Stochastic Structural Dynamics

Lecture-15

Random vibration analysis of MDOF systems-3

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Recall

$$\bullet\left[H(\omega)\right]=\left[-\omega^2 M+i\omega C+k\right]^{-1}=\left[\sum_{n=1}^{N}\frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2-\omega^2+i2\eta_n\omega_n\omega\right)}\right]$$

$$\bullet\left[h(t)\right]=\left[h_{rs}(t)\right]=\left[\sum_{n=1}^{N}\Phi_{rn}\Phi_{sn}\frac{1}{\omega_{dn}}\exp\left(-\eta_n\omega_n t\right)\sin\omega_{dn}t\right]$$

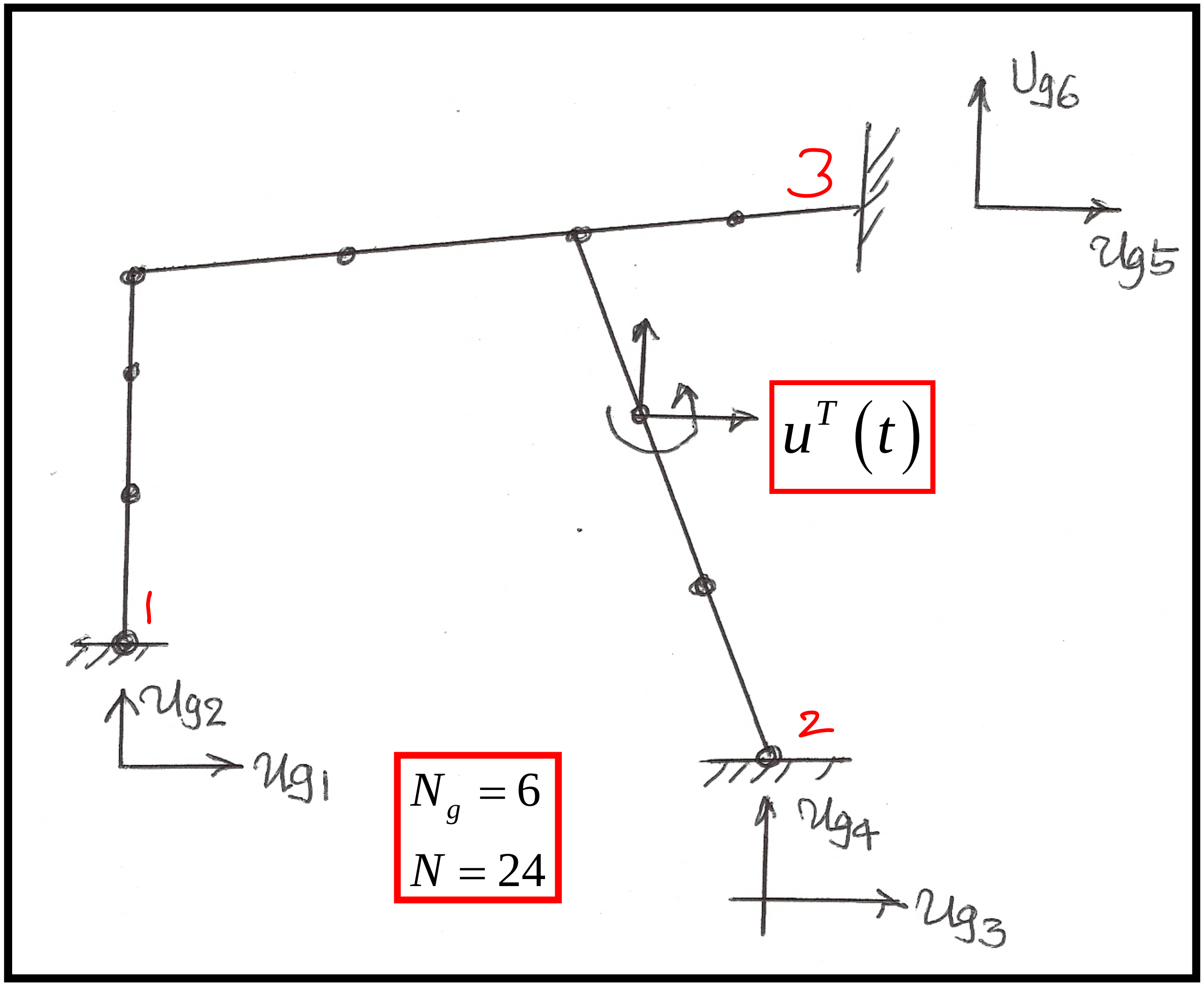
$$H_{ij}(\omega)=\int_{-\infty}^{\infty}h_{ij}(t)\exp(i\omega t)\,d\tau$$

$$h_{ij}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H_{ij}(\omega)\exp(-i\omega t)\,d\omega$$

$$S_{XX}(\omega)=H(\omega)S_{FF}(\omega)H^{*t}(\omega)$$

$$R_{XX}(t_1,t_2)=\int_0^{t_1}\int_0^{t_2}\left[h(t_1-\tau_1)\right]R_{FF}(\tau_2-\tau_1)\left[h(t_2-\tau_2)\right]^t d\tau_1 d\tau_2$$

Structures under differential support motions



$$\begin{bmatrix} M & M_g \\ M_g^t & M_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{u}^T \\ \ddot{u}_g \end{Bmatrix} + \begin{bmatrix} C & C_g \\ C_g^t & C_{gg} \end{bmatrix} \begin{Bmatrix} \dot{u}^T \\ \dot{u}_g \end{Bmatrix} + \begin{bmatrix} K & K_g \\ K_g^t & K_{gg} \end{bmatrix} \begin{Bmatrix} u^T \\ u_g \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_g(t) \end{Bmatrix}$$

$$\ddot{u}^T \sim N \times 1$$

$$\ddot{u}_g, p_g(t) \sim N_g \times 1; N_T = N + N_g$$

$$M, C, K \sim N \times N$$

$$M_g, C_g, K_g \sim N \times N_g$$

$$M_{gg}, C_{gg}, K_{gg} \sim N_g \times N_g$$

Pseudo-dynamic response

$$\begin{bmatrix} K & K_g \\ K_g^t & K_{gg} \end{bmatrix} \begin{Bmatrix} u^p \\ u_g \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_g^p(t) \end{Bmatrix}$$

$$Ku^p + K_g u_g = 0 \Rightarrow u^p = -K^{-1} K_g u_g(t) = \Gamma u_g(t)$$

$$\Gamma = -K^{-1} K_g$$

$$p_g^p(t) = K_g^t u^p + K_{gg} u_g = \left[-K_g^t K^{-1} K_g + K_{gg} \right] u_g(t)$$

