Stochastic Structural Dynamics

Lecture-14

Random vibration of MDOF systems - 2

Dr C S Manohar
Department of Civil Engineering
Professor of Structural Engineering
Indian Institute of Science
Bangalore 560 012 India
manohar@civil.iisc.ernet.in

Review of dynamics of mdof systems

- Coupling and non-diagonal nature of structural matrices
- Natural coordinates
- Normal modes & natural frequencies
- Orthogonality of normal modes
- Uncoupling of equations of motion
- Classical damping models
- Input-output relations in frequency domain

MDOF system with *s*-th dof driven by an unit harmonic force

$$M\ddot{X} + C\dot{X} + KX = F\exp(i\omega t)$$

$$F^t = \{0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0\}$$

$s -$ th entry

$X_{rs}(t) =$ response of the r-th coordinate due to unit harmonic driving at s-th coordinate.

$$\lim_{t\to\infty} X_{rs}(t) = ?$$

$$M\ddot{X} + C\dot{X} + KX = F\exp(i\omega t)$$

$$F^t = \{0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0\}$$

$$\lim_{t\to\infty} X(t) = X_0 \exp(i\omega t)$$

$$\Rightarrow \dot{X}(t) = X_0 i\omega \exp(i\omega t)$$

$$\ddot{X}(t) = -X_0\omega^2 \exp(i\omega t)$$

$$-MX_0\omega^2 \exp(i\omega t) + CX_0 i\omega \exp(i\omega t) + KX_0 \exp(i\omega t) = F\exp(i\omega t)$$

$$\left[-\omega^2 M + i\omega C + K\right] X_0 \exp(i\omega t) = F\exp(i\omega t)$$

$$\left[-\omega^2 M + i\omega C + K\right] X_0 = F$$

$$\left[-\omega^2 M + i\omega C + K\right] X_0 = F$$

$$X(t) = X_0 \exp(i\omega t) = \Phi Z_0 \exp(i\omega t)$$

$$\Phi^t M \Phi = I \;\&\; \Phi^t K \Phi = \Lambda$$

C is classical $\Rightarrow \Phi^t C \Phi = \Gamma$ (Diagonal) with $\Gamma_{nn} = 2\eta_n \omega_n$

$$\left[-\omega^2 M + i\omega C + K\right] \Phi Z_0 = F$$

$$\Phi^t \left[-\omega^2 M + i\omega C + K\right] \Phi Z_0 = \Phi^t F$$

$$\left[-\omega^2 \Phi^t M \Phi + i\omega \Phi^t C \Phi + \Phi^t K \Phi\right] Z_0 = \Phi^t F$$

$$\left[-\omega^2 I + i\omega \Gamma + \Lambda\right] Z_0 = \Phi^t F$$

Diagonal

$$\left[-\omega^2 I + i\omega\Gamma + \Lambda\right] Z_0 = \Phi^t F$$

$$Z_{0n} = \frac{\sum_{k=1}^{N} \Phi_{nk}^t F_k}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)} = \frac{\sum_{k=1}^{N} \Phi_{kn} F_k}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}$$

Recall

$F^t = \{0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0\}$ (s-th entry=1; rest=0)

$$\Rightarrow Z_{0n} = \frac{\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}$$

$$\lim_{t\to\infty} X(t) = \Phi Z_0 \exp(i\omega t) \Rightarrow X_r(t) = \sum_{n=1}^{N} \Phi_{rn} Z_{0n} \exp(i\omega t)$$

$$= \sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)} \exp(i\omega t)$$

$$X_{rs}(t) = H_{rs}(\omega) \exp(i\omega t)$$

$$H_{rs}(\omega) = \sum_{n=1}^{N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}$$

$$X_{rs}(t)=\sum_{n=1}^{N}\frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2-\omega^2+i2\eta_n\omega_n\omega\right)}\exp(i\omega t)$$

$$H_{rs}(\omega)=\sum_{n=1}^{N}\frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2-\omega^2+i2\eta_n\omega_n\omega\right)}$$

