Stochastic Structural Dynamics

Lecture-17

Failure of randomly vibrating systems-1

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Input-output relations for randomly driven systems

- Time and frequency domain analyses
- Stationary and non-stationary excitations
- Response mean, covariance & psd functions
  - SDOF systems
  - Discrete MDOF systems
  - Distributed parameter systems

pdf of the response process

$$m\ddot{x}+c\dot{x}+kx=f(t); x(0)=0; \dot{x}(0)=0$$

Let $f(t)$ be a zero mean Gaussian random process

$\Rightarrow$ $x(t)$ is also a Gaussian random process.

$\Rightarrow$

$$p_x(x;t)=\frac{1}{\sqrt{2\pi\sigma_x^2(t)}}\exp\left[-\frac{1}{2}\left\{\frac{x-m_x(t)}{\sigma_x(t)}\right\}^2\right]; -\infty<x<\infty$$

$$p_{xx}(x_1,x_2;t_1,t_2)\sim N\left[0 \quad \begin{bmatrix} R_{xx}(t_1,t_1) & R_{xx}(t_1,t_2) \\ R_{xx}(t_1,t_2) & R_{xx}(t_2,t_2) \end{bmatrix}\right]$$

$\vdots$

$$p_{\tilde{x}}(\tilde{x};\tilde{t})\sim N\left[0 \quad [\boldsymbol{R}]\right]$$

Problem of reliability analysis

$$P\left[x(t)\le\alpha\,\forall t\in(0,T)\right]=?$$

Select $\{t_i\}_{i=1}^{n}\in(0,T)$ such that $t_i=i\Delta t$ and $n\Delta t=T$.

Question: can we approximate the given probability by $\iint\cdots\int p_{\tilde{x}}(\tilde{x};\tilde{t})\,d\tilde{x}$ where the integration is carried over the region $\Omega=(x_1\le\alpha)\cap(x_2\le\alpha)\cap\cdots(x_n\le\alpha)$?

Not quite!

May be yes, as $n \to \infty$.

Even if this were to be acceptable, we still need to evaluate a multi-fold integral (with dimension $= n$ and set to become large) which by no means is a simple task.

How to proceed?

We need newer descriptions of $x(t)$.

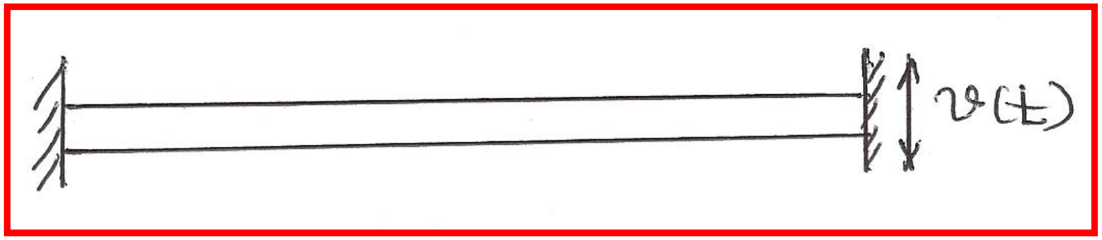
Failure of randomly driven vibrating systems

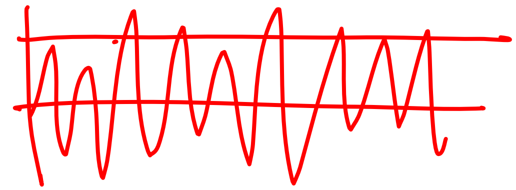
First passage failure
Response level exceeds a permissible threshold for the first time.

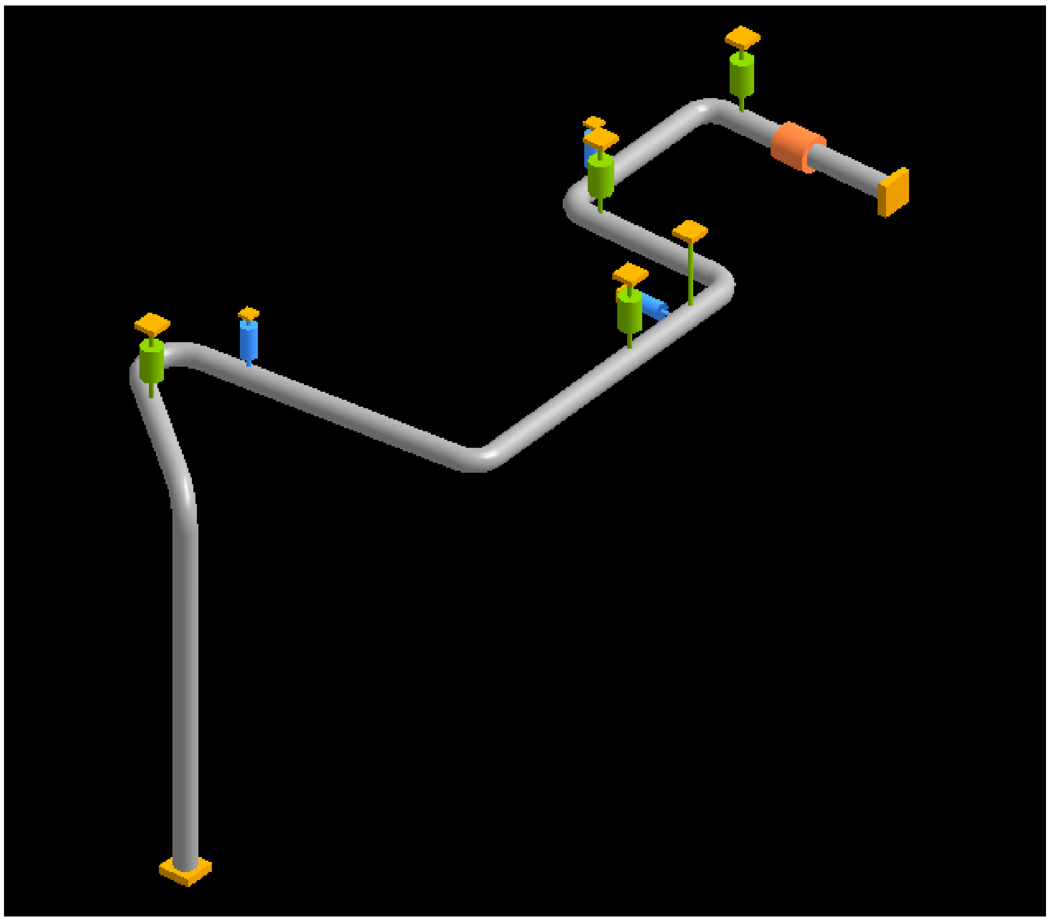
Fatigue failure
The accumulated fatigue damage exceeds a threshold value.

Loss of stability
The structure loses its stability under parametric random excitations