Total response= pseudo-dynamic response + dynamic response

$$\begin{Bmatrix} u^T \\ u_g \end{Bmatrix} = \begin{Bmatrix} u^p(t) \\ u_g(t) \end{Bmatrix} + \begin{Bmatrix} u(t) \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} M & M_g \\ M_g^t & M_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{u} + \ddot{u}^p(t) \\ \ddot{u}_g \end{Bmatrix} + \begin{bmatrix} C & C_g \\ C_g^t & C_{gg} \end{bmatrix} \begin{Bmatrix} \dot{u} + \dot{u}^p \\ \dot{u}_g \end{Bmatrix} +$$

$$\begin{bmatrix} K & K_g \\ K_g^t & K_{gg} \end{bmatrix} \begin{Bmatrix} u + u^p \\ u_g \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_g(t) \end{Bmatrix}$$

$$\Rightarrow$$

$$M\ddot{u} + C\dot{u} + Ku = p_{eff}(t)$$

$$p_{eff}(t) = -M\ddot{u}^p(t) - M_g\ddot{u}_g - C\dot{u}^p(t) - C_g\dot{u}_g$$

$$= -M\Gamma\ddot{u}_g(t) - M_g\ddot{u}_g - \Gamma C\dot{u}_g(t) - C_g\dot{u}_g$$

$$= -\left[M\Gamma + M_g\right]\ddot{u}_g(t) - \left[C\Gamma + C_g\right]\dot{u}_g(t)$$

$$M\ddot{u} + C\dot{u} + Ku = -\left[M\Gamma + M_g \right]\ddot{u}_g(t) - \left[C\Gamma + C_g \right]\dot{u}_g(t)$$

Special Case

Mass matrix is diagonal $\Rightarrow \mathrm{M}_g = 0$

C is proportional to K $(C = \alpha K)$

$$\Rightarrow \left[C\Gamma + C_g \right] = \alpha\left[K\Gamma + K_g \right] = \alpha\left[-KK^{-1}K_g + K_g \right] = 0$$

$\Rightarrow$

$$M\ddot{u} + C\dot{u} + Ku = -M\Gamma\ddot{u}_g(t)$$

$\Gamma \sim N \times N_g$

$\ddot{u}_g(t) \sim N_g \times 1$

Note: If all supports suffer the same motion, $N_g = 1$

$$\Gamma = \{1 \quad 1 \quad \cdots \quad 1\}^t$$

Random vibration analysis in frequency domain

$$M\ddot{u}+C\dot{u}+Ku=-\left[M\Gamma+M_g\right]\ddot{u}_g(t)-\left[C\Gamma+C_g\right]\dot{u}_g(t)=p(t)$$

$u_g(t)\sim N_g\times 1$: vector of stationary random process with zero mean and PSD matrix

$$S_{gg}\left(\omega\right)=\lim_{T\to\infty}\frac{1}{T}\left\langle U_{gT}\left(\omega\right)U_{gT}^{*t}\left(\omega\right)\right\rangle$$

$$p(t)=-\left[M\Gamma+M_g\right]\ddot{u}_g(t)-\left[C\Gamma+C_g\right]\dot{u}_g(t)$$

$$P_T\left(\omega\right)=\omega^2\left[M\Gamma+M_g\right]U_{gT}\left(\omega\right)-i\omega\left[C\Gamma+C_g\right]U_{gT}\left(\omega\right)$$

$$=\left[\omega^2\left[M\Gamma+M_g\right]-i\omega\left[C\Gamma+C_g\right]\right]U_{gT}\left(\omega\right)$$

$$P_T^{*t}\left(\omega\right)=U_{gT}^{*t}\left(\omega\right)\left[\omega^2\left[\Gamma^tM+M_g\right]+i\omega\left[\Gamma^tC+C_g\right]\right]$$

$$S_{pp}(\omega) = \lim_{T\to\infty} \frac{1}{T} < \left[\omega^2 \left[M\Gamma + M_g \right] - i\omega \left[C\Gamma + C_g \right] \right] U_{gT}(\omega)$$

$$U_{gT}^{*t}(\omega) \left[\omega^2 \left[\Gamma^t M + M_g \right] + i\omega \left[\Gamma^t C + C_g \right] \right] >$$

$$S_{pp}(\omega) = \left[\omega^2 \left[M\Gamma + M_g \right] - i\omega \left[C\Gamma + C_g \right] \right]$$

$$S_{gg}(\omega) \left[\omega^2 \left[\Gamma^t M + M_g \right] + i\omega \left[\Gamma^t C + C_g \right] \right]$$

$$\Rightarrow$$

$$S_{UU}(\omega) = H(\omega) S_{pp}(\omega) H^{*t}(\omega)$$

$$\left[H(\omega) \right] = \left[-\omega^2 M + i\omega C + k \right]^{-1} = \left[\sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega \right)} \right]$$

Pseudo-dynamic response

$$u^p = -K^{-1}K_g u_g(t) = \Gamma u_g(t)$$

$$\Gamma = -K^{-1}K_g$$

$$S_{u^p u^p}(\omega) = \Gamma S_{gg}(\omega)\Gamma^t$$

Total response

$$u^T(t) = u^p(t) + u(t)$$

$$= \Gamma u_g(t) + u(t)$$

$$U_T^T(\omega) = \Gamma U_{gT}(\omega) + U_T(\omega)$$

$$= \Gamma U_{gT}(\omega) + H(\omega) P_T(\omega)$$

$$P_T(\omega) = \left[\omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] U_{gT}(\omega)$$

$$\Rightarrow$$

$$U_T^T(\omega) = \left[\Gamma + \omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] U_{gT}(\omega)$$

Total response

$$U_T^T(\omega) = \left[\Gamma + \omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] U_{gT}(\omega)$$

$$\Rightarrow$$

$$S_{TT}(\omega) = \left[\Gamma + \omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right] S_{gg}(\omega)$$

$$\left[\Gamma + \omega^2\left[M\Gamma + M_g\right] - i\omega\left[C\Gamma + C_g\right]\right]^t$$

Variance of total response=

variance of pseudo-dynamic response+

variance of dynamic response+

contributions due to correlation between

pseudo-dynamic and dynamic responses

Random vibration analysis in time domain

$$M\ddot{u}+C\dot{u}+Ku=-\left[M\Gamma+M_g\right]\ddot{u}_g(t)-\left[C\Gamma+C_g\right]\dot{u}_g(t)=p(t)$$

$u_g(t)\sim N_g\times 1$: vector of stationary random process with zero mean and auto-covariance matrix

$$R_{gg}\left(t_1,t_2\right)=\left\langle u_g(t_1)u_g^t(t_2)\right\rangle=R_{gg}\left(t_1-t_2\right)$$

$$p(t_1)=-\left[M\Gamma+M_g\right]\ddot{u}_g(t_1)-\left[C\Gamma+C_g\right]\dot{u}_g(t_1)$$

$$p^t\left(t_2\right)=-\ddot{u}_g^t(t_2)\left[\Gamma^t M+M_g\right]-\dot{u}_g^t(t_2)\left[\Gamma^t C+C_g\right]$$

$$
\begin{aligned}
&\left\langle p(t_1)\,p^t\left(t_2\right)\right\rangle = R_{pp}\left(t_1,t_2\right)\\
&=\left[M\Gamma+M_g\right]\left\langle \ddot{u}_g(t_1)\ddot{u}_g^t(t_2)\right\rangle\left[\Gamma^t M+M_g\right]\\
&+\left[M\Gamma+M_g\right]\left\langle \ddot{u}_g(t_1)\dot{u}_g^t(t_2)\right\rangle\left[\Gamma^t C+MC_g\right]\\
&+\left[C\Gamma+C_g\right]\left\langle \dot{u}_g(t_1)\ddot{u}_g^t(t_2)\right\rangle\left[\Gamma^t M+M_g\right]\\
&+\left[C\Gamma+C_g\right]\left\langle \dot{u}_g(t_1)\dot{u}_g^t(t_2)\right\rangle\left[\Gamma^t C+MC_g\right]\\
&R_{UU}\left(t_1,t_2\right)=\int_0^{t_1}\int_0^{t_2}\left[h\left(t_1-\tau_1\right)\right]R_{pp}\left(\tau_2-\tau_1\right)\left[h\left(t_2-\tau_2\right)\right]^t d\tau_1 d\tau_2
\end{aligned}
$$

$$
\left[h\left(t\right)\right]=\left[h_{rs}\left(t\right)\right]=\left[\sum_{n=1}^{N}\Phi_{rn}\Phi_{sn}\,\frac{1}{\omega_{dn}}\exp\left(-\eta_n\omega_n t\right)\sin\omega_{dn}t\right]
$$

