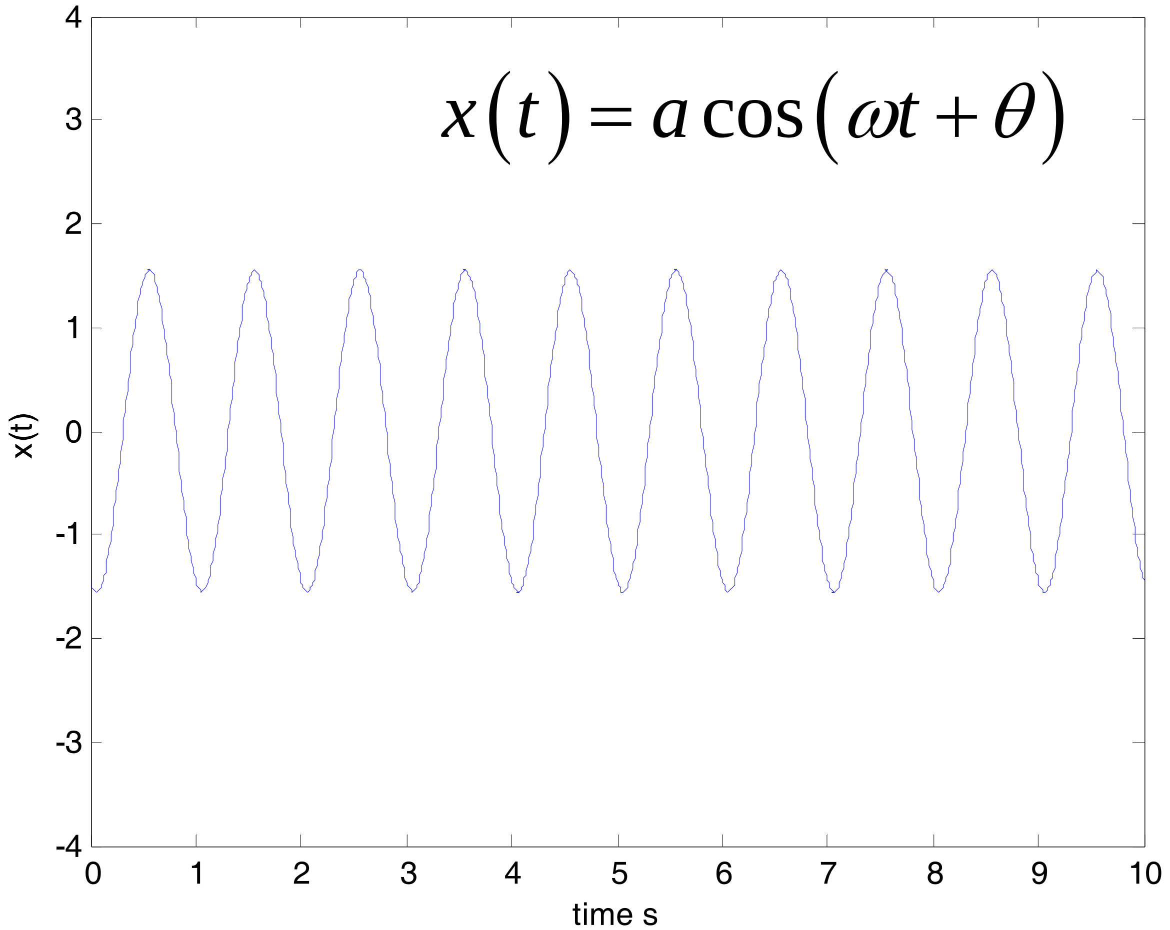

Ideal narrowband process
PSD
frequecny rad/s


Ideal broadband process (White noise)
PSD
frequecny rad/s


Realistic narrowband process
PSD
frequency rad/s

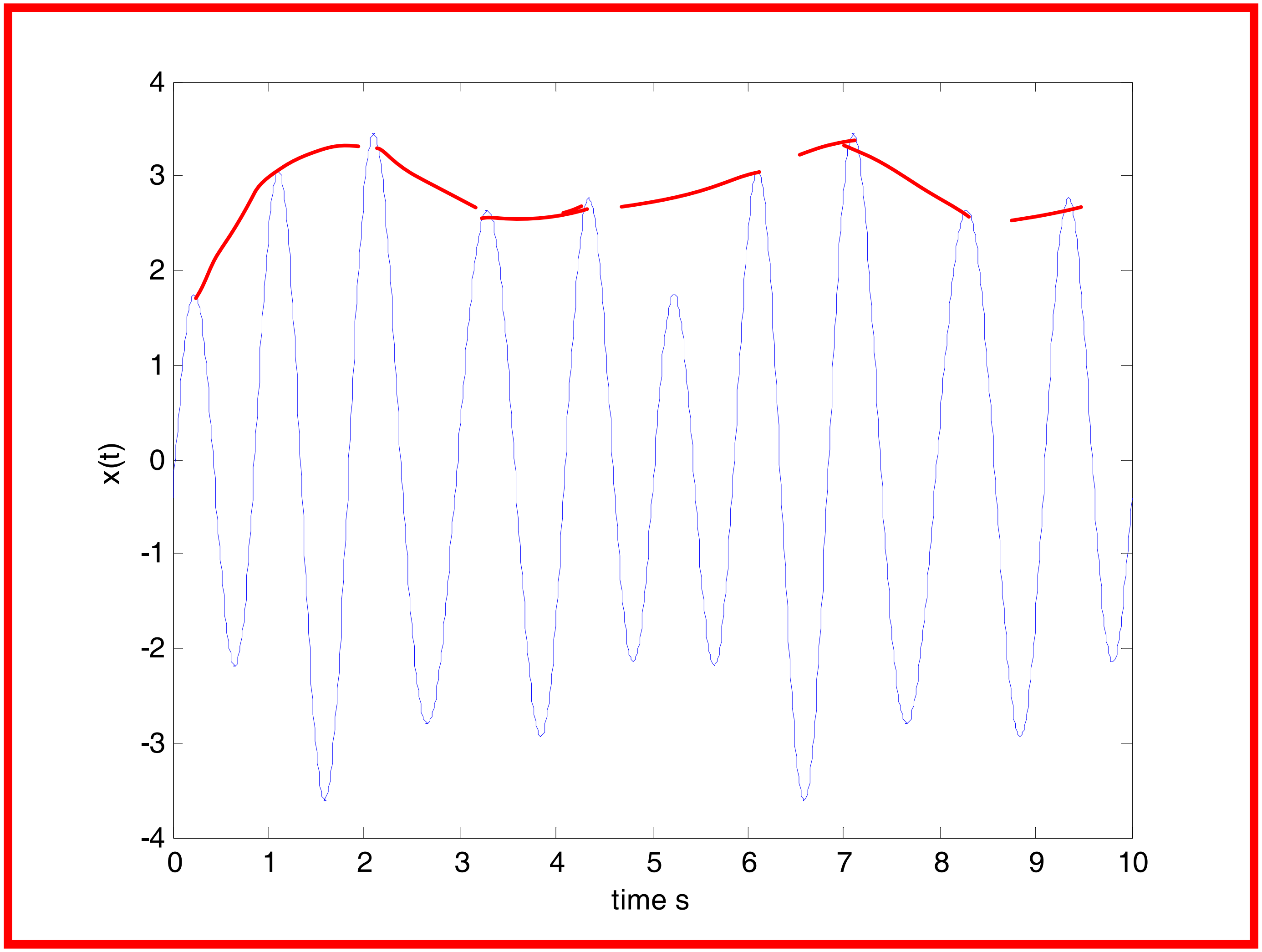

Realistic broadband process (band limited white noise)