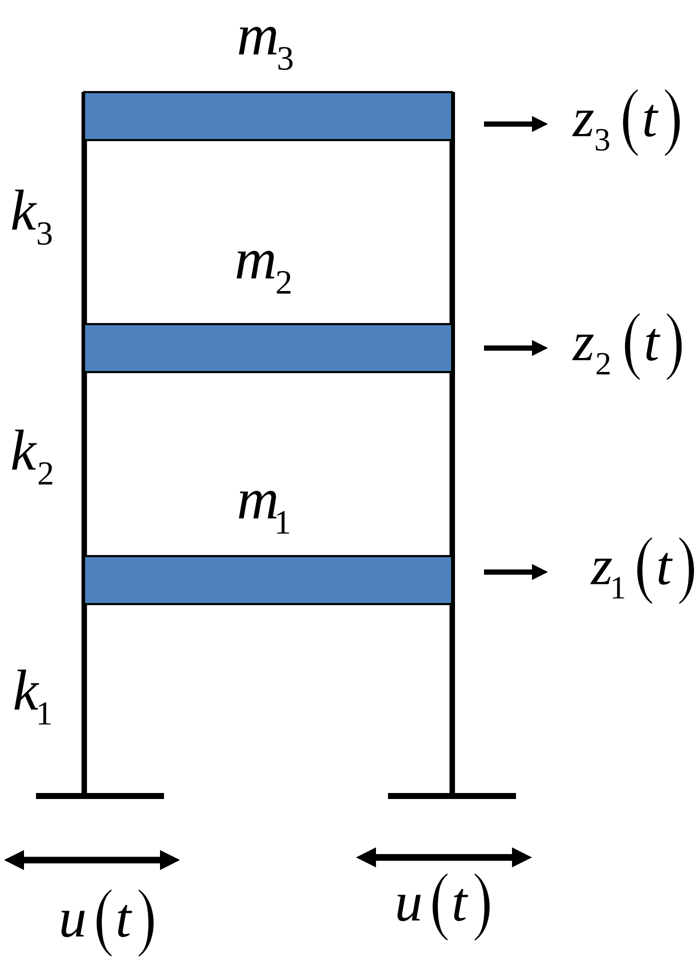
Starting point: the description of response process of interest has been obtained by using appropriate input-output relations (possibly based on the application of FEM)

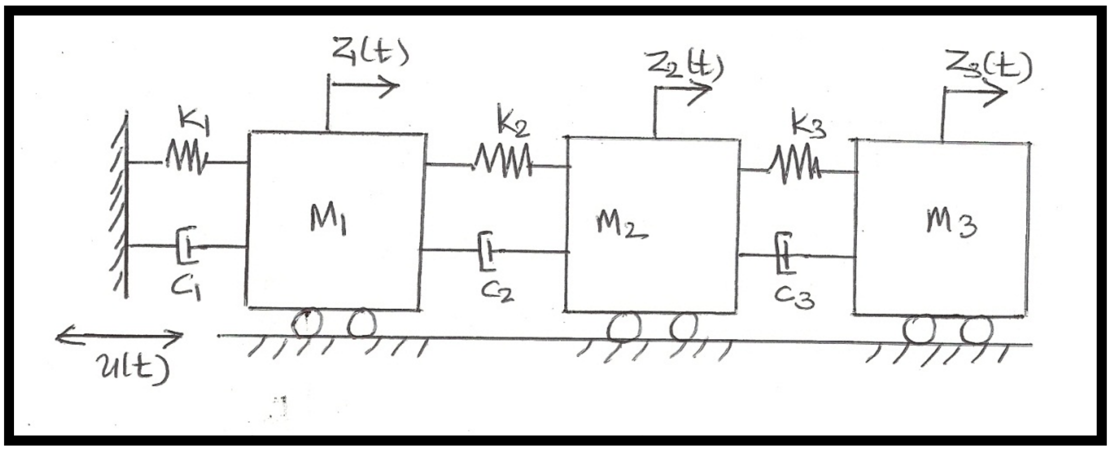

v(t)