Recall

$$\left\langle \left.\frac{d^n X}{dt^n}\right|_{t=t_1} \left.\frac{d^m X}{dt^m}\right|_{t=t_2} \right\rangle = \frac{\partial^{n+m} R_{XX}\left(t_1,t_2\right)}{\partial t_1^n \partial t_2^m}$$

$$\left\langle \frac{d^n X\left(t+\tau\right)}{dt^n}\frac{d^m Y\left(t\right)}{dt^m} \right\rangle = \left(-1\right)^m \frac{d^{n+m} R_{XY}\left(\tau\right)}{d\tau^{n+m}}$$

Total response= pseudo-dynamic response+dynamic response

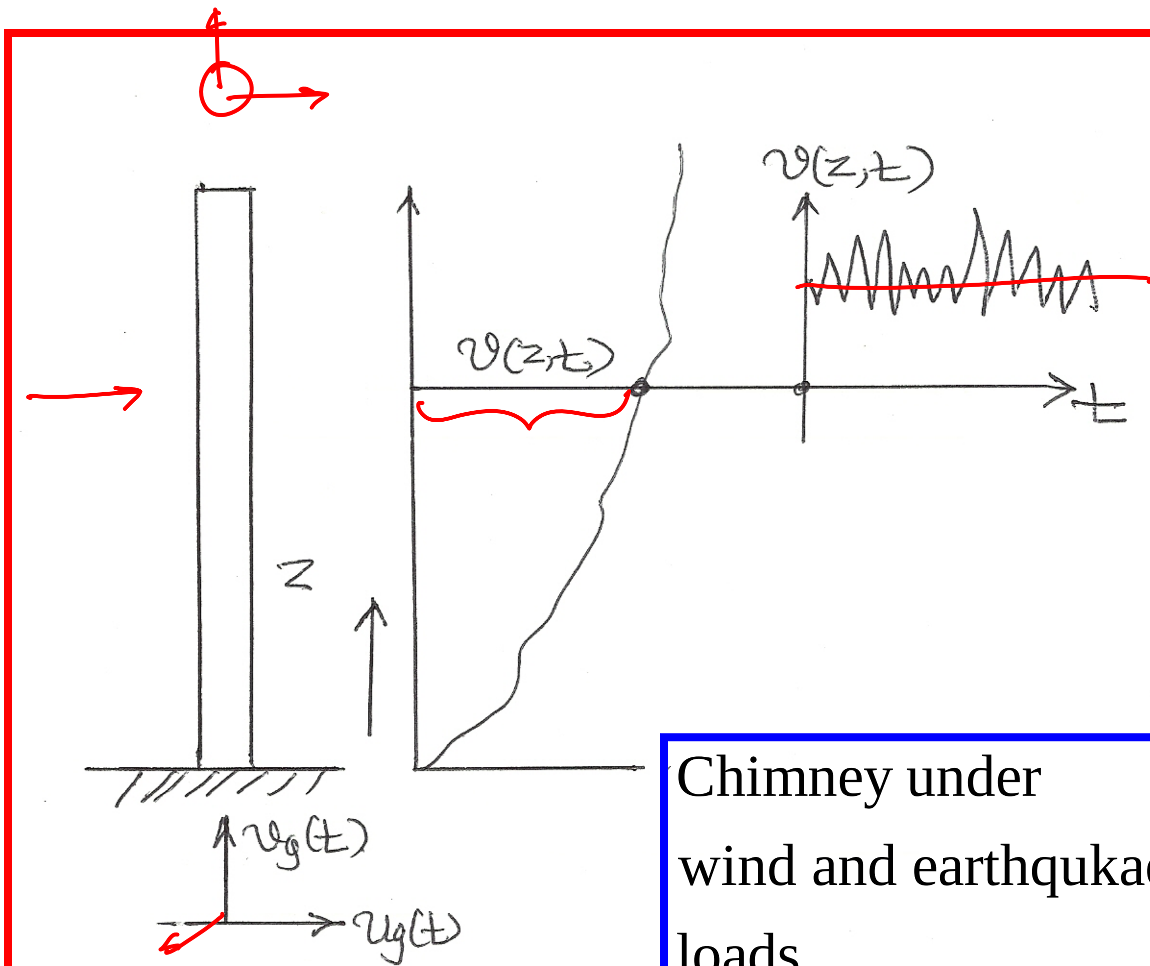
$$u^T(t) = u^p(t) + u(t)$$

$$= \Gamma u_g(t) + u(t)$$

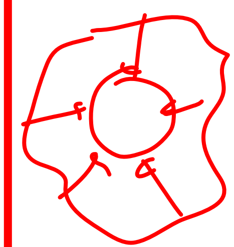
$$= \Gamma u_g(t) + \int_0^t \left[h(t-\tau)\right]\left\{-\left[M\Gamma + M_g\right]\ddot{u}_g(\tau) - \left[C\Gamma + C_g\right]\dot{u}_g(\tau)\right\} d\tau$$

Variance of total response=
variance of pseudo-dynamic response+
variance of dynamic response+
contributions due to correlation between
pseudo-dynamic and dynamic responses

RANDOM VIBRATION ANALYSIS OF CONTINUOUS MDOF SYSTEMS

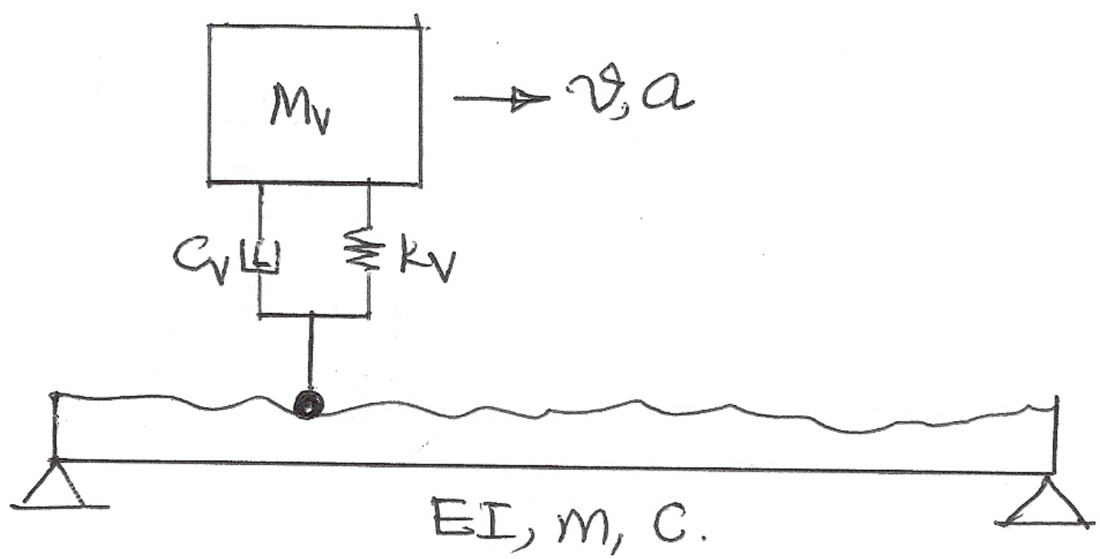


Chimney under wind and earthqukae loads



Vehicle structure interaction

- Guideway unevenness
- Vehicle motion
- Vehicle parameters



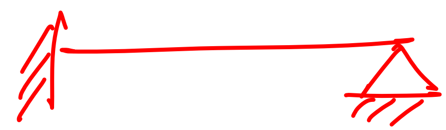
Review of dynamics of Euler-Bernoulli beams under deterministic excitations