PSD
frequecny rad/s

Sample of an ideal narrow band process

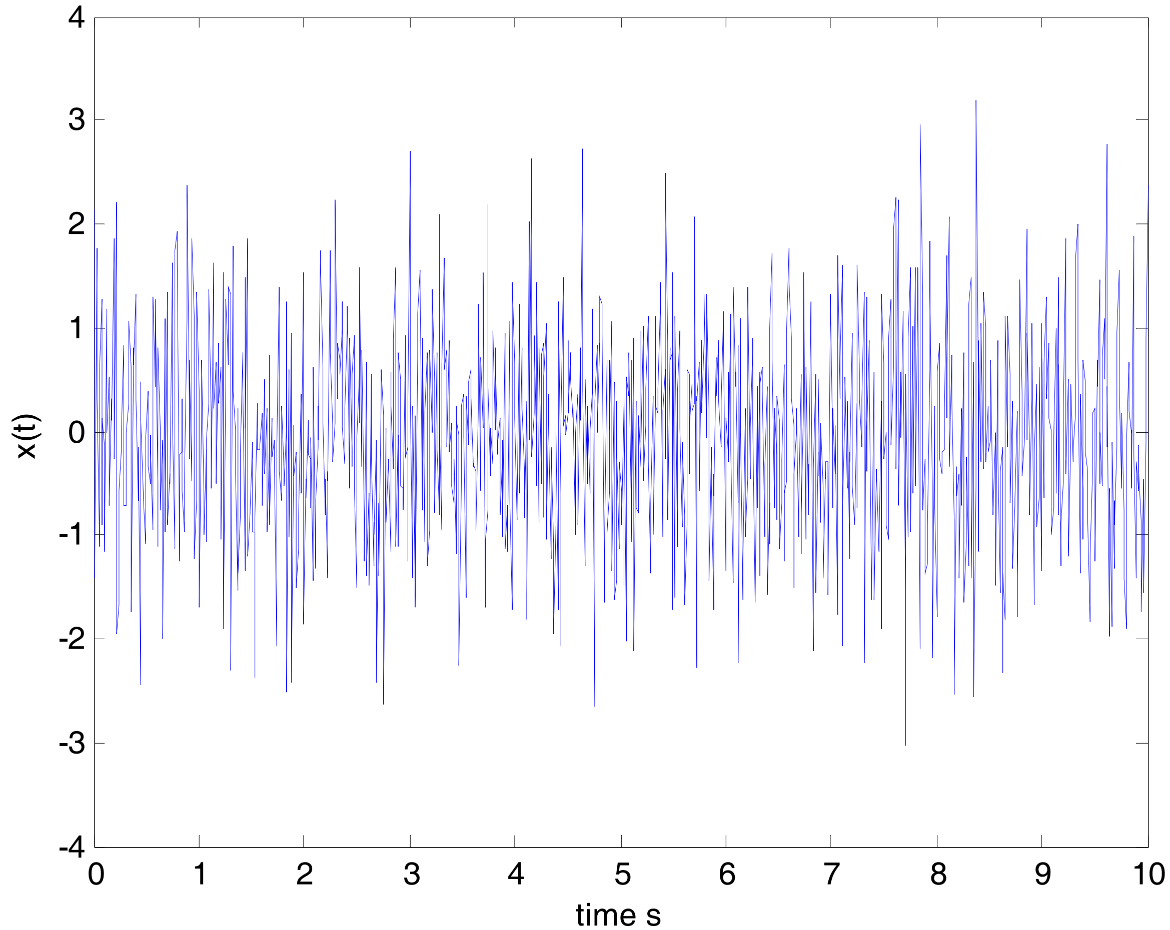


Sample of a realistic narrow band process

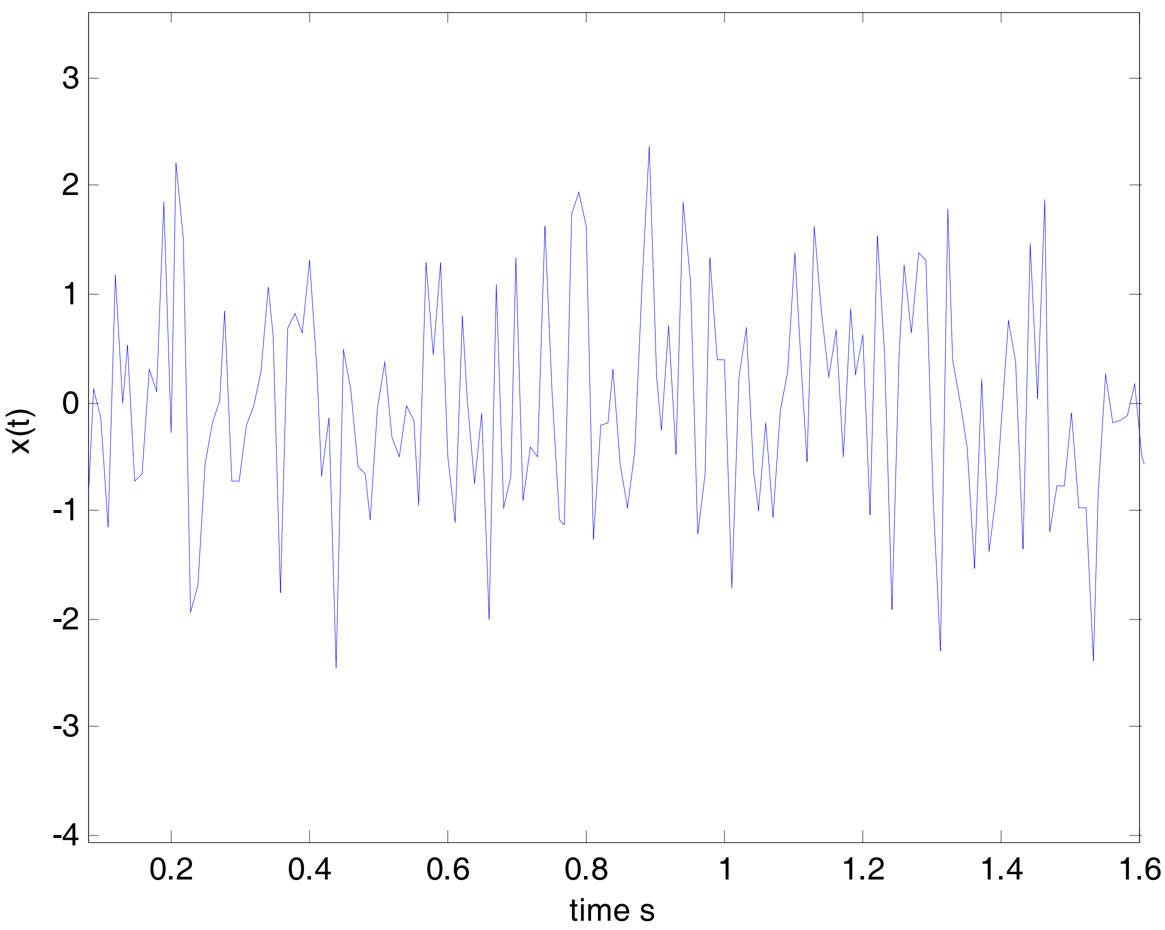
$$x(t) = a(t)\cos\left[\omega t + \theta(t)\right]$$