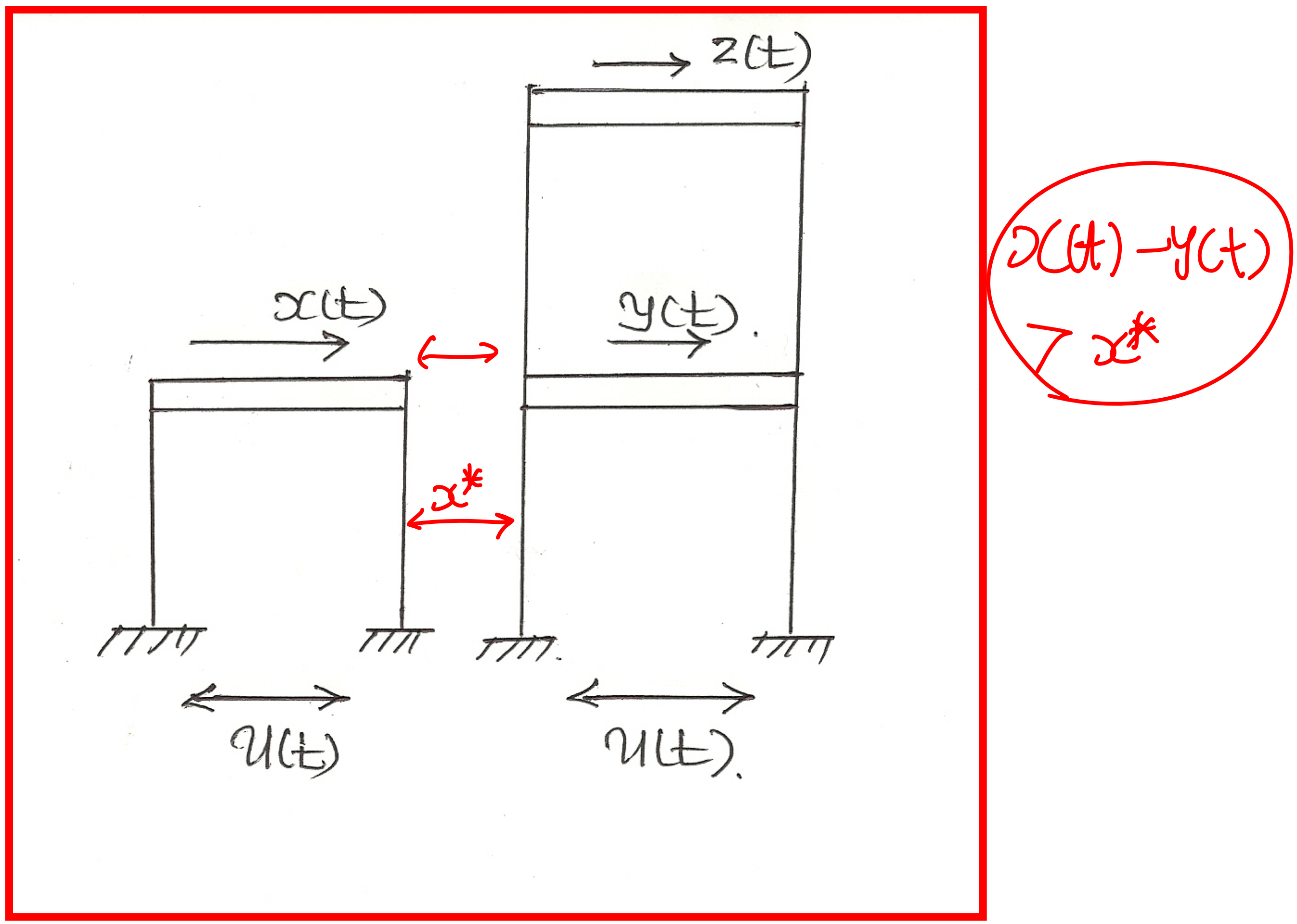


A building frame under random support motion





Pounding of adjacent buildings

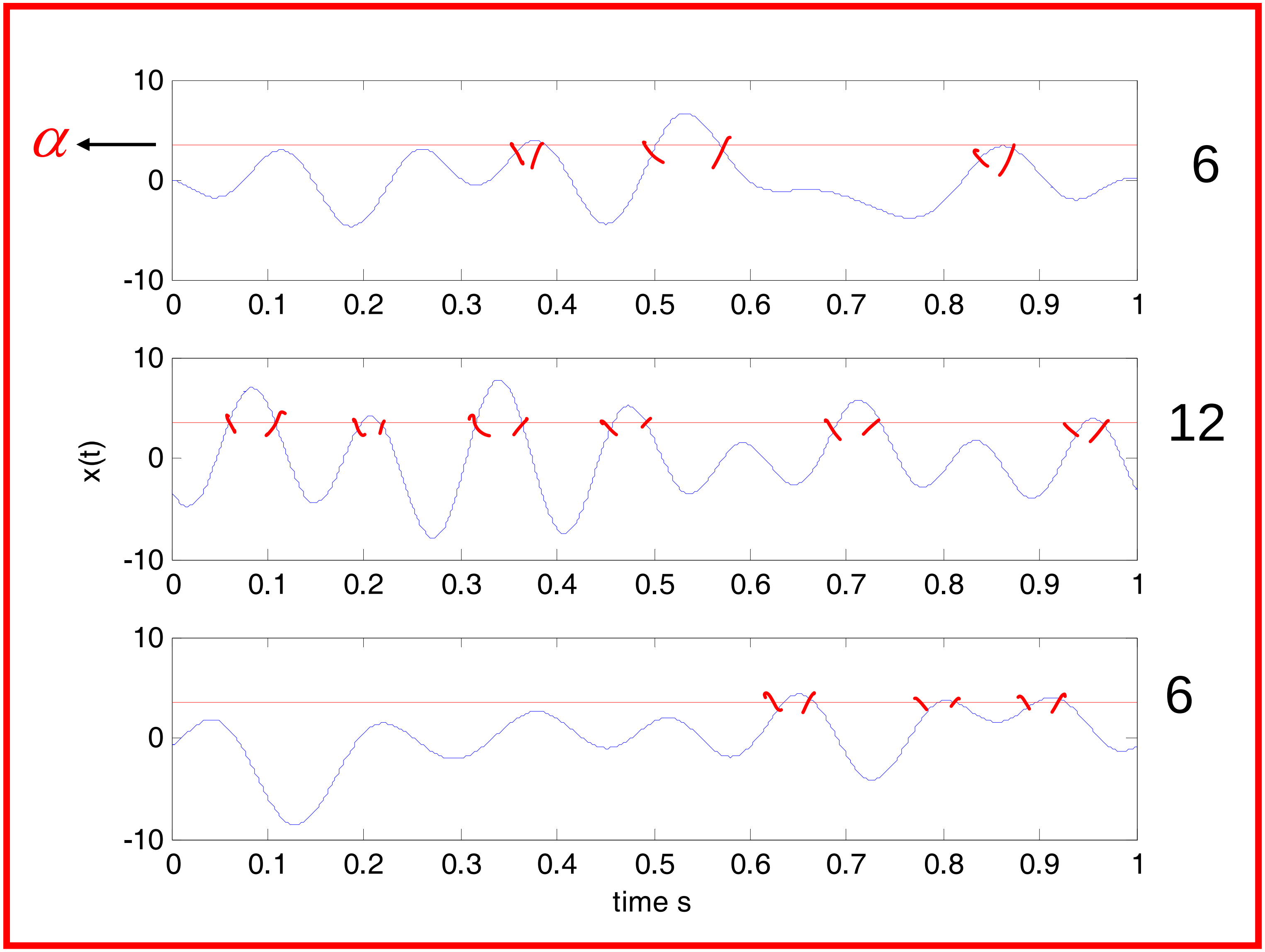


Fatigue damage accumulation

Descriptions

- The number of times the level α is crossed by $X(t)$ in interval 0 to T.
- The number of peaks of $X(t)$ above level α in interval 0 to T.
- Total time spent by $X(t)$ above level α in interval 0 to T.
- Time required by $X(t)$ to reach level α for the first time.
- The envelope and phase processes associated with $X(t)$
- The maximum value of $X(t)$ in interval 0 to T

The number of times the level α is crossed by $X(t)$ in interval 0 to T



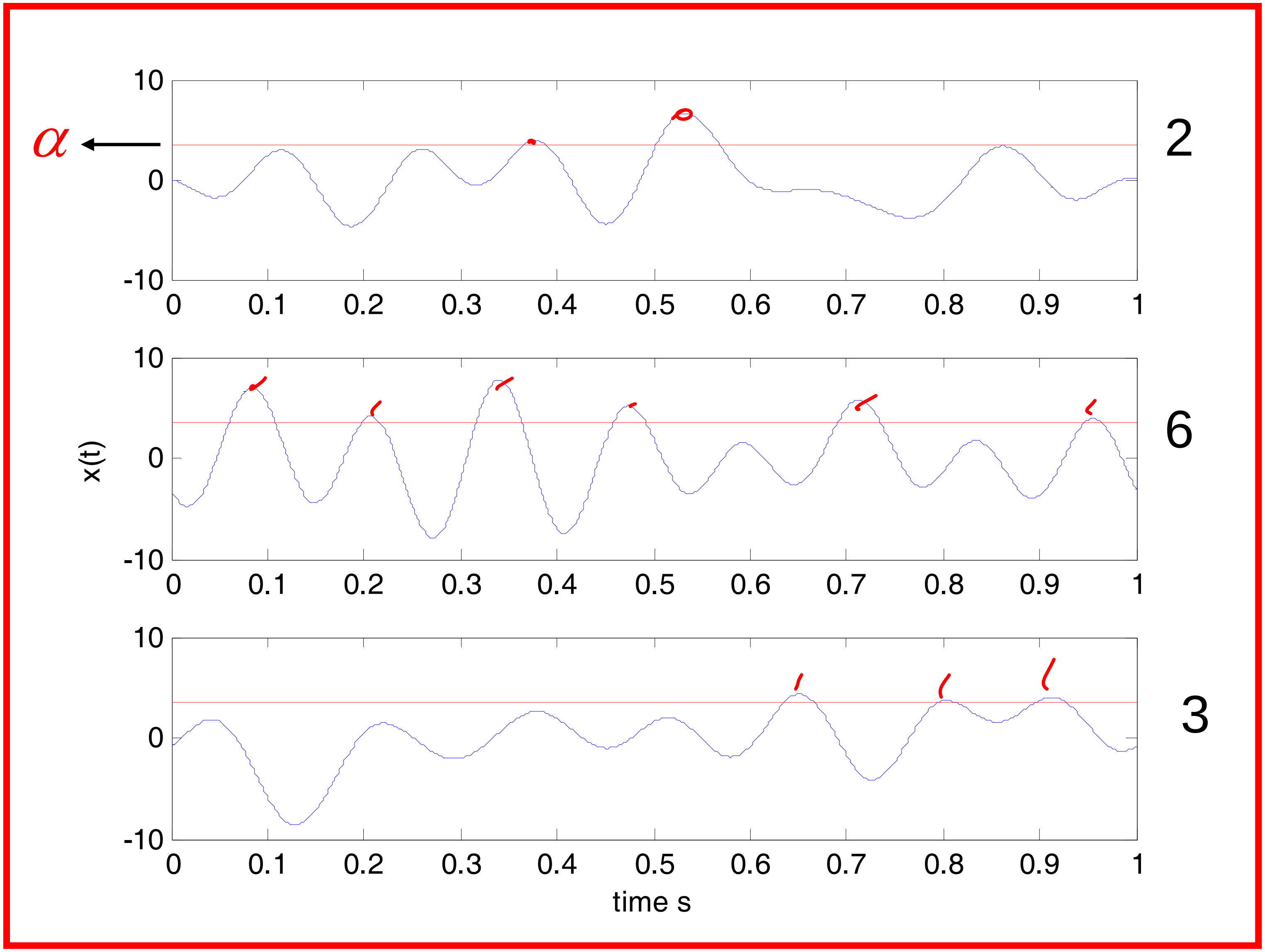
$N(\alpha, 0, T)$

- Number of times $X(t)$ crosses α in 0 to T
- A integer valued random variable
- Given the complete description of X(t),

can we characterize $N(\alpha, 0, T)$?

- This is known as the level crossing problem.