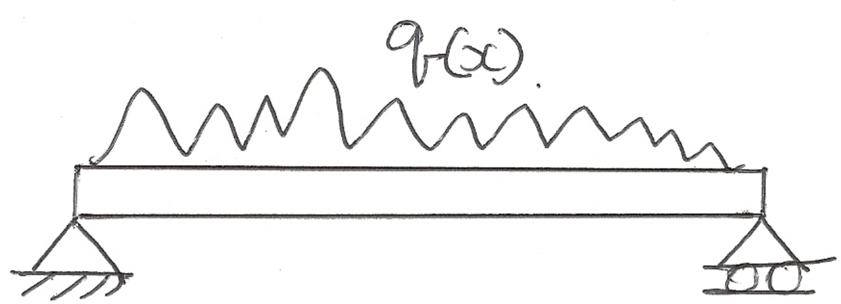
Statically loaded beam

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 y}{dx^2}\right] = q(x)$$

$$y(0) = 0;\ y(l) = 0$$

$$y'(0) = 0;\ EI\,\frac{d^2 y}{dx^2}(l) = 0$$





Dynamically loaded beam

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+a(x)\frac{\partial^3 y}{\partial x^2 \partial t}\right]+m(x)\frac{\partial^2 y}{\partial t^2}+c(x)\frac{\partial y}{\partial t}=q(x,t)$$

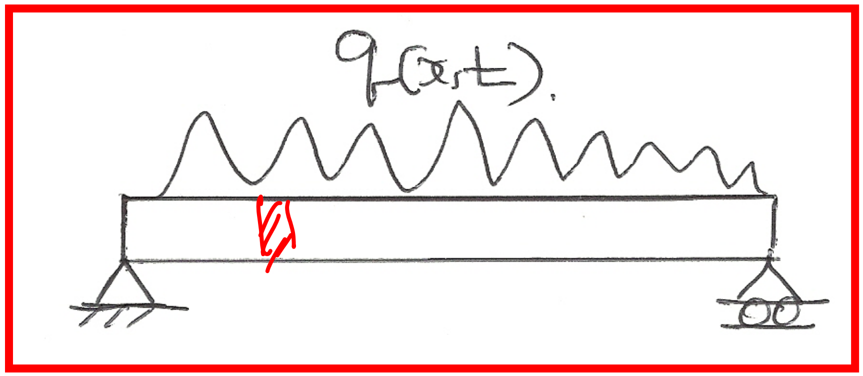
Strain rate dependent viscous damping

velocity dependent viscous damping

BCS

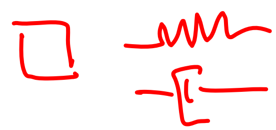
$$y(0,t)=0; y(l,t)=0$$

$$y'(0,t)=0; EI\frac{\partial^2 y}{\partial x^2}(l,t)=0$$



ICS

$$y(x,0)=y_0(x); \frac{\partial y}{\partial t}(x,0)=\dot{y}_0(x)$$



Undamped free vibration analysis

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]+m(x)\frac{\partial^2 y}{\partial t^2}=0$$

BCS

$$y(0,t)=0; y(l,t)=0$$

$$y'(0,t)=0; EI\,\frac{\partial^2 y}{\partial x^2}(l,t)=0$$

Seek the solution in the form

$$y(x,t)=\phi(x)T(t)$$

$$\Rightarrow$$

$$\left[EI\phi''(x)\right]''T(t)+\phi(x)m(x)\ddot{T}(t)=0$$

Note

$$(\ \ )'=\frac{\partial}{\partial x}\ \&\ \ddot{T}(t)=\frac{\partial^2 T}{\partial t^2}$$

$$\left[EI\phi''(x)\right]''T(t)+\phi(x)m(x)\ddot{T}(t)=0$$

$$\Rightarrow$$

$$\frac{\left[EI\phi''(x)\right]''T(t)}{\phi(x)T(t)}+\frac{\phi(x)m(x)\ddot{T}(t)}{\phi(x)T(t)}=0$$

$$\frac{\left[EI\phi''(x)\right]''}{\phi(x)m(x)}=-\frac{\ddot{T}(t)}{T(t)}=\text{constant}=\omega^2$$

$$\Rightarrow$$

$$\ddot{T}(t)+\omega^2T(t)=0\Rightarrow T(t)=A\cos(\omega t)+B\sin(\omega t)$$

$$\left[EI\phi''(x)\right]''-\omega^2m(x)\phi(x)=0$$

$$y(0,t)=0;y(l,t)=0;y'(0,t)=0;EI\,\frac{\partial^2 y}{\partial x^2}(l,t)=0$$

$$\Rightarrow\phi(0)=0;\phi(l)=0;\phi'(0)=0;EI\,\frac{d^2\phi}{dx^2}(l)=0$$

Eigenvalue problem

$$\left[EI\left(x\right)\phi''\left(x\right)\right]'' - \omega^2 m\left(x\right)\phi\left(x\right) = 0$$

$$\phi\left(0\right) = 0; \phi\left(l\right) = 0; \phi'\left(0\right) = 0; EI\frac{d^2\phi}{dx^2}(l) = 0$$

$\phi(x) = e^{sx}$

$s^4 - \lambda^4 = 0$

$s = \pm\lambda, \pm i\lambda$

Special case

$$EI\left(x\right) = EI; m(x) = m$$

$$EI\phi^{iv}\left(x\right) - \omega^2 m\phi\left(x\right) = 0$$

$\lambda^4 = \frac{m\omega^2}{EI}$

$$\phi^{iv} - \lambda^4\phi = 0$$

$$\phi\left(x\right) = a\left(\cos\lambda x + \cosh\lambda x\right) + b\left(\cos\lambda x - \cosh\lambda x\right)$$

$$+c\left(\sin\lambda x + \sinh\lambda x\right) + d\left(\sin\lambda x - \sinh\lambda x\right)$$

$$\phi'\left(x\right) = a\lambda\left(-\sin\lambda x + \sinh\lambda x\right) + b\lambda\left(-\sin\lambda x - \sinh\lambda x\right)$$

$$+c\lambda\left(\cos\lambda x + \cosh\lambda x\right) + d\lambda\left(\cos\lambda x - \cosh\lambda x\right)$$