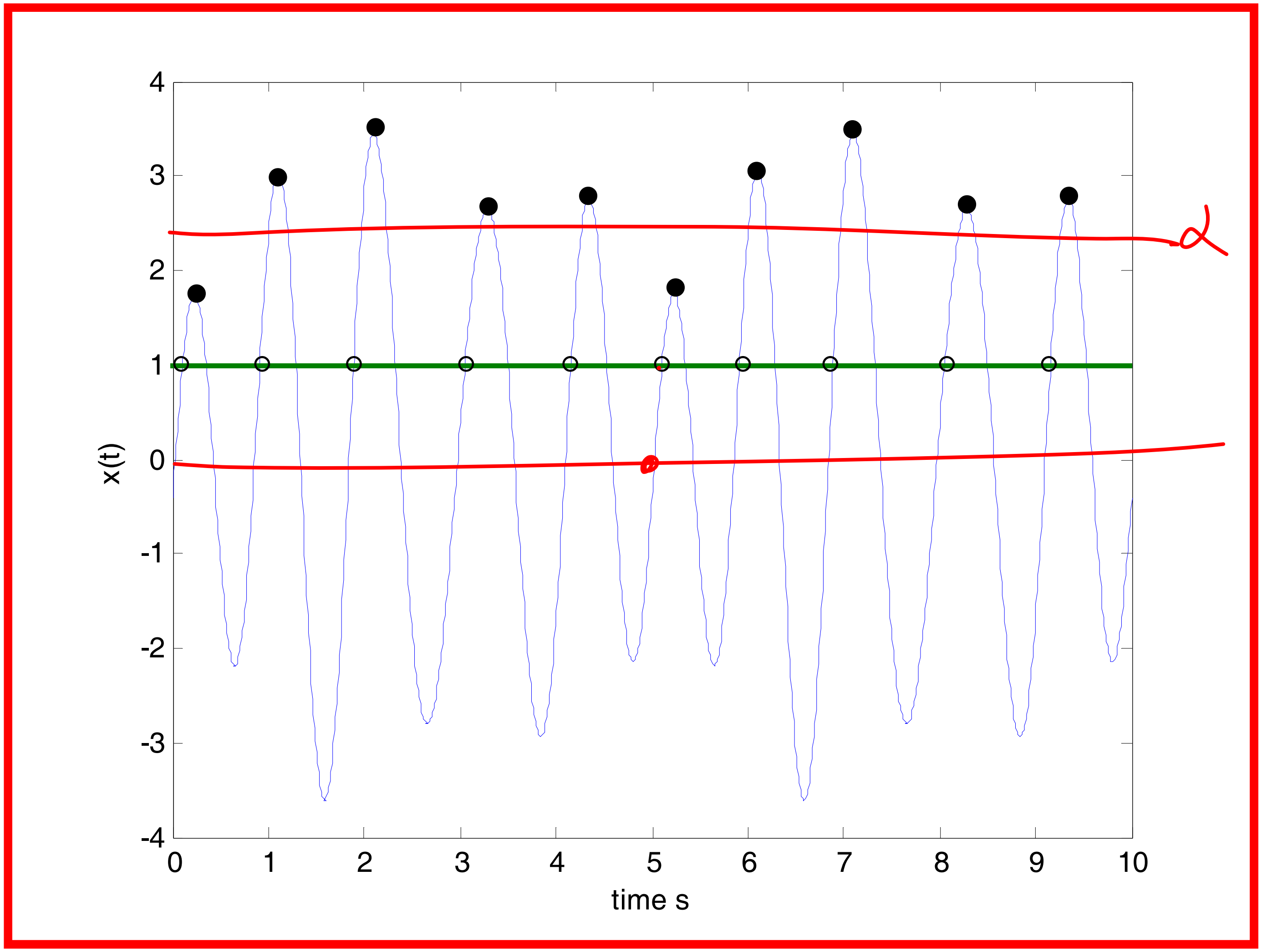
Sample of a band limited process



Sample of a band limited process

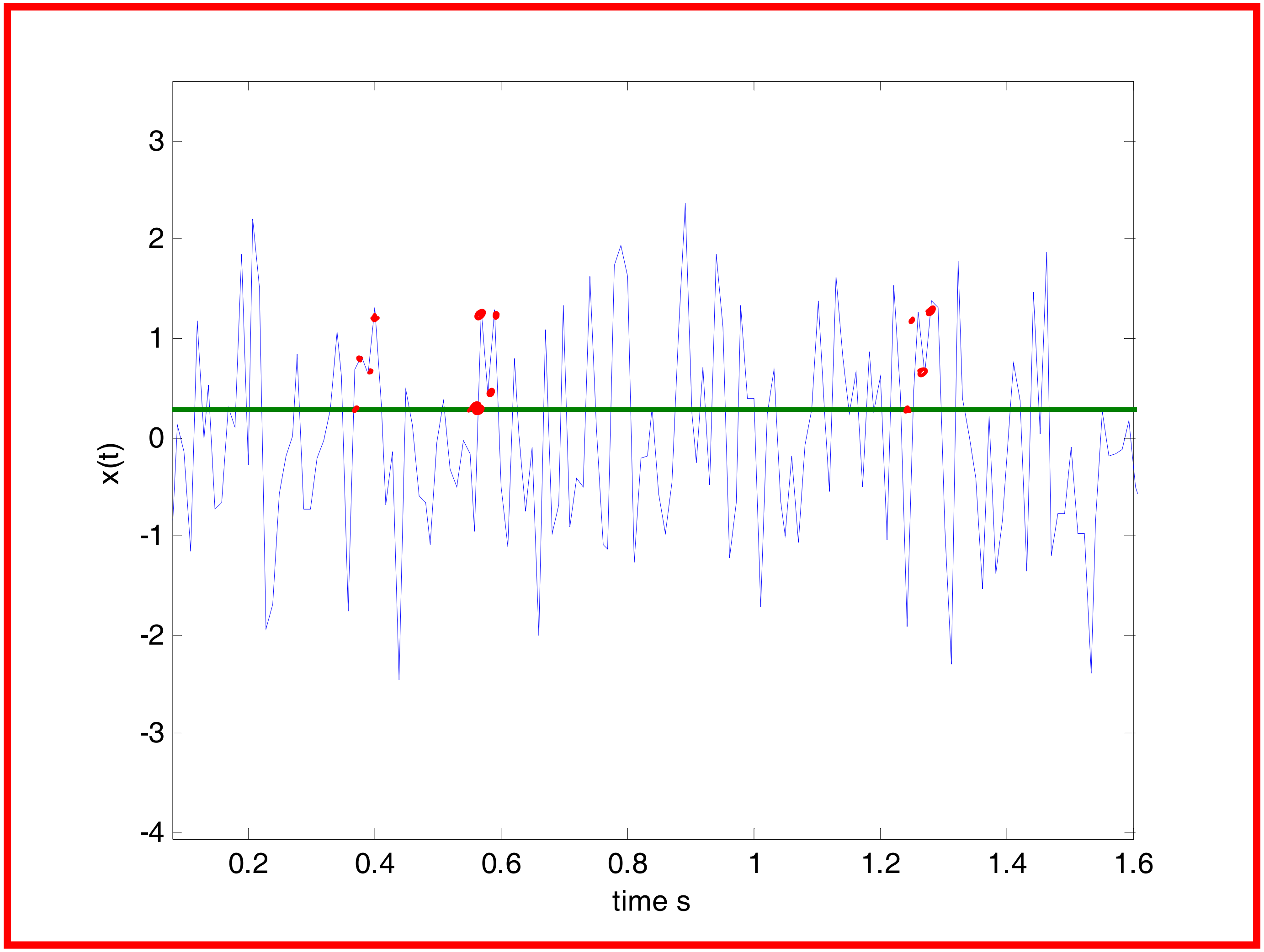


Narrow band process



Every zero crossing with +ve slope is followed by a peak

Broad band process



Crossing with +^{ve} slope can be followed by several extrema

Distribution of peaks for a narrow band process [Heuristic approach]

$X(t)=$ zero mean, stationary, narrow band,

Gaussian random process

Consider peaks above level α in the interval 0 to T.

$$P[\text{Peak} \le \alpha] = 1 - P[\text{Peak} > \alpha]$$

$$P[\text{Peak} > \alpha] = \frac{\text{Number of peaks above level } \alpha}{\text{Total number of peaks}}$$

[Relative frequency definition]

$$= \frac{\text{Total number of times the level } \alpha \text{ is crossed with positive slope in 0-T}}{\text{Total number of zero crossings with positive slope in 0 to T}}$$

[$X(t)$ is assumed to be a narrow band process]

$$P\left[\text{Peak}>\alpha\right]=$$

$$\frac{\text{Total number of times the level } \alpha \text{ is crossed with positive slope in 0-T}}{\text{Total number of zero crossings with positive slope in 0 to T}}$$

$$=\frac{N^{+}\left(\alpha,0,T\right)}{N^{+}\left(0,0,T\right)}$$

$$\approx\left\langle\frac{N^{+}\left(\alpha,0,T\right)}{N^{+}\left(0,0,T\right)}\right\rangle \qquad \left[\text{Ergodicity}\right]$$

$$\approx\frac{\left\langle N^{+}\left(\alpha,0,T\right)\right\rangle}{\left\langle N^{+}\left(0,0,T\right)\right\rangle} \qquad \left[\text{Ad hoc assumption}\right]$$

$$=\frac{T\left\langle n^{+}\left(\alpha,t\right)\right\rangle}{T\left\langle n^{+}\left(0,t\right)\right\rangle} \qquad \left[X\left(t\right)\text{ is stationary}\right]$$

$$P[\text{Peak} > \alpha] = \frac{\langle n^+(\alpha, t)\rangle}{\langle n^+(0, t)\rangle}$$

$$= \frac{\frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_x}\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right)}{\frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_x}}$$

$[X(t) \text{ is Gaussian}]$

$$= \exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right)$$

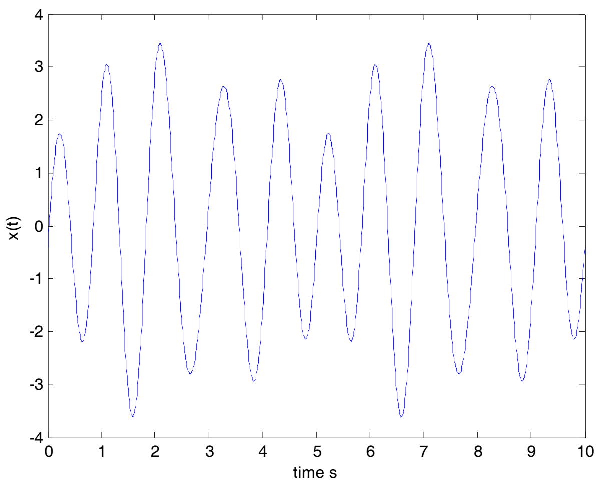
$\Rightarrow$

$$P_p(\alpha) = 1 - P[\text{Peak} > \alpha] = 1 - \exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right)$$

$$\Rightarrow p_p(\alpha) = \frac{\alpha}{\sigma_x^2}\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right); 0 < \alpha < \infty$$

Summary

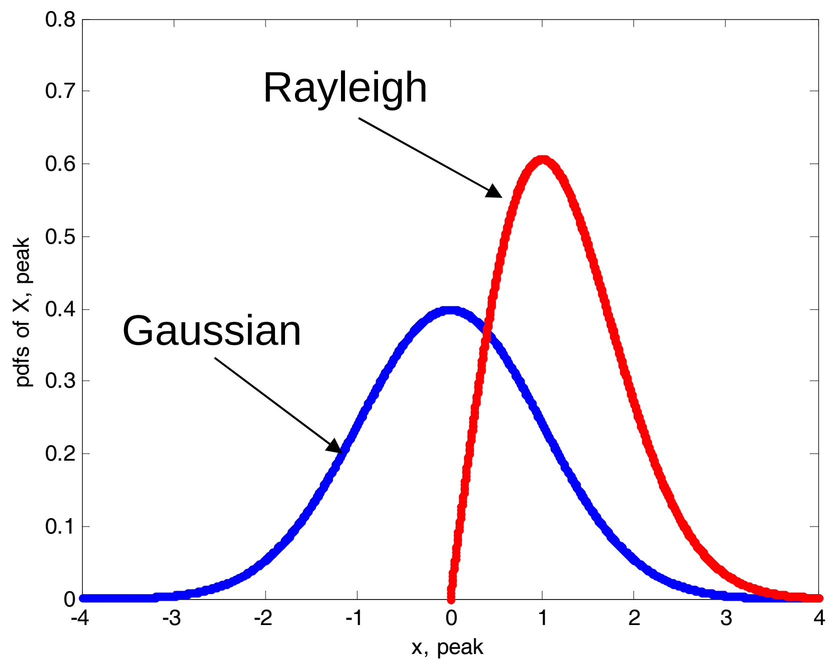
Gaussian narrow band process



$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x}\exp\left[-\frac{1}{2}\frac{x^2}{\sigma_x^2}\right]; -\infty < x < \infty$$

$$p_p(\alpha) = \frac{\alpha}{\sigma_x^2}\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right); 0 < \alpha < \infty$$

Heuristic basis



Peak distribution

Let *X(t)*= be a random process.

-Not necessarily Gaussian

-Not necessarily stationary

-Not necessarily narrow banded

M(α,0,T)=Number of peaks above level α in the time interval *0*-*T*.

What is the PDF of *M(α,0,T)*?