The number of peaks of $X(t)$ above level α in interval 0 to T



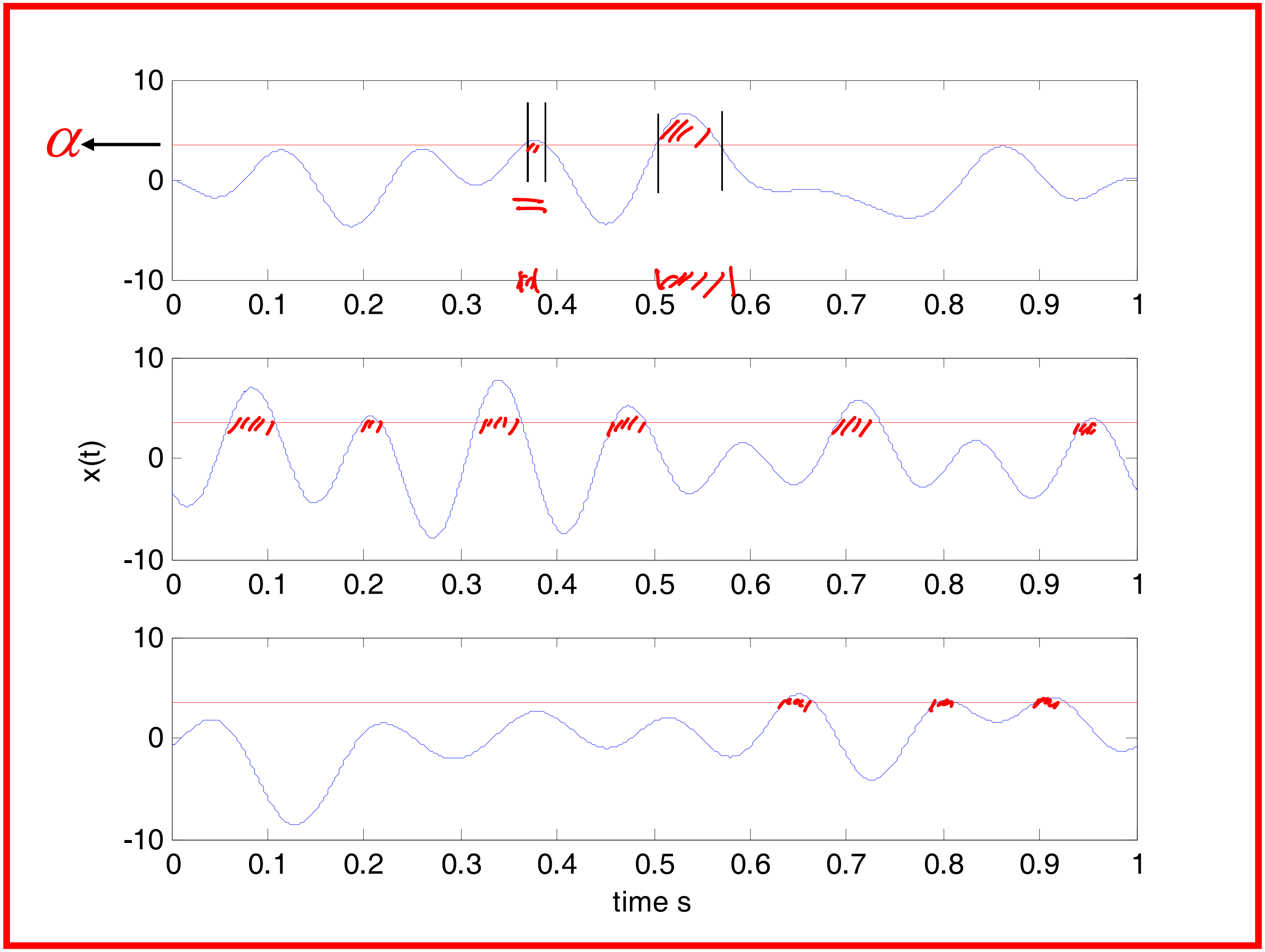
$M(\alpha, 0, T)$

- Number of peaks in $X(t)$ above the level α in 0 to T
- A integer valued random variable
- Given the complete description of $X(t)$,

can we characterize $M(\alpha, 0, T)$?

- This is the problem of determing peak statistics.

Total time spent by $X(t)$ above level α in interval 0 to T.



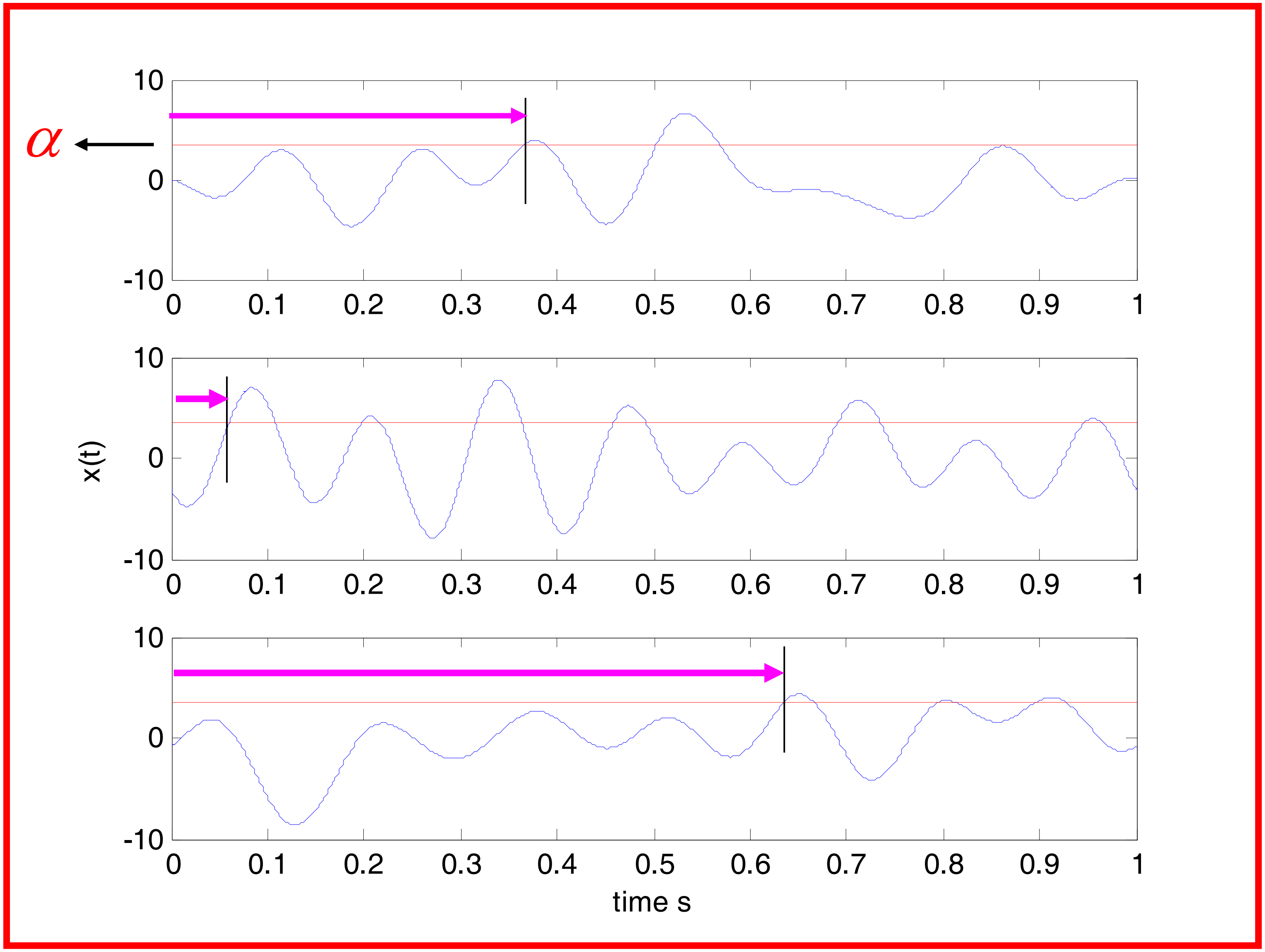
$\Gamma\left(\alpha,0,T\right)$

- Time spent by $X(t)$ above the level α in 0 to T
- A real valued random variable
- Given the complete description of $X(t)$,

can we characterize $\Gamma\left(\alpha,0,T\right)$?