$$\phi''\left(x\right) = a\lambda^2\left(-\cos\lambda x + \cosh\lambda x\right) + b\lambda^2\left(-\cos\lambda x - \cosh\lambda x\right)$$

$$+c\lambda^2\left(-\sin\lambda x + \sinh\lambda x\right) + d\lambda^2\left(-\sin\lambda x - \sinh\lambda x\right)$$

$$\phi(0)=0;\phi(l)=0;\phi'(0)=0;EI\,\frac{d^2\phi}{dx^2}(l)=0$$

$$\phi(x)=a(\cos\lambda x+\cosh\lambda x)+b(\cos\lambda x-\cosh\lambda x)$$

$$+c(\sin\lambda x+\sinh\lambda x)+d(\sin\lambda x-\sinh\lambda x)$$

$$\phi'(x)=a\lambda(-\sin\lambda x+\sinh\lambda x)+b\lambda(-\sin\lambda x-\sinh\lambda x)$$

$$+c\lambda(\cos\lambda x+\cosh\lambda x)+d\lambda(\cos\lambda x-\cosh\lambda x)$$

$$\phi''(x)=a\lambda^2(-\cos\lambda x+\cosh\lambda x)+b\lambda^2(-\cos\lambda x-\cosh\lambda x)$$

$$+c\lambda^2(-\sin\lambda x+\sinh\lambda x)+d\lambda^2(-\sin\lambda x-\sinh\lambda x)$$

$$\phi(0)=0\Rightarrow a=0$$

$$\phi'(0)=0\Rightarrow c=0$$

$$\phi(l)=0\Rightarrow b(\cos\lambda l-\cosh\lambda l)+d(\sin\lambda l-\sinh\lambda l)$$

$$\phi''(l)=0\Rightarrow b\lambda^2(-\cos\lambda l-\cosh\lambda l)+d\lambda^2(-\sin\lambda l-\sinh\lambda l)$$

$$\phi(l)=0 \Rightarrow b(\cos\lambda l-\cosh\lambda l)+d(\sin\lambda l-\sinh\lambda l)$$

$$\phi''(l)=0 \Rightarrow b\lambda^2(-\cos\lambda l-\cosh\lambda l)+d\lambda^2(-\sin\lambda l-\sinh\lambda l)$$

$$\Rightarrow$$

$$\begin{bmatrix} (\cos\lambda l-\cosh\lambda l) & (\sin\lambda l-\sinh\lambda l) \\ \lambda^2(-\cos\lambda l-\cosh\lambda l) & \lambda^2(-\sin\lambda l-\sinh\lambda l) \end{bmatrix} \begin{Bmatrix} b \\ d \end{Bmatrix} = 0$$

$$\Rightarrow$$

$$\begin{vmatrix} (\cos\lambda l-\cosh\lambda l) & (\sin\lambda l-\sinh\lambda l) \\ \lambda^2(-\cos\lambda l-\cosh\lambda l) & \lambda^2(-\sin\lambda l-\sinh\lambda l) \end{vmatrix} = 0$$

$$\Rightarrow \tan\lambda l=\tanh\lambda l \Rightarrow \{\lambda_n\}_{n=1}^{\infty}: \text{ characteristic values}$$

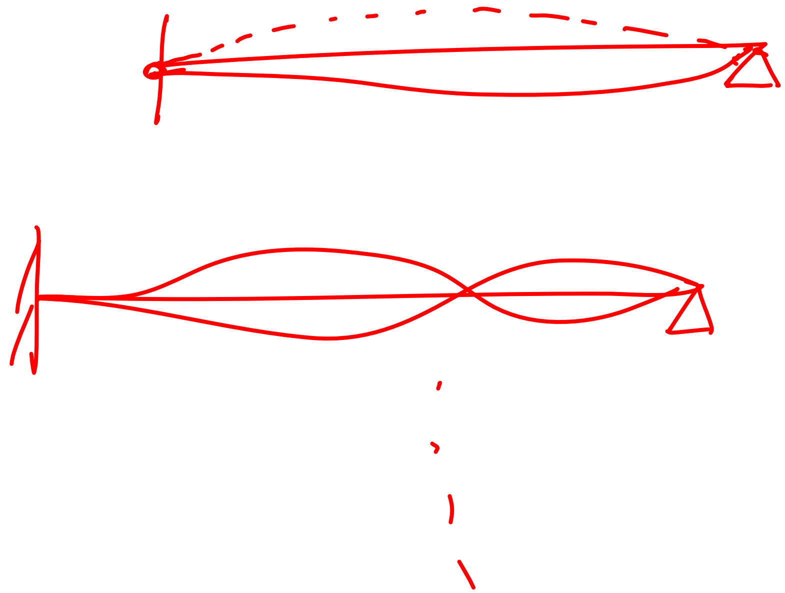
$$\phi_n(x)=(\cosh\lambda_n x-\cos\lambda_n x)+\sigma_n(\sinh\lambda_n x-\sin\lambda_n x)$$

$$\sigma_n=\frac{\cos\lambda_n l-\cosh\lambda_n l}{\sin\lambda_n l-\sinh\lambda_n l}$$

$$\omega_n = C_n \sqrt{\frac{EI}{ml^4}}; C_n = \left(\lambda_n l\right)^2$$

$$\left\{C_n\right\}_{n=1}^{5} = 15.4118, 49.9648, 104.2477, 178.2697, 272.0309$$

$$\left\{\sigma_n\right\}_{n=1}^{5} = 1.000777, 1.000001, 1.000000, 1.00000, 1.00000$$