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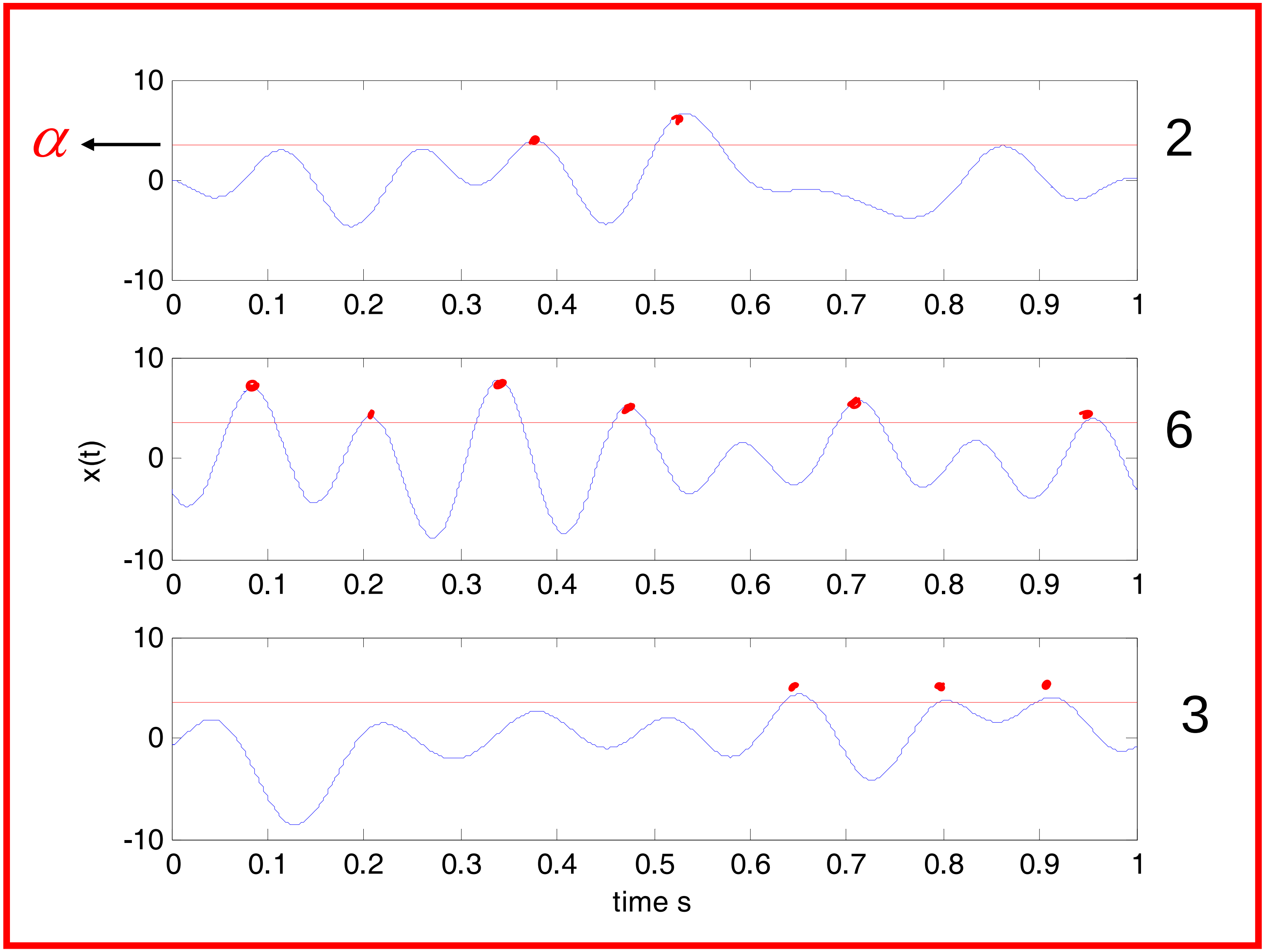
The number of peaks of $X(t)$ above level α in interval 0 to T

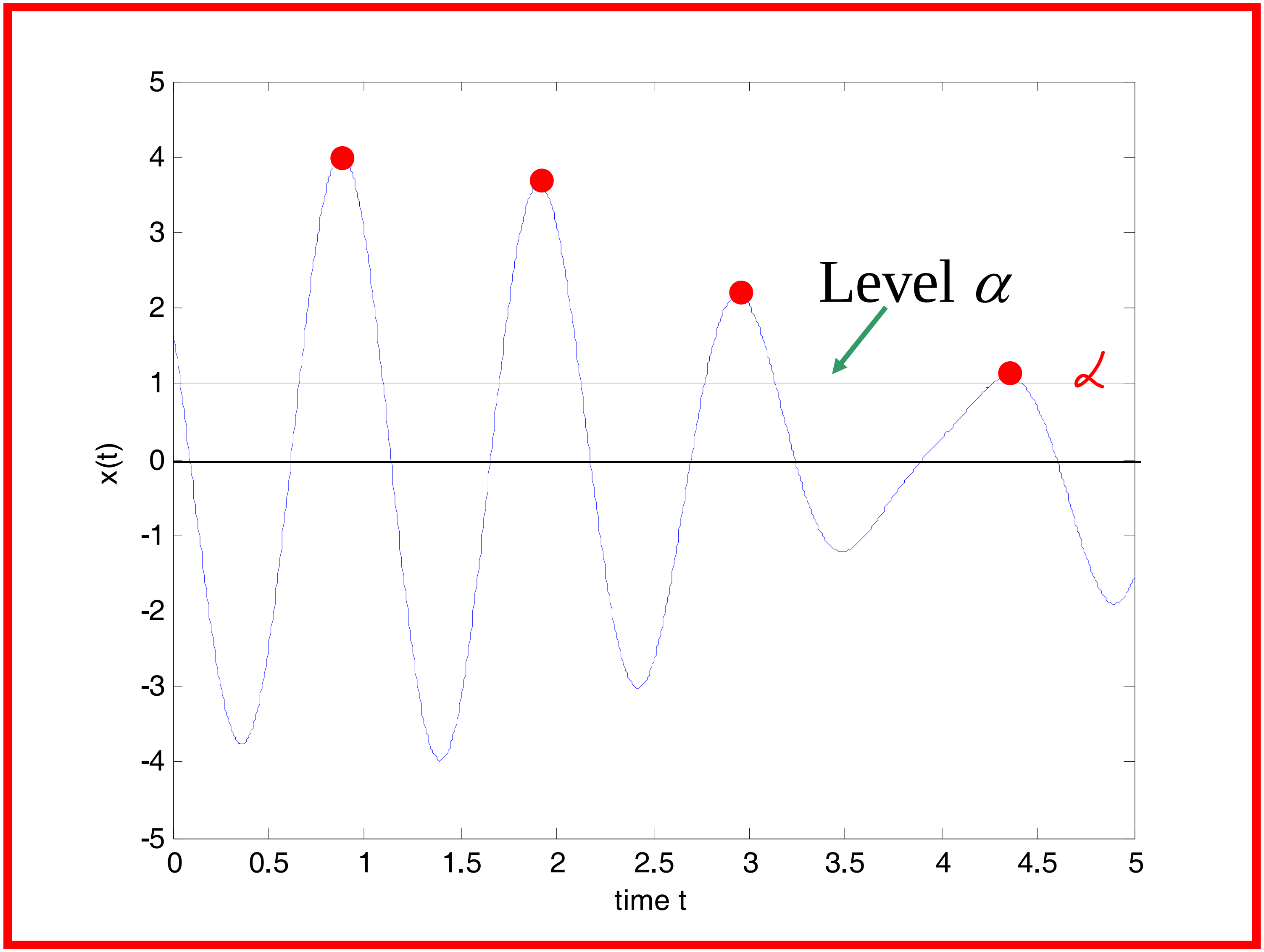


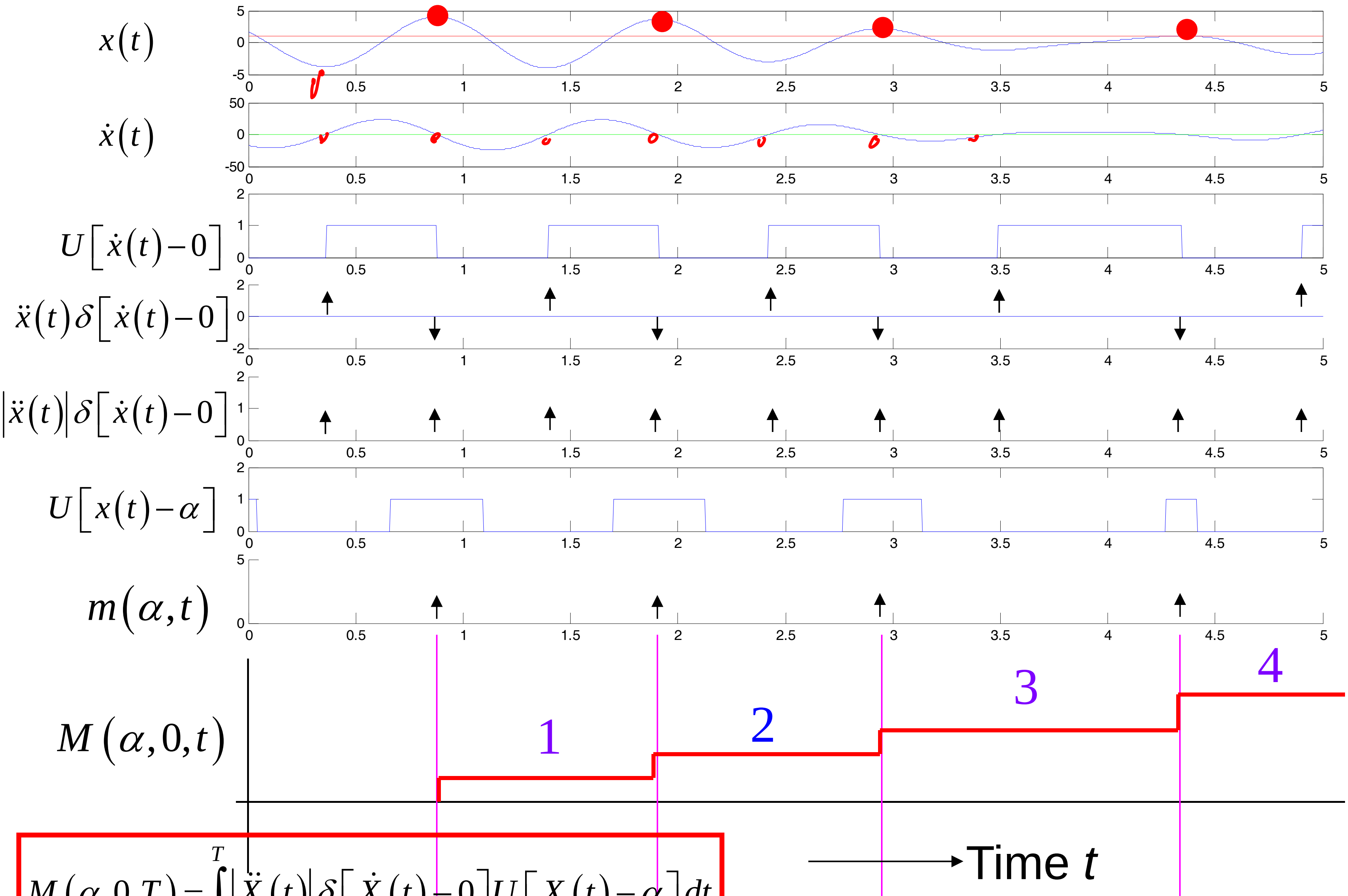
$M(\alpha, 0, T)$

- Number of peaks in $X(t)$ above the level α in 0 to T
- A integer valued random variable
- Given the complete description of $X(t)$,

can we characterize $M(\alpha, 0, T)$?