- $\dfrac{\Gamma\left(\alpha,0,T\right)}{T}$ is called the fractional occupation time
- This takes values in 0 to 1.
- The problem on hand consists of characterizing the fractional occupation time.

Time required by $X(t)$ to reach level α for the first time



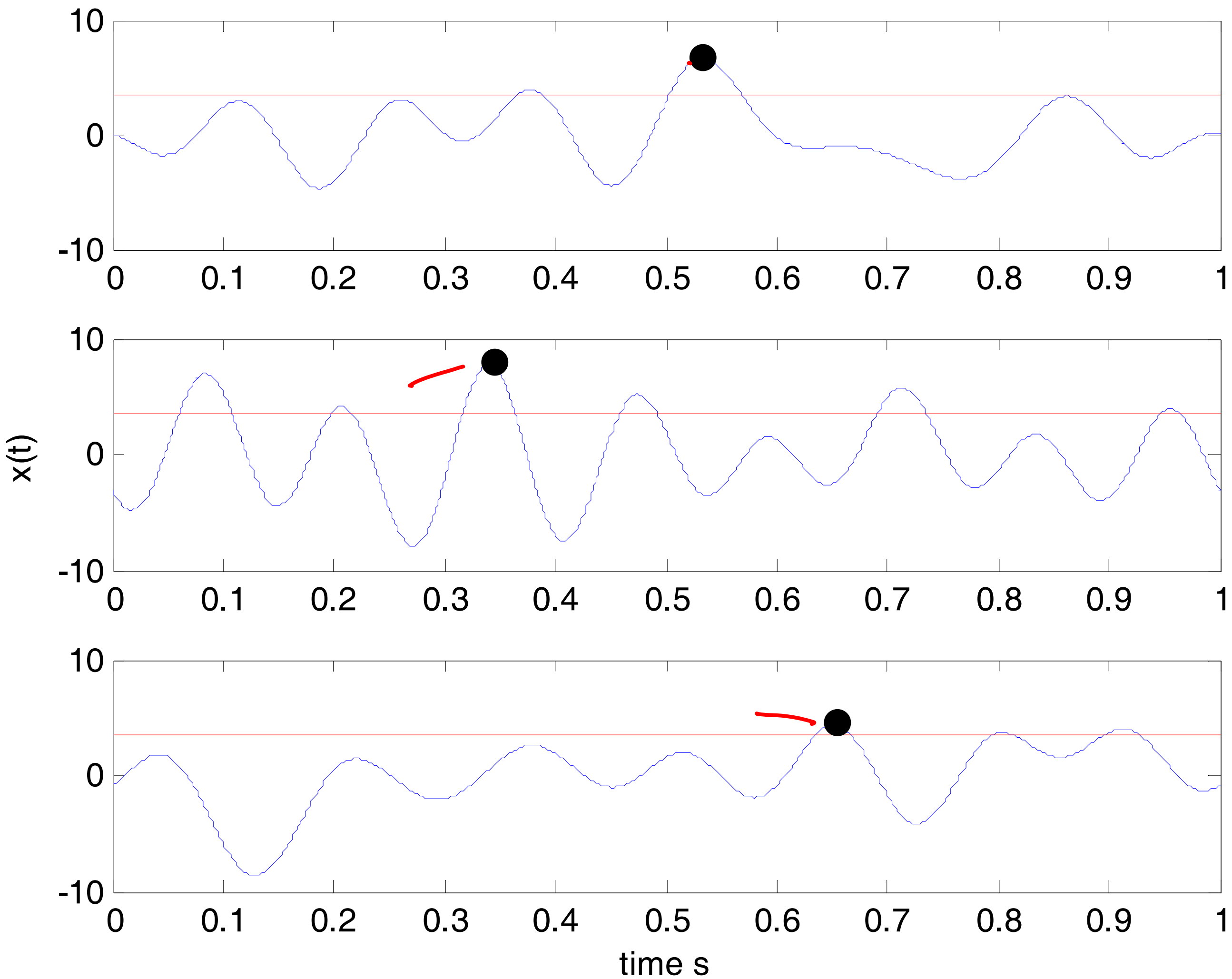
$T_f(\alpha)$

- The time required by $X(t)$ to cross level α for the first time
- A real valued random variable taking values in 0 to ∞
- Given the complete description of $X(t)$,

can we characterize $T_f(\alpha)$?

- This is known as the
  - First passage problem
  - Barrier crossing problem
  - Outcrossing problem

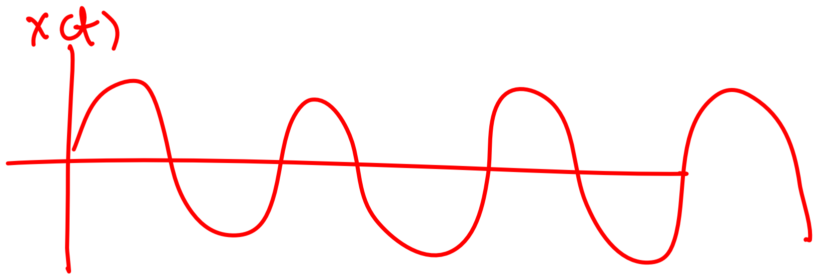
The maximum value of $X(t)$ in interval 0 to T



$$X_m = \max_{0<t<T} X(t)$$

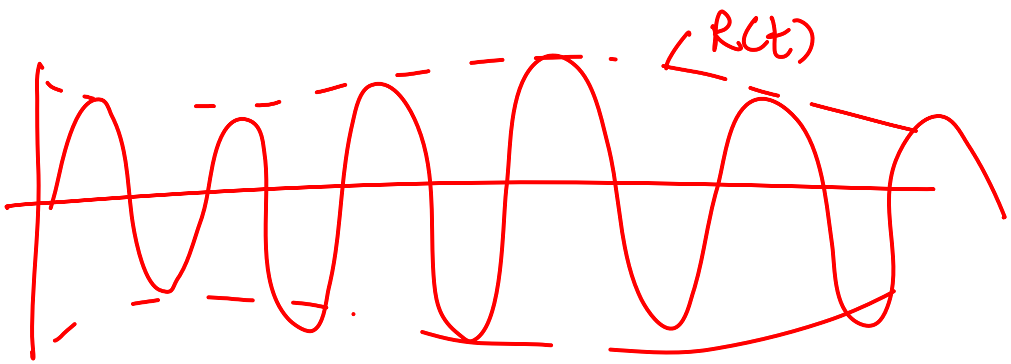
- The maximum value of $X(t)$ in 0 to T
- A real valued random variable
- Given the complete description of $X(t)$, can we characterize X_m?
- This is known as the problem of extreme value analysis

Envelope and phase



$R \sin(\lambda t + \theta)$

R, θ — constants



$R(t) \sin(\lambda t + \theta(t))$

Level Crossing problem

- Let $X(t)=$ be a random process.