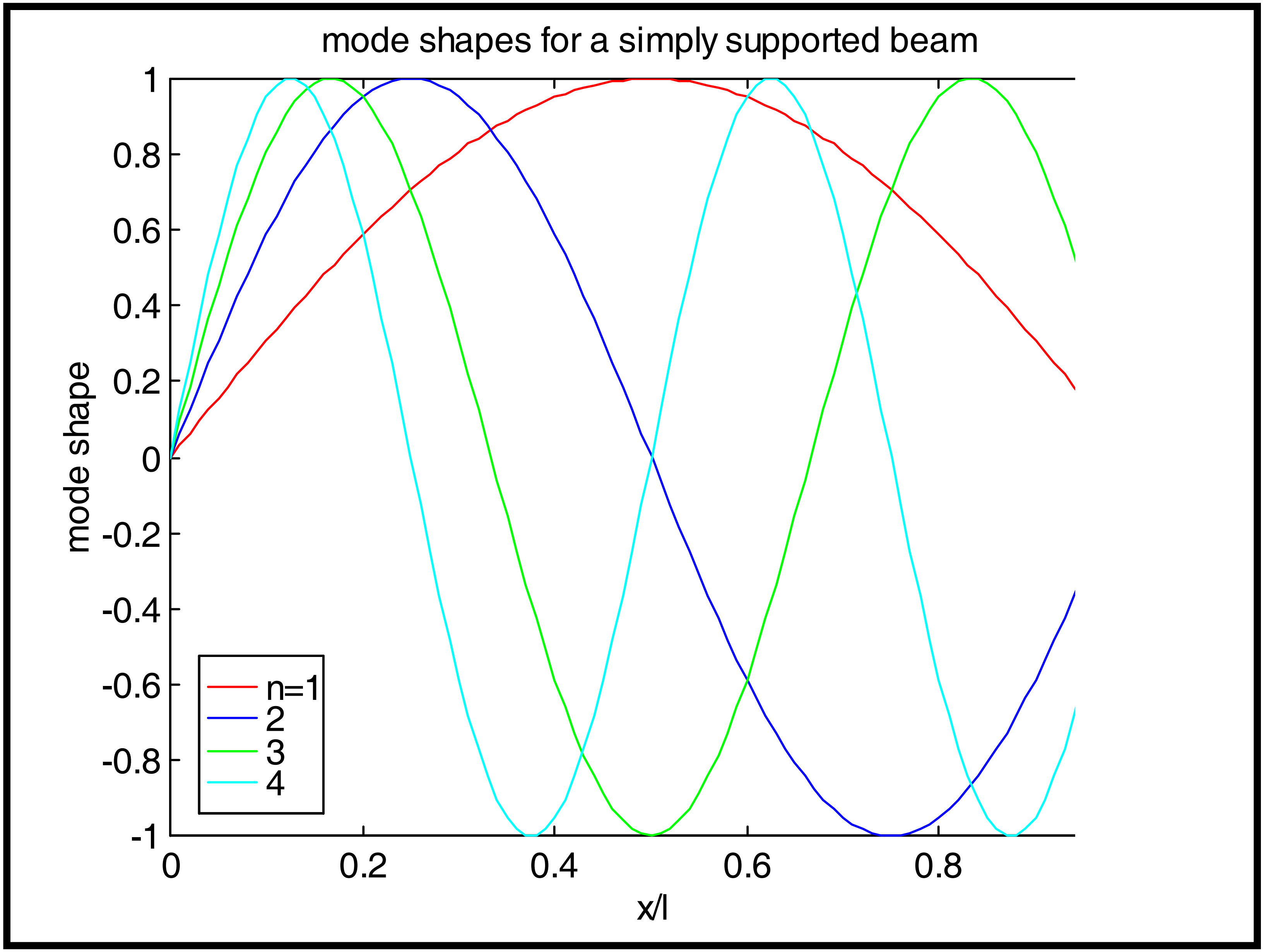


Homogeneous simply supported beam

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}} \qquad n = 1, 2, \cdots \infty$$

$$\varphi_n(x) = \sin\frac{n\pi x}{L}$$

Exercise


mode shapes for a simply supported beam
mode shape
x/l
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.2
0.4
0.6
0.8
n=1
2
3
4

Orthogonality conditions

$$[EI\phi_n'']'' = m\omega_n^2\phi_n \cdots\cdots(1)$$

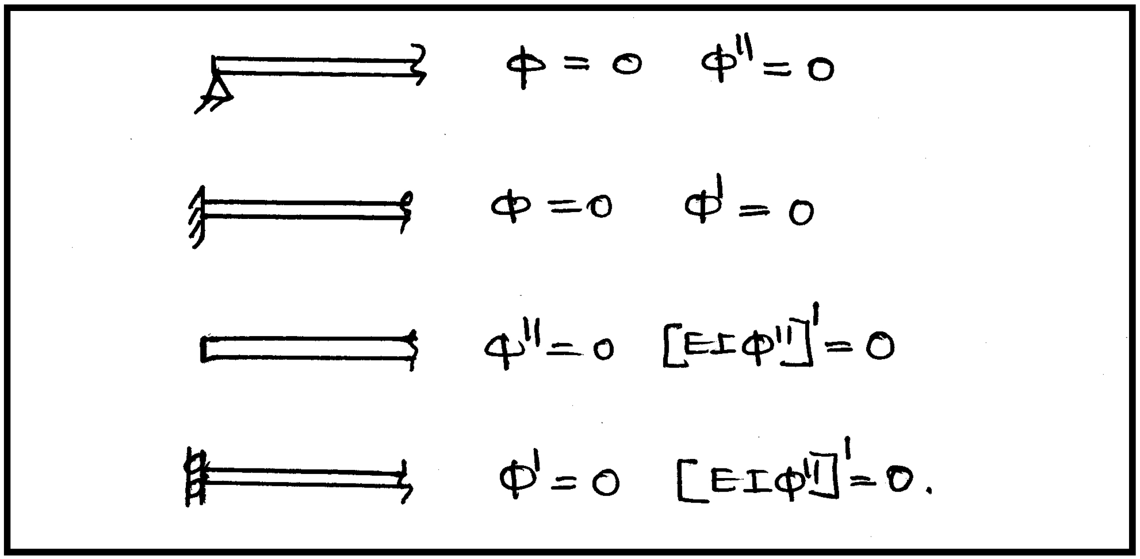
$$[EI\phi_k'']'' = m\omega_k^2\phi_k \cdots\cdots(2)$$

$$\int_0^L \phi_k[EI\phi_n'']''dx = \int_0^L m\omega_n^2\phi_k\phi_n dx \cdots\cdots(3)$$

$$\int_0^L \phi_n[EI\phi_k'']''dx = \int_0^L m\omega_k^2\phi_k\phi_n dx \cdots\cdots(4)$$

$$\int_0^L \phi_n [EI\phi_k'']'' dx = \{\phi_k [EI\phi_k'']'\}_0^L - \int_0^L \phi_n' [EI\phi_k'']' dx$$

$$= \{\phi_k [EI\phi_k'']'\}_0^L - \{\phi_n' [EI\phi_k'']\}_0^L + \int_0^L \phi_n'' EI \phi_k'' dx$$



$\Phi^t K \Phi$ being diagonal

$$\Rightarrow \int_0^L EI(x)\phi_n''(x)\phi_k''(x)dx = 0 \;\; for \; n \neq k$$

$$\int_0^L m(x)\phi_n(x)\phi_k(x)dx = 0 \; for \; n \neq k$$

lly to $\Phi^t M \Phi$ diagonal

Forced response analysis using eigenfunction expansion

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}+\varepsilon(x)\frac{\partial^3 y}{\partial x^2\partial t}\right]+m(x)\ddot{y}+c(x)\dot{y}=f(x,t)$$

ICS: $y_0(x)=y(x,0)\quad \dot{y}_0(x)=\dot{y}(x,0)$ & BCS as appropriate.

Known

$$y(x,t)=\sum_{n=1}^{\infty}a_n(t)\varphi_n(x)$$

Unknowns (Gen-co-ordinates

$$\left[EI\varphi_n''\right]''=m\omega_n^2\varphi_n(x)$$

$$\int_0^L EI\varphi_n''\varphi_k''dx=0\quad n\neq k\qquad \int_0^L m\varphi_n\varphi_k dx=0\quad n\neq k$$

$$[EI\sum_{n=1}^{\infty} a_n(t)\varphi_n'' + \varepsilon\sum_{n=1}^{\infty}\dot{a}_n(t)\varphi_n'']'' + c(x)\sum_{n=1}^{\infty}\dot{a}_n(t)\varphi_n + m(x)\sum_{n=1}^{\infty}\ddot{a}_n(t)\varphi_n = f(x,t)$$

$$\sum_{n=1}^{\infty} a_n(t)[EI\varphi_n'']'' + \sum_{n=1}^{\infty}\dot{a}_n(t)[\varepsilon(x)\varphi_n'']'' + c(x)\sum_{n=1}^{\infty}\dot{a}_n(t)\varphi_n + m(x)\sum_{n=1}^{\infty}\ddot{a}_n(t)\varphi_n = f(x,t)$$

$$\sum_{n=1}^{\infty} a_n(t)m\omega_n^2\varphi_n + \sum_{n=1}^{\infty}\dot{a}_n(t)[\varepsilon(x)\varphi_n'']'' + c(x)\sum_{n=1}^{\infty}\dot{a}_n(t)\varphi_n + m(x)\sum_{n=1}^{\infty}\ddot{a}_n(t)\varphi_n = f(x,t)$$