- This is the problem of determing peak statistics.


Level α
α
x(t)
time t
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



$$M(\alpha,0,T)=\int_0^T\left|\ddot{X}(t)\right|\delta\left[\dot{X}(t)-0\right]U\left[X(t)-\alpha\right]dt$$

$$M(\alpha,0,T)=\int_0^T\left|\ddot{X}(t)\right|\delta\left[\dot{X}(t)-0\right]U\left[X(t)-\alpha\right]dt$$

$$=\int_0^T m(\alpha,t)\,dt$$

$$m(\alpha,t)=\left|\ddot{X}(t)\right|\delta\left[\dot{X}(t)-0\right]U\left[X(t)-\alpha\right]$$

Remarks

- For a fixed value of T, $M(0,\alpha,T)$ is an integer valued random variable
- $m(\alpha,t)=$ rate of peaks above level α
- $m(\alpha,t)$ is a random variable for a fixed value ot t
- Finding PDF of $M(0,\alpha,T)$ or $m(\alpha,t)$ is difficult
- We can try finding moments

Mean value

$$M(\alpha,0,T)=\int_0^T m(\alpha,t)\,dt$$

$$\langle M(\alpha,0,T)\rangle=\int_0^T \langle m(\alpha,t)\rangle\,dt$$

$$\langle m(\alpha,t)\rangle=\left\langle \left|\ddot{X}(t)\right|\delta\left[\dot{X}(t)-0\right]U\left[X(t)-\alpha\right]\right\rangle$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\ddot{x}|\,\delta[\dot{x}-0]\,U[x-\alpha]\,p_{X\dot{X}\ddot{X}}(x,\dot{x},\ddot{x};t)\,dx\,d\dot{x}\,d\ddot{x}$$

$$=\int_{\alpha}^{\infty}\int_{-\infty}^{\infty}|\ddot{x}|\,p_{X\dot{X}\ddot{X}}(x,0,\ddot{x};t)\,dx\,d\ddot{x}$$

Mean square value

$$\left\langle M^2(\alpha,0,T)\right\rangle = \int_0^T\int_0^T \left\langle m(\alpha,t_1)\,m(\alpha,t_2)\right\rangle dt_1 dt_2$$

$$\left\langle m(\alpha,t_1)\,m(\alpha,t_2)\right\rangle = \int_\alpha^\infty\int_\alpha^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty |\ddot{x}_1||\ddot{x}_2|\,p_{X\dot{X}\ddot{X}X\dot{X}\ddot{X}}\left(x_1,0,\ddot{x}_1,x_2,0,\ddot{x}_2;t_1,t_2\right) dx_1 d\ddot{x}_1 dx_2 d\ddot{x}_2$$

Remarks

- Suppose we are interested only in peaks (i.e., maxima), we need to restrict 2^{nd} derivative to take negative values.

$$\Rightarrow \left\langle m(\alpha,t)\right\rangle = \int\limits_{\substack{\alpha \\ (x)}}^{\infty} \int\limits_{\substack{-\infty \\ (\ddot{x})}}^{0} |\ddot{x}| p_{X\dot{X}\ddot{X}}(x,0,\ddot{x};t)\,dx\,d\ddot{x}$$

- Average rate of extrema in $x(t)$ (i.e., $\alpha = -\infty$) = Average rate of zero crossings in $\dot{x}(t)$.

$\Rightarrow$

$$\left\langle m(-\infty,t)\right\rangle = \int\limits_{\substack{-\infty \\ (x)}}^{\infty} \int\limits_{\substack{-\infty \\ (\ddot{x})}}^{\infty} |\ddot{x}| p_{X\dot{X}\ddot{X}}(x,0,\ddot{x};t)\,dx\,d\ddot{x}$$

$$= \int_{-\infty}^{\infty} |\ddot{x}| p_{\dot{X}\ddot{X}}(0,\ddot{x};t)\,d\ddot{x}$$

$$= \left\langle n_{\dot{x}}(0,t)\right\rangle \qquad \text{(ok)}$$

Example

Let $X(t)$ be a stationary Gaussian random process with zero mean. Determine $\langle M(\alpha, 0, T) \rangle$.

We need $p_{X\dot{X}\ddot{X}}(x, \dot{x}, \ddot{x}; t)$.

Recall

$$p_{\tilde{X}}(\tilde{x}; \tilde{t}) = \frac{1}{(2\pi)^{\frac{n}{2}} |S|^{\frac{1}{2}}} \exp\left[-\frac{1}{2} \tilde{x}^t S^{-1} \tilde{x}\right]; -\infty < x_i < \infty; i = 1, 2, \cdots, n$$

$$S = \begin{bmatrix} \langle X^2(t) \rangle & \langle X(t)\dot{X}(t) \rangle & \langle X(t)\ddot{X}(t) \rangle \\ \langle \dot{X}(t) X(t) \rangle & \langle \dot{X}^2(t) \rangle & \langle \dot{X}(t)\ddot{X}(t) \rangle \\ \langle \ddot{X}(t) X(t) \rangle & \langle \ddot{X}(t)\dot{X}(t) \rangle & \langle \ddot{X}^2(t) \rangle \end{bmatrix}$$

$X(t)$ is stationary $\Rightarrow$

$$\left\langle X(t)\dot{X}(t)\right\rangle = 0 \;\&\; \left\langle \dot{X}(t)\ddot{X}(t)\right\rangle = 0$$

$$\left\langle \frac{d^n X(t)}{dt^n}\frac{d^m X(t+\tau)}{dt^m}\right\rangle = (-1)^m \frac{d^{m+n}R_{XX}(\tau)}{d\tau^{m+n}}$$

$$\left\langle X(t)\ddot{X}(t+\tau)\right\rangle = \frac{d^2 R_{XX}(\tau)}{d\tau^2} = -\left\langle \dot{X}(t)\dot{X}(t+\tau)\right\rangle$$

$$\sigma_1^2 = \left\langle X^2(t)\right\rangle; \sigma_2^2 = \left\langle \dot{X}^2(t)\right\rangle; \sigma_3^2 = \left\langle \ddot{X}^2(t)\right\rangle;$$

$$S = \begin{bmatrix} \sigma_1^2 & 0 & -\sigma_2^2 \\ 0 & \sigma_2^2 & 0 \\ -\sigma_2^2 & 0 & \sigma_3^2 \end{bmatrix}$$

$$|S| = \sigma_1^2\sigma_2^2\sigma_3^2 + \sigma_3^2\sigma_2^4$$ ✓

$$= \sigma_3^2\left(\sigma_1^2\sigma_2^2 + \sigma_2^4\right)$$ ✓

$$S^{-1} = \frac{1}{|S|}\begin{bmatrix} \sigma_2^2\sigma_3^2 & 0 & \sigma_2^4 \\ 0 & \sigma_1^2\sigma_3^2 + \sigma_2^4 & 0 \\ \sigma_2^4 & 0 & \sigma_1^2\sigma_2^2 \end{bmatrix}$$

$$x^T S^{-1} x = \left\{ x \quad 0 \quad \ddot{x} \right\} \frac{1}{|S|}\begin{bmatrix} \sigma_2^2\sigma_3^2 & 0 & \sigma_2^4 \\ 0 & \sigma_1^2\sigma_3^2 + \sigma_2^4 & 0 \\ \sigma_2^4 & 0 & \sigma_1^2\sigma_2^2 \end{bmatrix} \begin{Bmatrix} x \\ 0 \\ \ddot{x} \end{Bmatrix}$$

$$= \frac{1}{|S|}\left(\sigma_2^2\sigma_3^2 x^2 + 2\sigma_2^4 x\ddot{x} + \sigma_1^2\sigma_2^2\ddot{x}^2\right)$$

$$p_{X\dot{X}\ddot{X}}\left(x, 0, \ddot{x}; t\right) = \frac{1}{(2\pi)^{\frac{3}{2}}|S|} \exp\left[-\frac{1}{2|S|}\left(\sigma_2^2\sigma_3^2 x^2 + 2\sigma_2^4 x\ddot{x} + \sigma_1^2\sigma_2^2\ddot{x}^2\right)\right]$$

$$p_{X\dot{X}\ddot{X}}\left(x,0,\ddot{x};t\right)=\frac{1}{(2\pi)^{\frac{3}{2}}\left|S\right|}\exp\left[-\frac{1}{2\left|S\right|}\left(\sigma_2^2\sigma_3^2x^2+2\sigma_2^4x\ddot{x}+\sigma_1^2\sigma_2^2\ddot{x}^2\right)\right]$$

$$\left\langle m\left(\alpha,t\right)\right\rangle=\int_{\alpha}^{\infty}\int_{-\infty}^{\infty}\left|\ddot{x}\right|p_{X\dot{X}\ddot{X}}\left(x,0,\ddot{x};t\right)dxd\ddot{x}$$

$$=\frac{1}{(2\pi)^{\frac{3}{2}}\sigma_1^2\sigma_2^2}\int_{\alpha}^{\infty}[\left|S\right|^{\frac{1}{2}}\exp\left(-\frac{\sigma_2^2\sigma_3^2x^2}{2\left|S\right|}\right)+\frac{\sigma_2^3}{\sigma_1}x\sqrt{\frac{\pi}{2}}\exp\left(-\frac{x^2}{2\sigma_1^2}\right)\left\{1+erf\left(\frac{\sigma_3^2x}{\sigma_1\sqrt{2\left|S\right|}}\right)\right\}]dx$$

$$\left\langle M\left(\alpha,0,T\right)\right\rangle=\int_0^T\left\langle m\left(\alpha,t\right)\right\rangle dt=T\left\langle m\left(\alpha,t\right)\right\rangle$$

Remarks

• $\langle m(\alpha,t)\rangle$ here is not a function of time t.