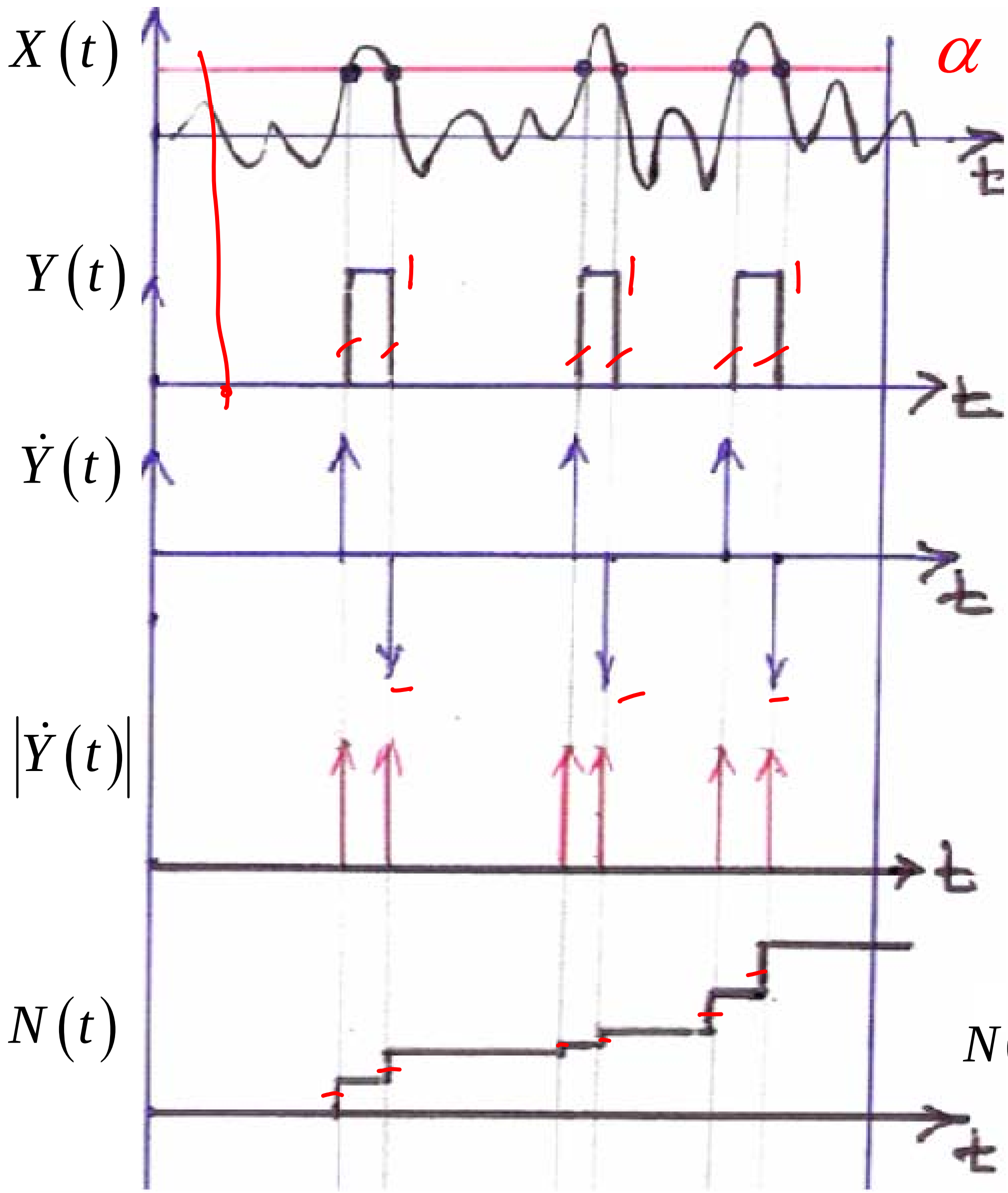
  -Not necessarily Gaussian

  -Not necessarily stationary

$N(\alpha,0,T)$=Number of times the level α is crossed in the time interval 0-T.

$N(\alpha,0,T)$ *is an integer valued random variable.*

What is the PDF of $N(\alpha,0,T)$?



$$Y(t) = U\left[X(t) - \alpha\right]$$

$= 1 \quad x(t) > \alpha$

$= 0 \quad x(t) < \alpha$

$$\dot{Y}(t) = \dot{X}(t)\,\delta\left[X(t) - \alpha\right]$$

$$\left|\dot{Y}(t)\right| = \left|\dot{X}(t)\right|\delta\left[X(t) - \alpha\right]$$

$$N(T) = \int_0^T \left|\dot{X}(t)\right|\delta\left[X(t) - \alpha\right]dt$$

Notes

- $\text{U}\left[X(t)-\alpha\right]=1 \text{ for } X(t)>\alpha$

  $=0 \text{ for } X(t)<\alpha$

- $\frac{d}{dt}U(t-\tau)=\delta(t-\tau)$

$$N(0,\alpha,T) = N(T) = \int_0^T \left|\dot{X}(t)\right| \delta\left[X(t)-\alpha\right] dt$$

$\alpha, 0, T$

$$= \int_0^T n(\alpha,t)\,dt$$

$$n(\alpha,t) = \left|\dot{X}(t)\right| \delta\left[X(t)-\alpha\right]$$

$= F[x(t), \dot{x}(t)]$

Remarks

- For a fixed value of T, $N(0,\alpha,T)$ is an integer valued random variable

  $\alpha, 0, T$

- $n(\alpha,t) =$ rate of crossing of level α
- For a fixed value of t, $n(\alpha,t)$ is an integer valued random variable

$$N(T) = \int_0^T \left|\dot{X}(t)\right| \delta\left[X(t) - \alpha\right] dt$$

- **Finding PDF of *N(α,0,T)* is difficult.**
- **Here, given the highly nonlinear nature of transformation, the rules of transformation of random variables are difficult to apply.**
- **Can we find moments of *N(α,0,T)*?**

$$N(T) = \int_0^T \left|\dot{X}(t)\right| \delta\left[X(t) - \alpha\right] dt$$

$$\langle N(T) \rangle = \left\langle \int_0^T \left|\dot{X}(t)\right| \delta\left[X(t) - \alpha\right] dt \right\rangle$$

$$= \int_0^T \left\langle \left|\dot{X}(t)\right| \delta\left[X(t) - \alpha\right] \right\rangle dt$$

$$= \int_0^T \langle n(\alpha, t) \rangle dt$$

$$\langle n(\alpha,t)\rangle = \langle |\dot{X}(t)|\,\delta[X(t)-\alpha]\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{x}|\,\delta(x-\alpha)\,p_{X\dot{X}}(x,\dot{x};t)\,dx\,d\dot{x}$$

$$= \int_{-\infty}^{\infty} |\dot{x}|\,p_{X\dot{X}}(\alpha,\dot{x};t)\,d\dot{x}$$

This integral can be evaluated.

How about higher order moments?

$$N(T)=\int_0^T\left|\dot{X}(t)\right|\delta\left[X(t)-\alpha\right]dt$$

$$N^2(T)=\int_0^T\int_0^T\left|\dot{X}(t_1)\right|\delta\left[X(t_1)-\alpha\right]\left|\dot{X}(t_2)\right|\delta\left[X(t_2)-\alpha\right]dt_1dt_2$$

$$\left\langle N^2(T)\right\rangle=\int_0^T\int_0^T\left\langle\left|\dot{X}(t_1)\right|\delta\left[X(t_1)-\alpha\right]\left|\dot{X}(t_2)\right|\delta\left[X(t_2)-\alpha\right]\right\rangle dt_1dt_2$$

$$\left\langle\left|\dot{X}(t_1)\right|\delta\left[X(t_1)-\alpha\right]\left|\dot{X}(t_2)\right|\delta\left[X(t_2)-\alpha\right]\right\rangle=$$