Proportional damping model

$C = \alpha M + \beta K$

$$\varepsilon(x) = \nu EI(x)$$

(Damping force proportional to time rate of bending strain)

$$c(x) = \alpha m(x)$$

(Damping force is proportional to velocity)

$$\sum_{n=1}^{\infty} a_n(t) m\omega_n^2\varphi_n + v\sum_{n=1}^{\infty}\dot{a}_n(t)[EI(x)\varphi_n'']'' + \alpha m(x)\sum_{n=1}^{\infty}\dot{a}_n(t)\varphi_n + m(x)\sum_{n=1}^{\infty}\ddot{a}_n(t)\varphi_n = f(x,t)$$

$$\sum_{n=1}^{\infty} a_n(t) m\omega_n^2\varphi_n + v\sum_{n=1}^{\infty}\dot{a}_n(t) m\omega_n^2\varphi_n(x) + \alpha m(x)\sum_{n=1}^{\infty}\dot{a}_n(t)\varphi_n + m(x)\sum_{n=1}^{\infty}\ddot{a}_n(t)\varphi_n = f(x,t)$$

$$\sum_{n=1}^{\infty} a_n(t)\omega_n^2\int_0^L m(x)\phi_k\phi_n dx + v\sum_{n=1}^{\infty}\dot{a}_n(t)\omega_n^2\int_0^L m(x)\phi_k\phi_n dx$$

$$+\sum_{n=1}^{\infty}\dot{a}_n(t)\int_0^L c(x)\phi_k\phi_n dx + \sum_{n=1}^{\infty}\ddot{a}_n(t)\int_0^L m(x)\phi_k\phi_n dx = \int_0^l \phi_k(x) f(x,t) dx$$

$\alpha m(x)$

$$\Rightarrow m_n \ddot{a}_n + m_n(\alpha + \nu\omega_n^2)\dot{a}_n + m_n\omega_n^2 a_n = \overline{p}_n(t)$$

$$\Rightarrow \ddot{a}_n + (\alpha + \nu\omega_n^2)\dot{a}_n + \omega_n^2 a_n = \frac{\overline{p}_n(t)}{m_n} = p_n(t)$$

$n = 1, 2, \cdots \infty$

$m_n = \int_0^L m(x)\phi_n^2(x)\,dx$

$$\ddot{a}_n + 2\eta_n\omega_n\dot{a}_n + \omega_n^2 a_n = p_n(t);$$

$$2\eta_n\omega_n = (\alpha + \nu\omega_n^2);$$

$$p_n(t) = \frac{\int_0^L \varphi_n(x) f(x,t)\,dx}{\int_0^L \varphi_n^2(x) m(x)\,dx} \qquad n = 1, 2, \cdots \infty$$

Initial conditions

$$y(x,t) = \sum_{k=1}^{\infty} a_k(t)\phi_k(x)$$

$$\Rightarrow y(x,0) = \sum_{k=1}^{\infty} a_k(0)\phi_k(x)$$

$$\Rightarrow M_n a_n(0) = \int_0^L m(x)\phi_n(x)y(x,0)dx \quad M_n \dot{a}_n(0) = \int_0^L m(x)\phi_n(x)\dot{y}(x,0)dx$$

Final solution

$$y(x,t) = \sum_{n=1}^{\infty} \phi_n(x) \left\{ \exp(-\eta_n \omega_n t)[A_n \cos \omega_{dn} t + B_n \sin \omega_{dn} t] + \int_0^t h_n(t-\tau) p_n(\tau) d\tau \right\}$$

σ

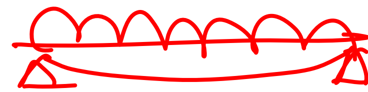
$$\theta(x,t) = \frac{\partial y}{\partial x} \quad BM = EI \frac{\partial^2 y}{\partial x^2}$$

Remark:

$$SF = \frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right)$$

Once the displacement field is found, we can easily find stress resultants (BM and SF) and the required bending/shear stresses

Example 1: **An undamped simply supported beam carries an udl. The load is suddenly removed. Determine the ensuing vibrations.**



Solution:

The initial deflection of the beam satisfies the equation

$$EIy_0^{iv} = q$$

Assume that the initial velocity of the beam is zero.

$$y(x,t)=\sum_{n=1}^{\infty}a_n(t)\varphi_n(x) \qquad \ddot{a}_n+\omega_n^2 a_n=0$$

$$\Rightarrow a_n(t)=A_n\cos\omega_n t+B_n\sin\omega_n t$$

$$\dot{y}(x,0)=0\Rightarrow \dot{a}_n(0)=0\Rightarrow y(x,t)=\sum_{n=1}^{\infty}A_n\cos\omega_n t\sin\frac{n\pi x}{L}$$

$$\Rightarrow y(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}=y_0(x)$$

$$EIy_0^{iv}=q\Rightarrow\sum_{n=1}^{\infty}EI\left(\frac{n\pi}{L}\right)^4 A_n\sin\frac{n\pi x}{L}=q\Rightarrow A_n=\frac{4L^4}{n^5\pi^5}\left(\frac{q}{EI}\right)$$

$$y(x,t)=\sum_{n=1,3,5,\cdots}^{\infty}\frac{4L^4}{n^5\pi^5}\left(\frac{q}{EI}\right)\cos\omega_n t\sin\frac{n\pi x}{L}$$

<u>Example 2</u>: An udl is suddenly applied on an undamped simply supported beam. Determine the ensuing vibrations. Assume that the beam is at rest at *t=0*.