• For α=0 further simplifications are possible

• If $X(t)$ is nonstationary, the matrix S would be fully populated and expressions for $\langle m(\alpha,t)\rangle$ differs. This expression can be obtained by using the approach similar to the one outlined just now.

In this case $\langle m(\alpha,t)\rangle$ would be a function of time t and evaluation of $\langle M(\alpha,0,T)\rangle$ would be more involved.

• $\sigma_1^2, \sigma_2^2, \& \sigma_3^2$ can be expressed in terms of the PSD of $X(t)$ as

$$\sigma_1^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega; \quad \sigma_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S(\omega)d\omega; \quad \sigma_3^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^4 S(\omega)d\omega$$

Approximate evaluation of pdf of peaks above level α in 0 to T

$$P(\text{peak} \le \alpha) = 1 - P(\text{peak} > \alpha)$$

$$P(\text{peak} > \alpha) = \frac{\text{Number of peaks above level } \alpha \text{ in 0 to T}}{\text{Total number of peaks above } \sout{\text{level } \alpha \text{ in 0 to T}} \text{ 0 to T}}$$

$$= \frac{M(\alpha,0,T)}{M(-\infty,0,T)}$$

$$\approx \left\langle \frac{M(\alpha,0,T)}{M(-\infty,0,T)} \right\rangle$$

$$\approx \frac{\langle M(\alpha,0,T)\rangle}{\langle M(-\infty,0,T)\rangle}$$

$$= \frac{\langle m(\alpha,t)\rangle T}{\langle m(-\infty,t)\rangle T} = \frac{\langle m(\alpha,t)\rangle}{\langle m(-\infty,t)\rangle}$$

$$P\left(\text{peak} > \alpha\right) = \frac{\left\langle m(\alpha, t)\right\rangle}{\left\langle m(-\infty, t)\right\rangle}$$

$$= \frac{\dfrac{1}{(2\pi)^{\frac{3}{2}}\sigma_1^2\sigma_2^2}\displaystyle\int_\alpha^\infty [|S|^{\frac{1}{2}}\exp\left(-\frac{\sigma_2^2\sigma_3^2x^2}{2|S|}\right) + \frac{\sigma_2^3}{\sigma_1}x\sqrt{\frac{\pi}{2}}\exp\left(-\frac{x^2}{2\sigma_1^2}\right)\left\{1 + erf\left(\frac{\sigma_3^2 x}{\sigma_1\sqrt{2|S|}}\right)\right\}]dx}{\dfrac{1}{2\pi}\dfrac{\sigma_3}{\sigma_2}}$$

$$\text{Define } \varepsilon = \frac{\text{Average number of zero crossings with } +^{ve} \text{ slope per unit time}}{\text{Average number of peaks per unit time}}$$

$$= \frac{\left\langle n^+(0,t)\right\rangle}{\left\langle m(0,t)\right\rangle} = \frac{\dfrac{1}{2\pi}\dfrac{\sigma_2}{\sigma_1}}{\dfrac{1}{2\pi}\dfrac{\sigma_3}{\sigma_3}} = \frac{\sigma_2^2}{\sigma_1\sigma_3} = \text{Band width parameter}$$

Band width parameter $\varepsilon = \dfrac{\sigma_2^2}{\sigma_1 \sigma_3}$

Remarks

- $0 \le \varepsilon \le 1$
- $X(t)$ is broad banded $\Rightarrow \varepsilon = 0$
- $X(t)$ is narrow banded $\Rightarrow \varepsilon = 1$
- $\varepsilon = 0$ need not mean that $X(t)$ is broad banded
- $\varepsilon = 1$ need not mean that $X(t)$ is narrow banded
- ε can be expressed in terms of the spectral moments

$$\sigma_1^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega; \quad \sigma_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S(\omega)d\omega; \quad \sigma_3^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^4 S(\omega)d\omega$$

$$p_p(\alpha)=\frac{\left(1-\varepsilon^2\right)^{\frac{1}{2}}}{\sqrt{2\pi}\sigma_1}\exp\left[-\frac{\alpha^2}{2\sigma_1^2\sqrt{2\left(1-\varepsilon^2\right)}}\right]+\frac{\varepsilon\alpha}{2\sigma_1^2}\left\{1+erf\left(\frac{\varepsilon\alpha}{\sigma_1\sqrt{2\left(1-\varepsilon^2\right)}}\right)\right\}$$

Remarks

- For broad banded process $(\varepsilon=0)$ one gets

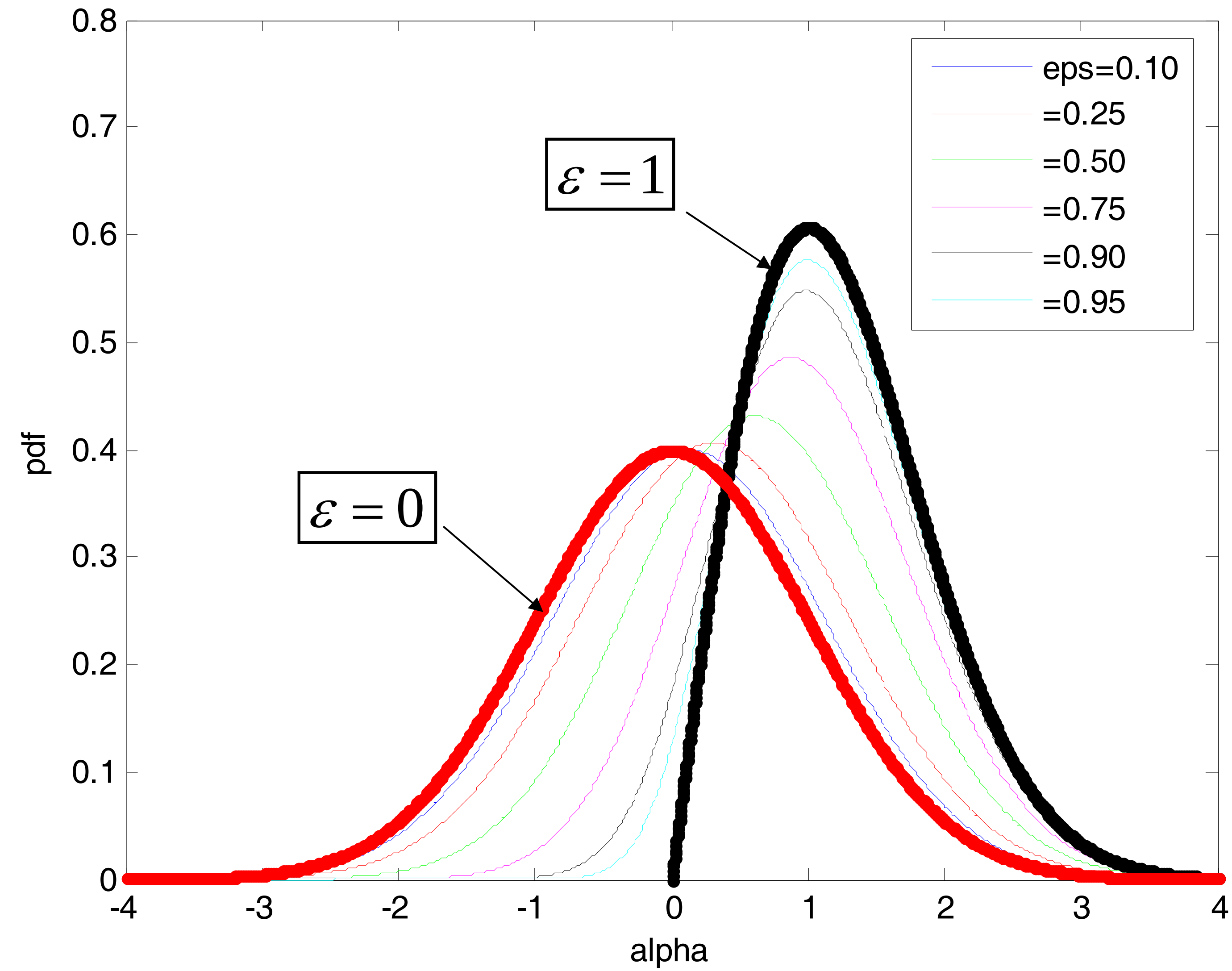
$$p_p(\alpha)=\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{\alpha^2}{2\sigma_1^2}\right);-\infty<\alpha<\infty$$

Here we get a Gaussian model.

- For narrow banded process $(\varepsilon=1)$ one gets

$$p_p(\alpha)=\frac{\alpha}{\sigma_1^2}\exp\left(-\frac{\alpha^2}{2\sigma_1^2}\right);0<\alpha<\infty$$

Here we get a Rayleigh model and this agrees with the result obtained earlier.