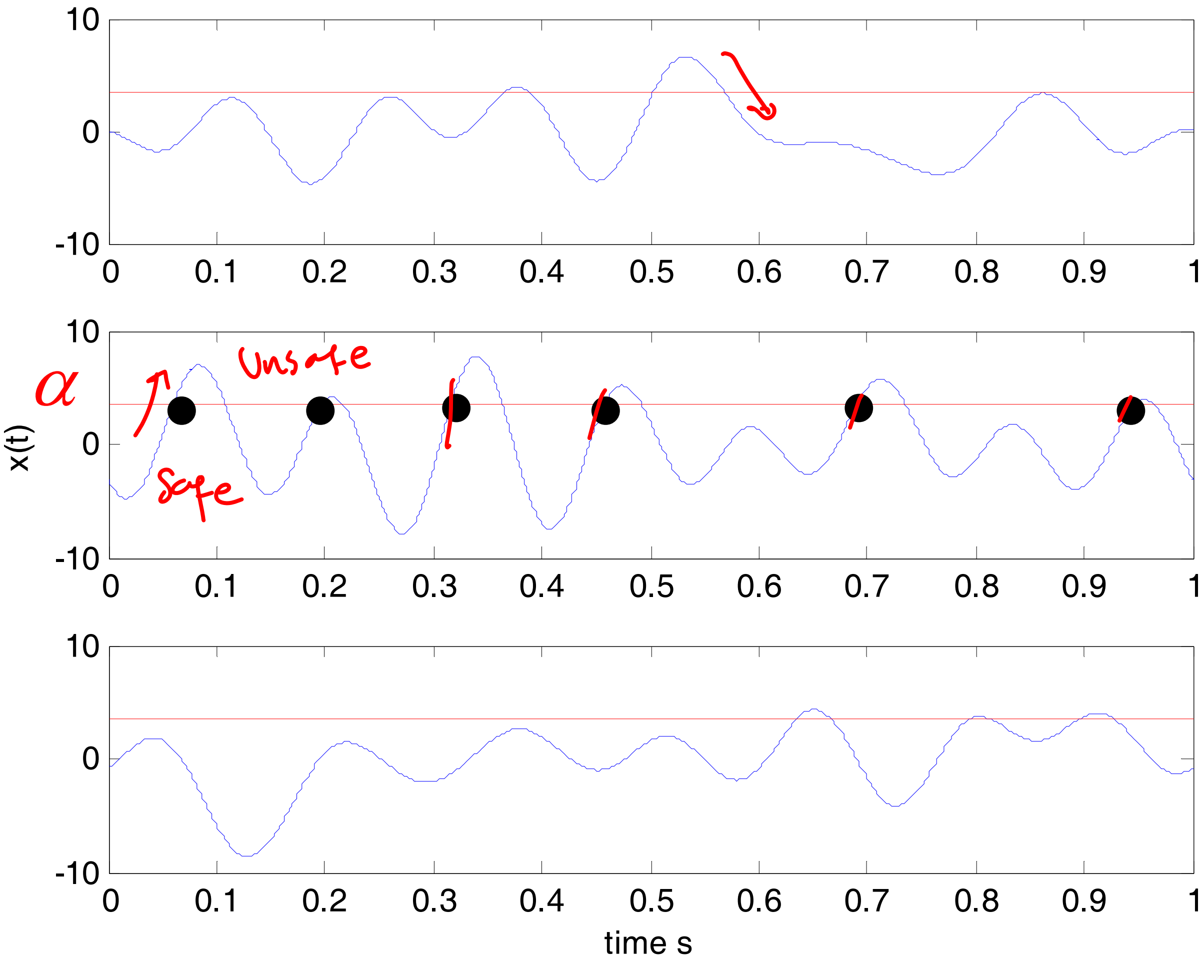
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\dot{x}_1||\dot{x}_2|\delta(x_1-\alpha)\delta(x_2-\alpha)\,p_{XX\dot{X}\dot{X}}(x_1,x_2,\dot{x}_1,\dot{x}_2;t_1,t_2,t_1,t_2)\,dx_1dx_2d\dot{x}_1d\dot{x}_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\dot{x}_1||\dot{x}_2|p_{XX\dot{X}\dot{X}}(\alpha,\alpha,\dot{x}_1,\dot{x}_2;t_1,t_2,t_1,t_2)\,d\dot{x}_1d\dot{x}_2$$

With some effort, this integral can also be evaluated.

Remarks

•Suppose we are interested in crossing of level α with postive slopes



$$N(T) = \int_0^T \dot{X}(t)\, U\left[\dot{X}(t) - 0\right] \delta\left[X(t) - \alpha\right] dt$$

$$= \int_0^T n^+(\alpha, t)\, dt$$

$$\left\langle n^+(\alpha, t) \right\rangle = \left\langle \dot{X}(t)\, U\left[\dot{X}(t) - 0\right] \delta\left[X(t) - \alpha\right] \right\rangle$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\dot{x}|\, \delta(x - \alpha)\, U(\dot{x} - 0)\, p_{X\dot{X}}(x, \dot{x}; t)\, dx\, d\dot{x}$$

$$= \int_0^{\infty} \dot{x}\, p_{X\dot{X}}(\alpha, \dot{x}; t)\, d\dot{x}$$

$-\infty$ →

Average rate of zero crossings with positive slope

$$\left\langle n^{+}(0,t)\right\rangle = \left\langle \dot{X}(t)U\left[\dot{X}(t)-0\right]\delta\left[X(t)-0\right]\right\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\dot{x}|\delta(x-0)U(\dot{x}-0)\,p_{X\dot{X}}(x,\dot{x};t)\,dx\,d\dot{x}$$

$$= \int_{0}^{\infty} \dot{x}\,p_{X\dot{X}}(0,\dot{x};t)\,d\dot{x}$$

Example 1:

$X(t)$ is a stationary Gaussian random process

with zero mean and covariance $R_{XX}(\tau)$ and PSD function

$S_{XX}(\omega)$. Determine the average rate of crossing of level α.

Given $\langle X(t)\rangle = 0; \langle X(t)X(t+\tau)\rangle = R_{XX}(\tau) \Leftrightarrow S_{XX}(\omega)$.

We need the jpdf of $X(t)$ & $\dot{X}(t^*)$ at $t^* = t$.

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$R_{X\dot{X}}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)(-i\omega)\exp(-i\omega\tau)\,d\omega$$

$$R_{X\dot{X}}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)(-i\omega)\,d\omega = 0$$

$\because S_{XX}(\omega) = S_{XX}(-\omega)$

That is, the process and its time derivative at the same time are uncorrelated.

This is a property of stationary random processes.

Since $X(t)$ is given to be Gaussian, we have

$$p_{X\dot{X}}\left(x,\dot{x};t\right)=\frac{1}{2\pi\sigma_x\sigma_{\dot{x}}}\exp\left\{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2}+\frac{\dot{x}^2}{\sigma_{\dot{x}}^2}\right]\right\};-\infty<x,\dot{x}<\infty$$

$$\left\langle n(\alpha,t)\right\rangle=\int_{-\infty}^{\infty}|\dot{x}|p_{X\dot{X}}\left(\alpha,\dot{x};t\right)d\dot{x}$$

$$=\int_{-\infty}^{\infty}|\dot{x}|\frac{1}{2\pi\sigma_x\sigma_{\dot{x}}}\exp\left\{-\frac{1}{2}\left[\frac{\alpha^2}{\sigma_x^2}+\frac{\dot{x}^2}{\sigma_{\dot{x}}^2}\right]\right\}d\dot{x}$$

$$=\frac{1}{2\pi\sigma_x\sigma_{\dot{x}}}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\int_{-\infty}^{\infty}|\dot{x}|\exp\left\{-\frac{1}{2}\frac{\dot{x}^2}{\sigma_{\dot{x}}^2}\right\}d\dot{x}$$

That is, the process and its time derivative at the same time are uncorrelated.