$$EIy^{iv} + m\ddot{y} = qU(t)$$

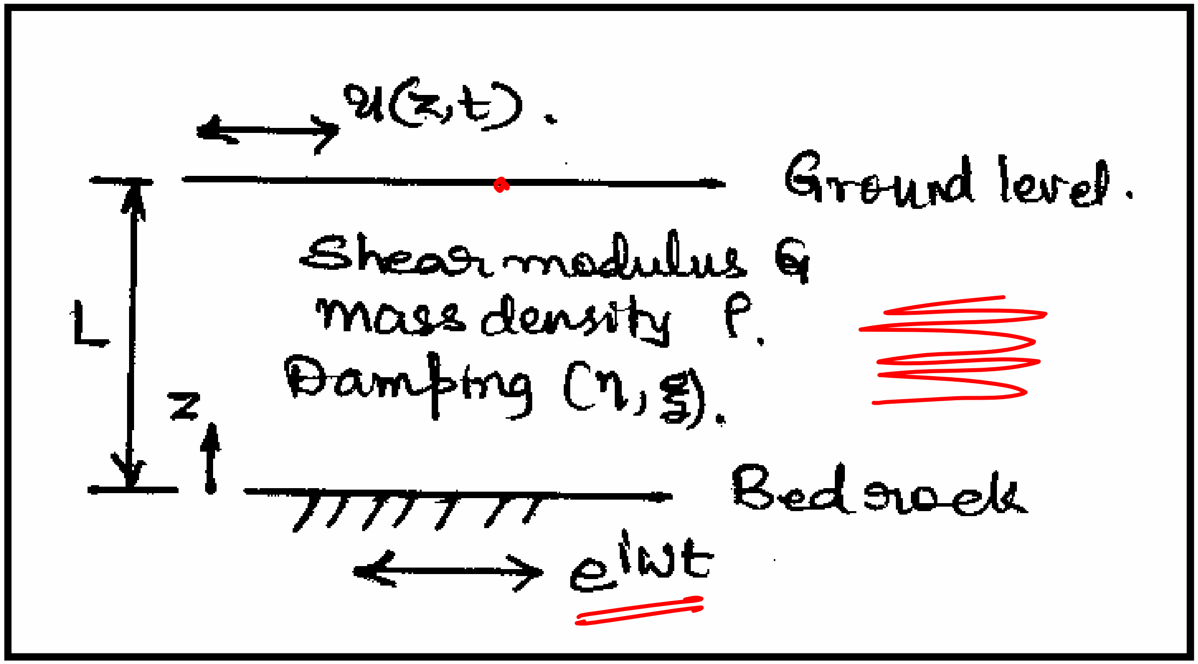
$$y(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{L}$$

$$\ddot{a}_n + \omega_n^2 a_n = \frac{q}{m_n}\int_0^L U(t) q \sin\frac{n\pi x}{L}\,dx$$

Ex

$$y(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{2qL}{n\pi m_n \omega_n^2}(1-\cos\omega_n t)\sin\frac{n\pi x}{L}$$

Seismic wave amplification through soil layers



$$\rho \frac{\partial^2 u}{\partial t^2} = G \frac{\partial^2 u}{\partial z^2} + \eta \frac{\partial^3 u}{\partial z^2 \partial t}$$

$$u(0,t) = \exp(i\omega t); \frac{\partial u}{\partial z}(L,t) = 0$$

$$u(z,t) = \phi(z)\exp(i\omega t)$$

$$\Rightarrow \quad -\rho\omega^2\phi\exp(i\omega t) = G\phi''\exp(i\omega t) + i\eta\omega\phi''\exp(i\omega t)$$

$$\Rightarrow \quad \phi''(G + i\eta\omega) + \rho\omega^2\phi = 0$$

$$\Rightarrow \quad \phi'' + \lambda^2\phi = 0; \quad \lambda^2 = \frac{\rho\omega^2}{(G + i\eta\omega)}$$

$\phi(0) = 1$

$\phi'(L) = 0$

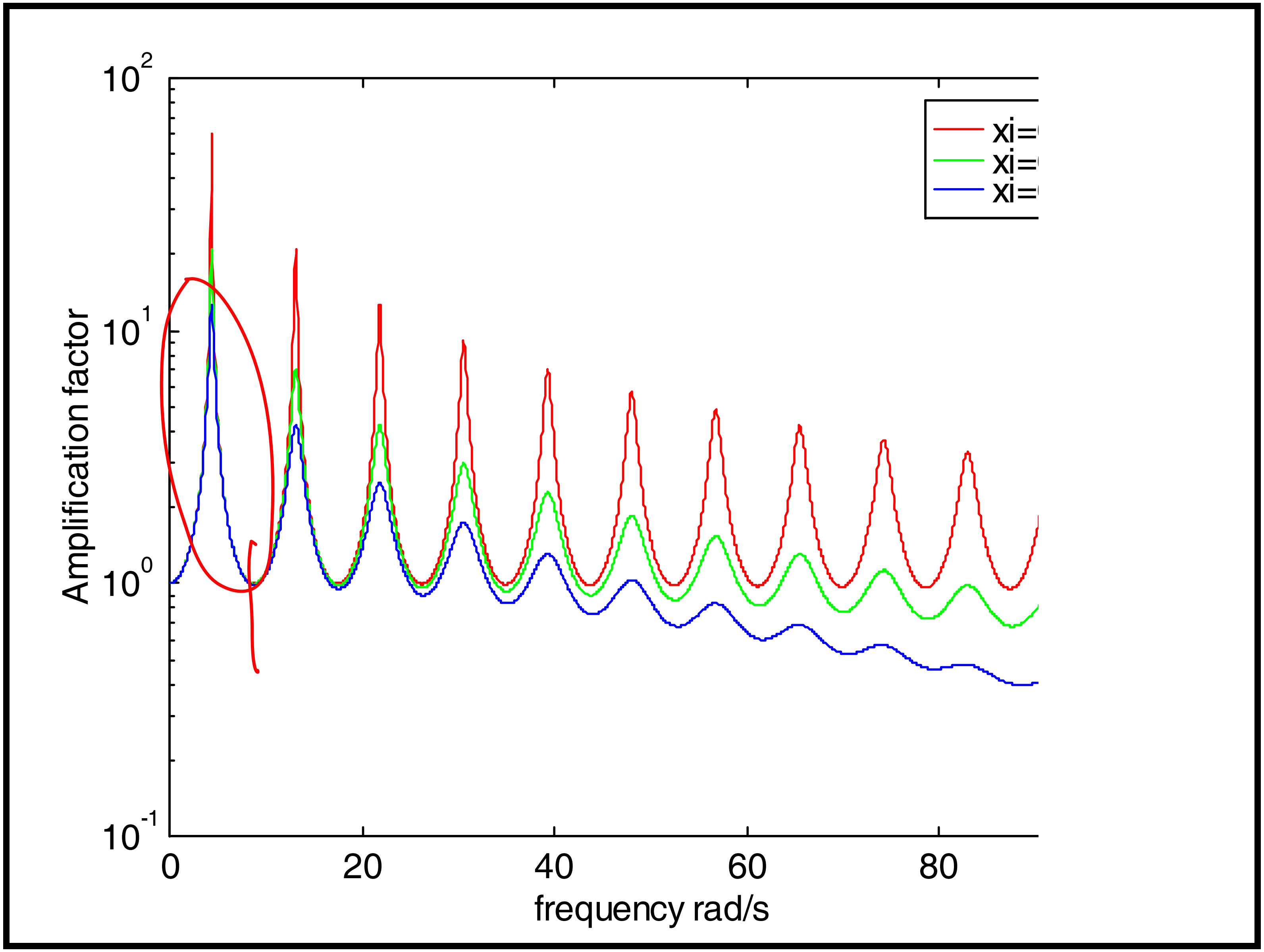
$$\phi(z) = A\cos\lambda z + B\sin\lambda z$$

$$\phi(0) = 1 \quad \phi'(L) = 0$$

$$\Rightarrow \quad \phi(x) = \cos\lambda z + \tan\lambda L\sin\lambda z$$

$$\phi(L) = \frac{1}{\cos\lambda L} = \frac{1}{\cos\left(\frac{\omega L}{\nu^*}\right)}$$

$$\nu^* = \sqrt{\frac{G^*}{\rho}} = \sqrt{\frac{G(1+i\omega\eta)}{\rho}} = \sqrt{\frac{G(1+2i\xi)}{\rho}}$$



10^2
10^1
10^0
10^-1
Amplification factor
0
20
40
60
80
frequency rad/s
xi=
xi=
xi=