0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
pdf
-4
-3
-2
-1
0
1
2
3
4
alpha
eps=0.10
=0.25
=0.50
=0.75
=0.90
=0.95
$\varepsilon = 1$
$\varepsilon = 0$

Fractional occupation time

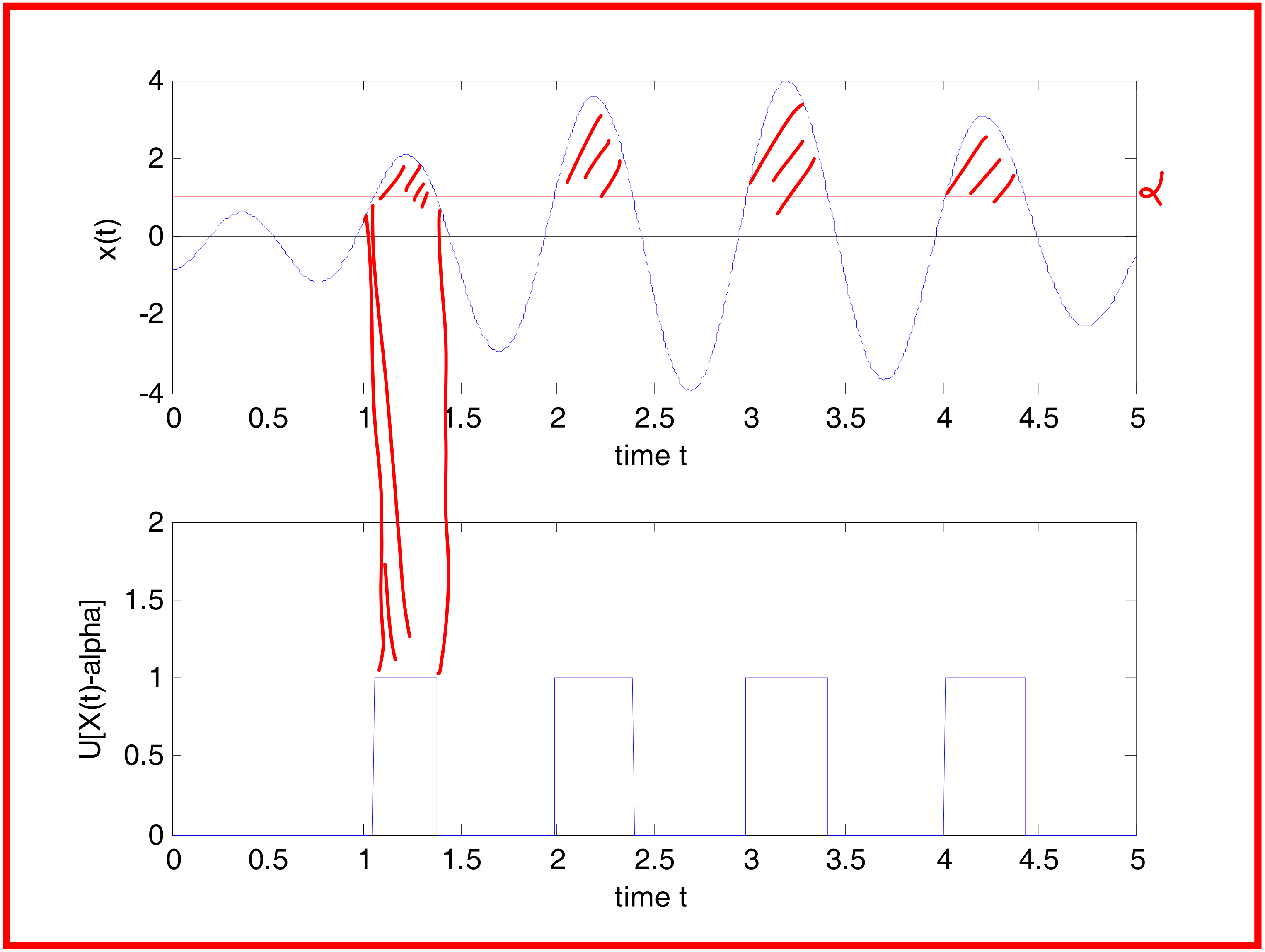
$\Gamma\left(\alpha,0,T\right)$

- Time spent by $X(t)$ above the level α in 0 to T
- A real valued random variable
- Given the complete description of $X(t)$,

can we characterize $\Gamma\left(\alpha,0,T\right)$?

- $\dfrac{\Gamma\left(\alpha,0,T\right)}{T}$ is called the fractional occupation time
- This takes values in 0 to 1.
- The problem on hand consists of characterizing the

fractional occupation time.



x(t)
time t
U[X(t)-alpha]
time t

Define

$$y(\alpha,T)=\frac{1}{T}\int_0^T U\left[X(t)-\alpha\right]dt$$

Finding pdf of $y(\alpha,T)$ is difficult.

Can we find its moments?

$$\left\langle y(\alpha,T)\right\rangle=\left\langle\frac{1}{T}\int_0^T U\left[X(t)-\alpha\right]dt\right\rangle$$

$$=\frac{1}{T}\int_0^T\left\langle U\left[X(t)-\alpha\right]\right\rangle dt$$

$$\left\langle U\left[X(t)-\alpha\right]\right\rangle=\int_{-\infty}^{\infty}U(x-\alpha)\,p_X(x;t)\,dx$$

$$\left\langle U\left[X(t)-\alpha\right]\right\rangle = \int_{-\infty}^{\infty} U(x-\alpha)\, p_X(x;t)\, dx$$

$$= \int_{\alpha}^{\infty} p_X(x;t) dx = 1 - \int_{-\infty}^{\alpha} p_X(x;t)$$

Let $X(t)$ be Gaussian process with zero mean.

$$\left\langle U\left[X(t)-\alpha\right]\right\rangle = 1 - \int_{-\infty}^{\alpha} p_X(x;t) dx$$

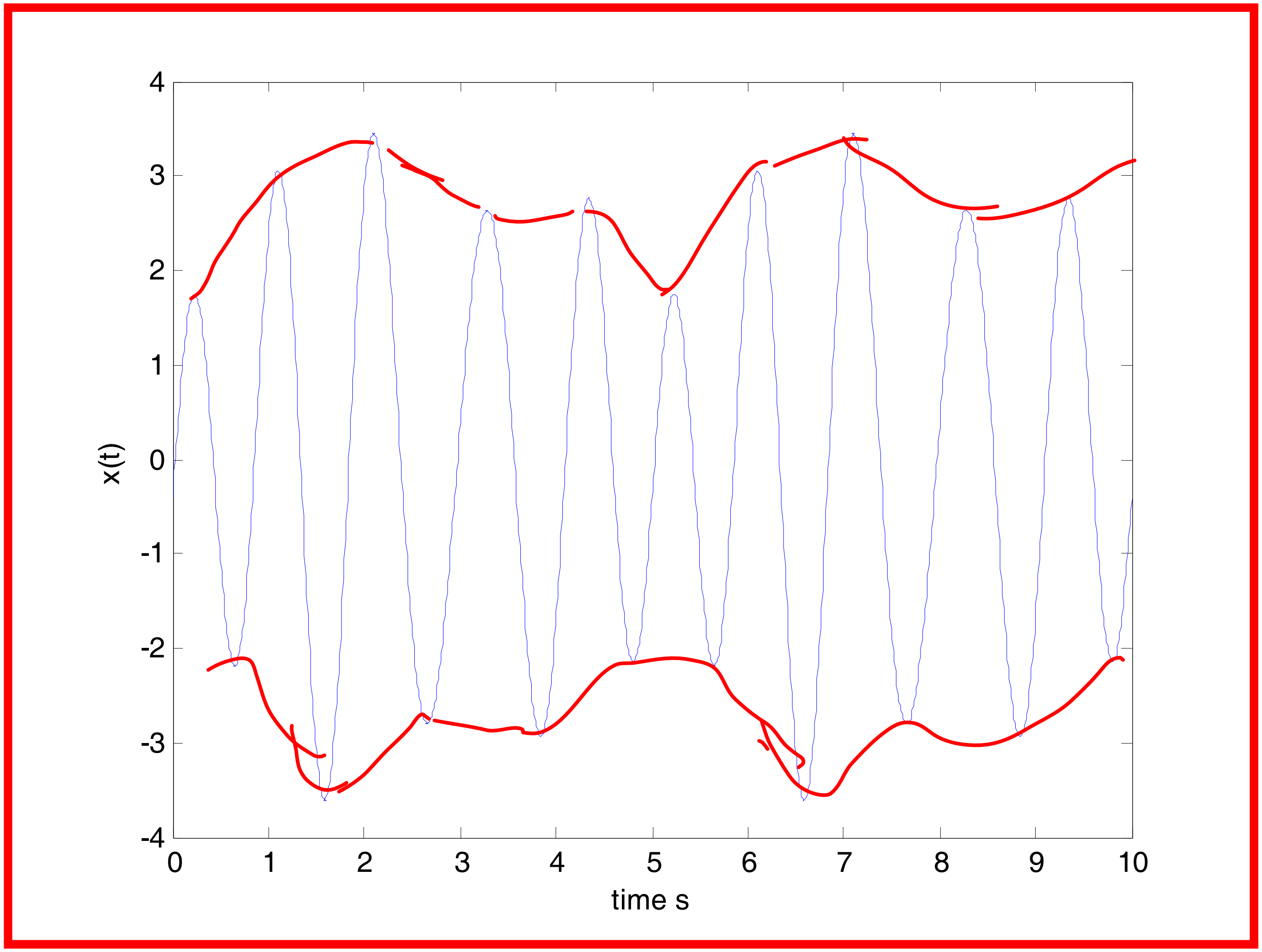
$$= 1 - \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(-\frac{x^2}{2\sigma_x^2}\right) dx$$

$$= \frac{1}{2}\left[1 - erf\left(\frac{\alpha}{\sigma_x}\right)\right]$$

$$\left\langle y(\alpha,t)\right\rangle = \frac{1}{2T}\int_0^T \left[1-erf\left(\frac{\alpha}{\sigma_x(t)}\right)\right]dt$$

$$= \frac{1}{2}\left[1-erf\left(\frac{\alpha}{\sigma_x}\right)\right]$$ [if X(t) is stationary]