This is a property of stationary random processes.

$$\langle n(\alpha,t)\rangle = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\int_{-\infty}^{\infty}|\dot{x}|\exp\left\{-\frac{1}{2}\frac{\dot{x}^2}{\sigma_{\dot{x}}^2}\right\}d\dot{x}$$

$$= \frac{2}{2\pi\sigma_x\sigma_{\dot{x}}}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\int_{0}^{\infty}\dot{x}\exp\left\{-\frac{1}{2}\frac{\dot{x}^2}{\sigma_{\dot{x}}^2}\right\}d\dot{x}$$

$$= \frac{2}{2\pi\sigma_x\sigma_{\dot{x}}}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}\int_{0}^{\infty}\exp\{-u\}\sigma_{\dot{x}}^2 du$$

$$\frac{\dot{x}^2}{2\sigma_{\dot{x}}^2} = u$$

$$= \frac{\sigma_{\dot{x}}}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\langle n(\alpha,t)\rangle = \frac{\sigma_{\dot{x}}}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\langle N(T)\rangle = \int_0^T \langle n(\alpha,t)\rangle dt$$

$$\int_0^T \frac{\sigma_{\dot{x}}}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}dt$$

$$= \frac{\sigma_{\dot{x}}T}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\langle n(\alpha,t)\rangle = \frac{\sigma_{\dot{x}}}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\sigma_x^2 = \int_0^\infty S_{XX}(\omega)\,d\omega$$

$$\sigma_{\dot{x}}^2 = \int_0^\infty \omega^2 S_{XX}(\omega)\,d\omega$$

Spectral moments

$$\lambda_n = \int_0^\infty \omega^n S_{XX}(\omega)\,d\omega$$

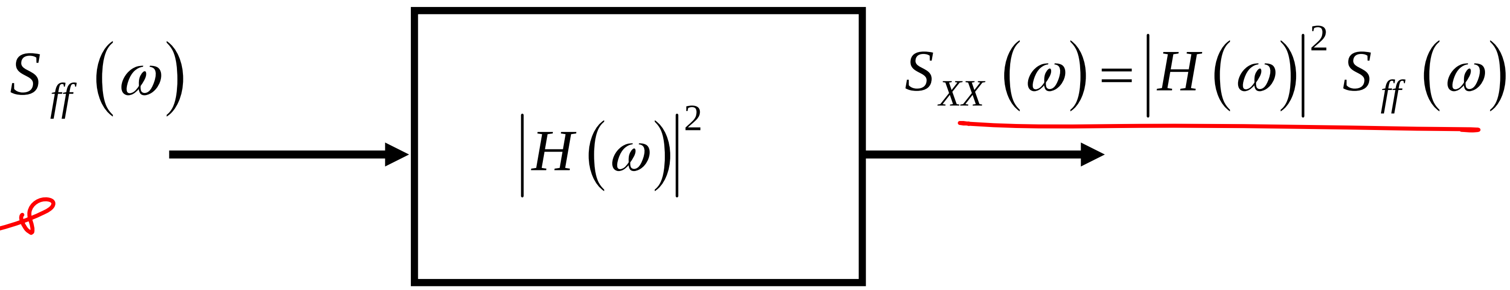
$$\langle n(\alpha,t)\rangle = \frac{1}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\lambda_2}\right\}$$

Zero-crossing rates

$$\left\langle n\left(0,t\right)\right\rangle = \frac{\sigma_{\dot{x}}}{\pi\sigma_x} = \frac{1}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}$$

$$\left\langle n^+\left(0,t\right)\right\rangle = \frac{\sigma_{\dot{x}}}{2\pi\sigma_x} = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}$$

RECALL



$$M\ddot{X} + C\dot{X} + KX = F(t);\ X(0) = 0;\ \dot{X}(0) = 0$$

$$S_{XX}(\omega) = H(\omega) S_{FF}(\omega) H^{*t}(\omega)$$

$$S_{YY}(x, \xi, \omega) = |G(x, \xi, \omega)|^2 S_{FF}(\omega)$$

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Example

Let $X(t)$ be the steady state response of a sdof system under stationary, zero mean Gaussian random excitation. Determine the mean rate of crossing of level α by the response process $X(t)$ in the steady state.

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\langle f(t) \rangle = 0; \langle f(t) f(t+\tau) \rangle = R_{ff}(\tau) \Leftrightarrow S_{ff}(\omega)$$

In the steady state $x(t)$ and $\dot{x}(t)$ would be uncorrelated.

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega)$$

$$H(\omega) = \frac{1}{-m\omega^2 + i\omega c + k}$$

$$\left\langle n(\alpha,t)\right\rangle = \frac{\sigma_{\dot{x}}}{\pi\sigma_x}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_x^2}\right\}$$

$$\sigma_x^2 = \int_0^\infty S_{xx}(\omega)d\omega = \int_0^\infty \left|H(\omega)\right|^2 S_{ff}(\omega)d\omega$$

$$\sigma_{\dot{x}}^2 = \int_0^\infty \omega^2 S_{xx}(\omega)d\omega = \int_0^\infty \omega^2\left|H(\omega)\right|^2 S_{ff}(\omega)d\omega$$

Spectral moments

$$\lambda_n = \int_0^\infty \omega^n S_{xx}(\omega)d\omega = \int_0^\infty \omega^n\left|H(\omega)\right|^2 S_{ff}(\omega)d\omega$$

$$\left\langle n(\alpha,t)\right\rangle = \frac{1}{\pi}\sqrt{\frac{\lambda_2}{\lambda_0}}\exp\left\{-\frac{1}{2}\frac{\alpha^2}{\lambda_2}\right\}$$

Example

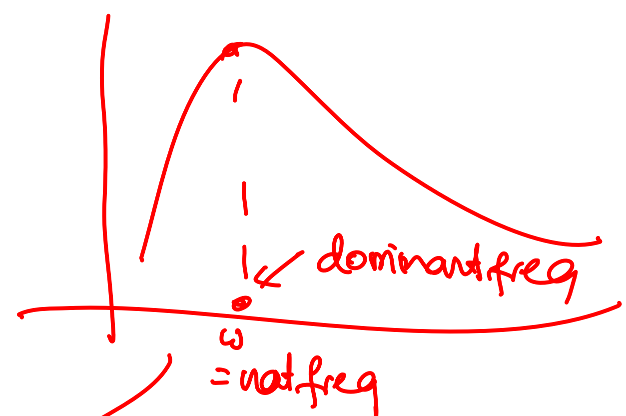
Find average rate of zero crossing with positive slope of $x(t)$ when $f(t)$ is a zero mean Gaussian white noise process. Consider response in the steady state.

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega)$$

$$= I\ |H(\omega)|^2$$

Recall

$$\sigma_x^2 = \frac{I}{4\eta\omega^3} \ \& \ \sigma_{\dot{x}}^2 = \frac{I}{4\eta\omega}$$



$\Rightarrow$

$$\langle n^+(0) \rangle = \frac{\omega}{2\pi}$$

Example

Let $X(t)$ be a nonstationary, zero mean, Gaussian random process with autocovariance function $R_{XX}(t_1, t_2)$.

Determine the average rate of crossing of level α by the process $X(t)$.

$$\langle n(\alpha, t)\rangle = \left\langle \left|\dot{X}(t)\right| \delta\left[X(t) - \alpha\right]\right\rangle = \int_{-\infty}^{\infty} |\dot{x}| p_{X\dot{X}}(\alpha, \dot{x}; t)\, d\dot{x}$$

We need the jpdf of $X(t)$ and $\dot{X}(t)$.

$$p(x, \dot{x}; t) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}\sqrt{\left(1 - r^2\right)}} \exp\left[-\frac{1}{2\left(1 - r^2\right)}\left\{\frac{x^2}{\sigma_x^2} + \frac{\dot{x}^2}{\sigma_{\dot{x}}^2} - \frac{2rx\dot{x}}{\sigma_x\sigma_{\dot{x}}}\right\}\right]$$

$$-\infty < x, \dot{x} < \infty$$

$$\langle n^{+}(\alpha,t)\rangle = \frac{1}{2\pi}\frac{\sigma_{\dot{x}}}{\sigma_x}\left(1-r^2\right)[\exp\left(-\frac{\alpha^2}{2\sigma_x^2\left(1-r^2\right)}\right)+$$

$$\frac{\alpha r}{\sigma_x}\exp\left(-\frac{\alpha^2}{2\sigma_x^2}\right)\left\{1-erf\left(\frac{\alpha r}{\sigma_x\sqrt{2\left(1-r^2\right)}}\right)\right\}]$$

$$N(T)=\int_0^T \langle n^{+}(\alpha,t)\rangle dt$$

Note

- The quantitites $r, \sigma_x, \& \sigma_{\dot{x}}$ are time varying.
- If $r=0$ and $\sigma_x, \& \sigma_{\dot{x}}$ are time invariant, the above expression reduces to the expression for the case when $X(t)$ is stationary.

This is what is expected.