Envelope and phase processes



Recall

$$\ddot{x}+\omega^2 x=0$$

$$x(0)=x_0;\dot{x}(0)=\dot{x}_0$$

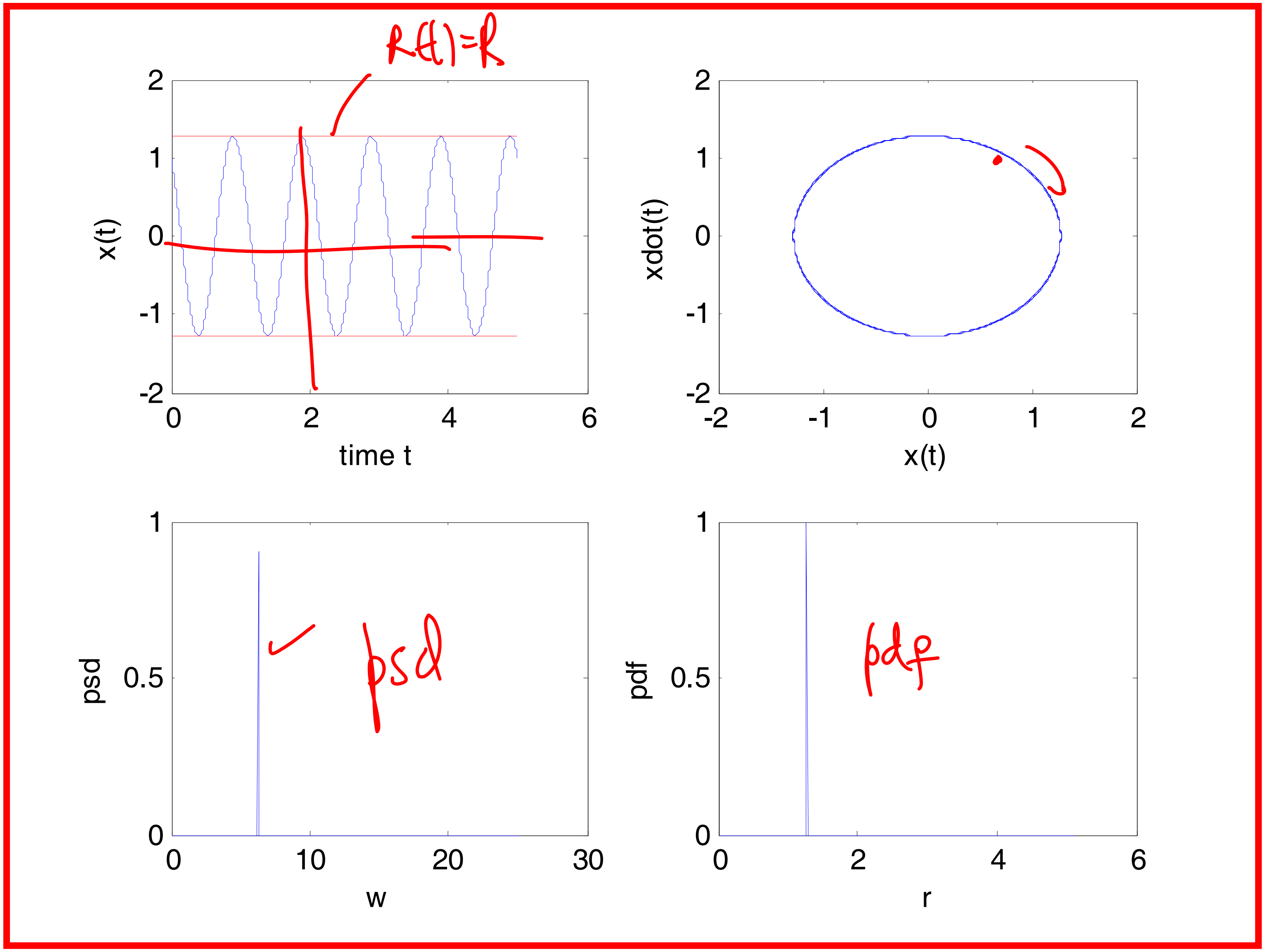
$$x(t)=x_0\cos\omega t+\frac{\dot{x}_0}{\omega}\sin\omega t$$

$$x_0=R\cos\theta;\frac{\dot{x}_0}{\omega}=R\sin\theta$$

$$\Rightarrow x(t)=R\cos(\omega t-\theta)$$

$$R=\sqrt{x_0^2+\left(\frac{\dot{x}_0}{\omega}\right)^2};\theta=\tan^{-1}\left(\frac{\dot{x}_0}{\omega x_0}\right)$$

- $R\geq|x(t)|\,\forall t$


R(t)=R
x(t)
time t
xdot(t)
x(t)
psd
psd
w
pdf
pdf
r

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = 0$$

$$x(0) = x_0; \dot{x}(0) = \dot{x}_0$$

$$x(t) = \exp(-\eta\omega t)\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

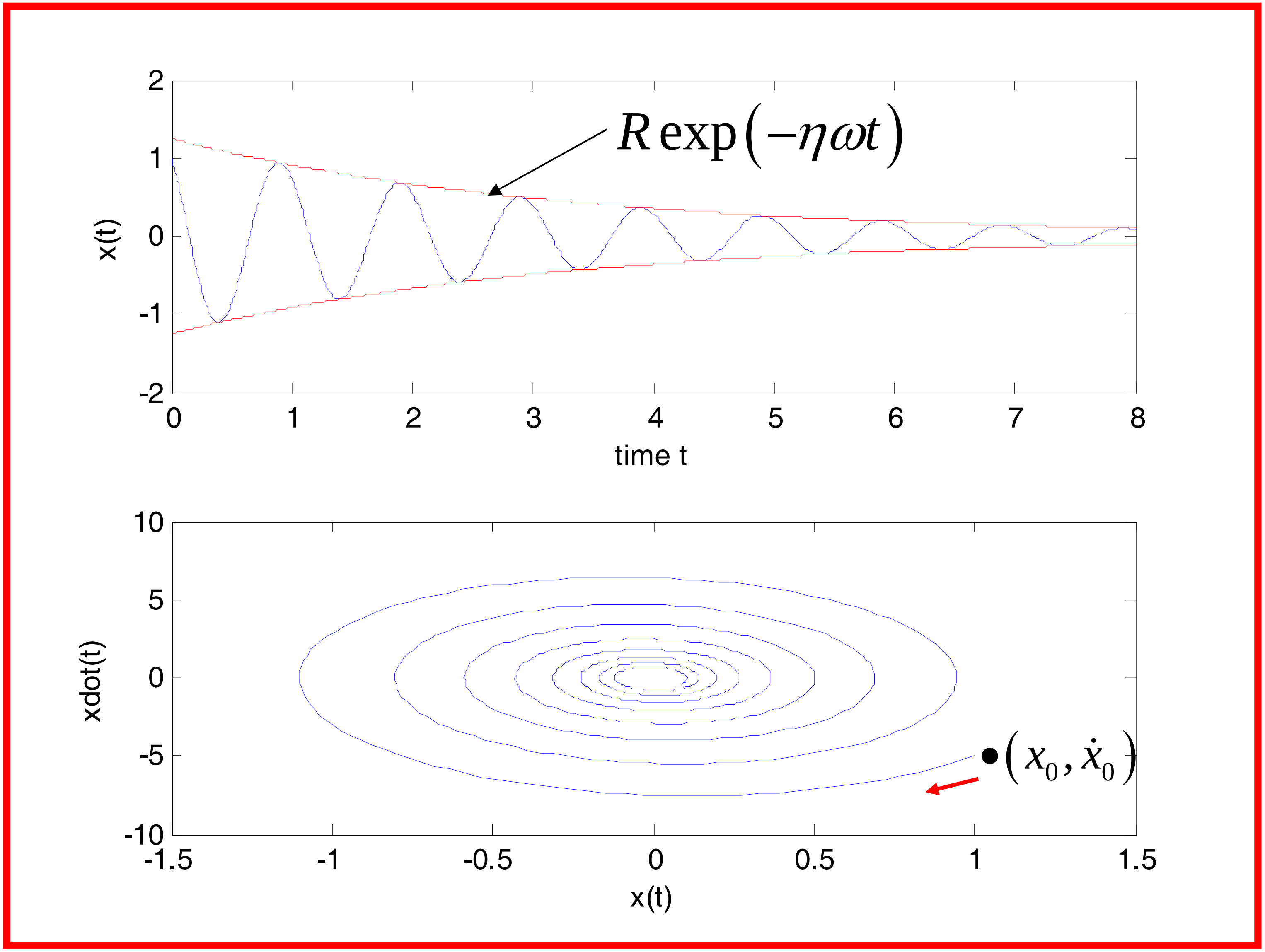
$$\dot{x}(t) = -\eta\omega\exp(-\eta\omega t)\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

$$+\exp(-\eta\omega t)\left(-\omega_d A\sin\omega_d t + \omega_d B\cos\omega_d t\right)$$

$$A = x_0; B = \frac{\dot{x}_0 + \eta\omega x_0}{\omega_d}$$

$$A = R\cos\theta; B = R\sin\theta$$

$$x(t) = \exp(-\eta\omega t) R\cos\left(\omega_d t - \theta\right)$$


$R\exp(-\eta\omega t)$
x(t)
time t
0
1
2
3
4
5
6
7
8
2
1
0
-1
-2
xdot(t)
x(t)
10
5
0
-5
-10
-1.5
-1
-0.5
0
0.5
1
1.5
$(x_0, \dot{x}_0)$

$$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x = \frac{P}{m}\cos\lambda t$$

$$x(0) = x_0; \dot{x}(0) = \dot{x}_0$$

$$\lim_{t\to\infty} x(t) = X_{st}(DMF)\cos(\omega_d t - \theta)$$

$$X_{st} = \frac{P}{k}; DMF = \frac{1}{\sqrt{(1-r^2)^2 + (2\eta r)^2}}$$