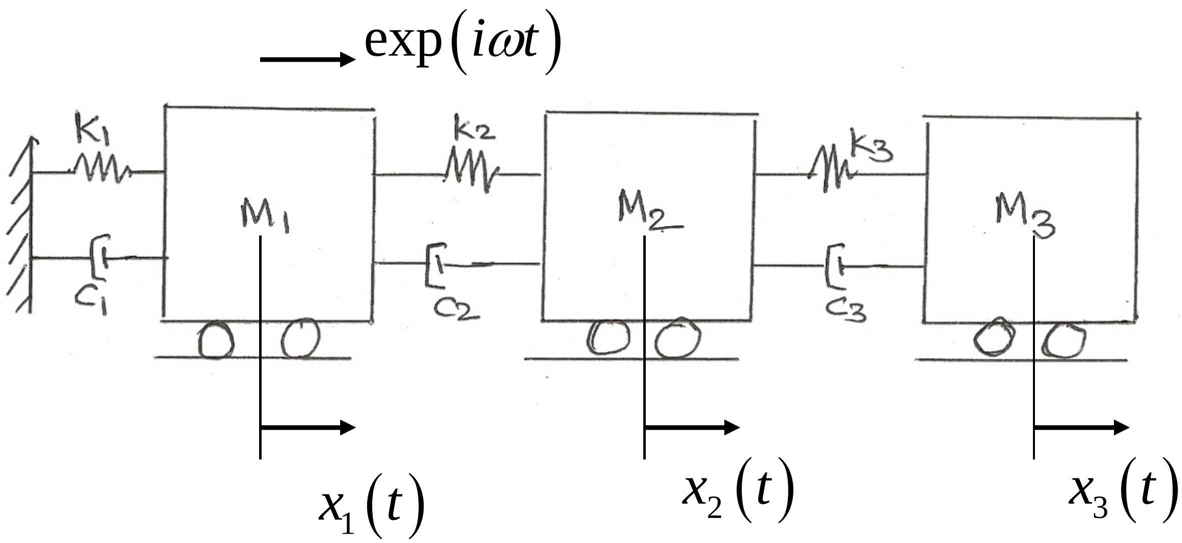
Remarks

- $X_{rs}(t)=X_{sr}(t)$
- $H_{rs}(\omega)=H_{sr}(\omega)$
- $\left[H(\omega)\right]=\left[H_{rs}(\omega)\right]$
- $\left[H(\omega)\right]$ is symmetric but not Hermitian
- $\left[H(\omega)\right]=\left[-\omega^2M+i\omega C+k\right]^{-1}=\left[\sum_{n=1}^{N}\frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2-\omega^2+i2\eta_n\omega_n\omega\right)}\right]$

- $\left[H(\omega)\right] = \left[-\omega^2 M + i\omega C + k\right]^{-1}$
  - Conceptually simple
  - Computationally difficult to implement

- $\left[H(\omega)\right] = \left[\sum_{n=1}^{N^* \le N} \frac{\Phi_{rn}\Phi_{sn}}{\left(\omega_n^2 - \omega^2 + i2\eta_n\omega_n\omega\right)}\right]$
  - Computationally easier to implement
  - Not all modes need to be included

    (Nor it is advisable to include all modes)



$$X_{rs}(t) = H_{rs}(\omega)\exp(i\omega t)$$

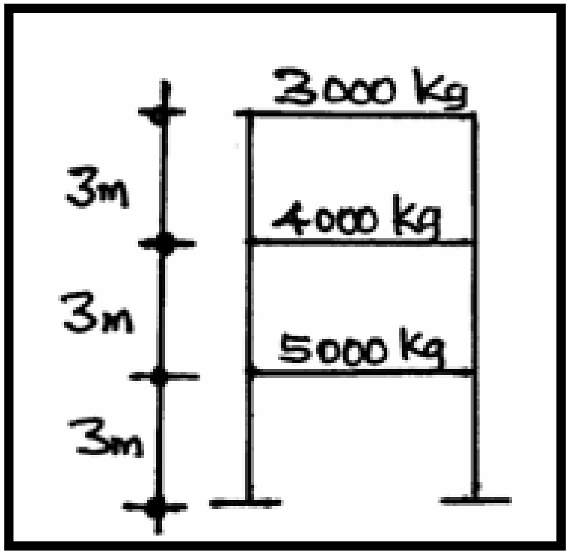
$$x_1(t) = X_{11}(t) = H_{11}(\omega)\exp(i\omega t)$$

$$x_2(t) = X_{12}(t) = H_{12}(\omega)\exp(i\omega t)$$

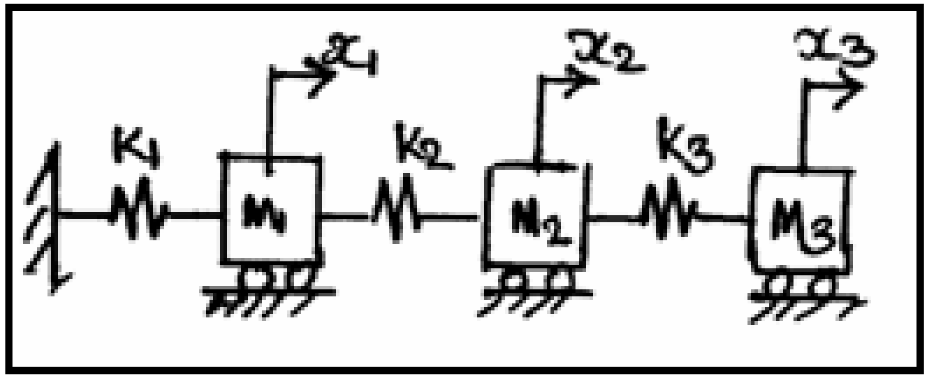
$$x_3(t) = X_{13}(t) = H_{13}(\omega)\exp(i\omega t)$$

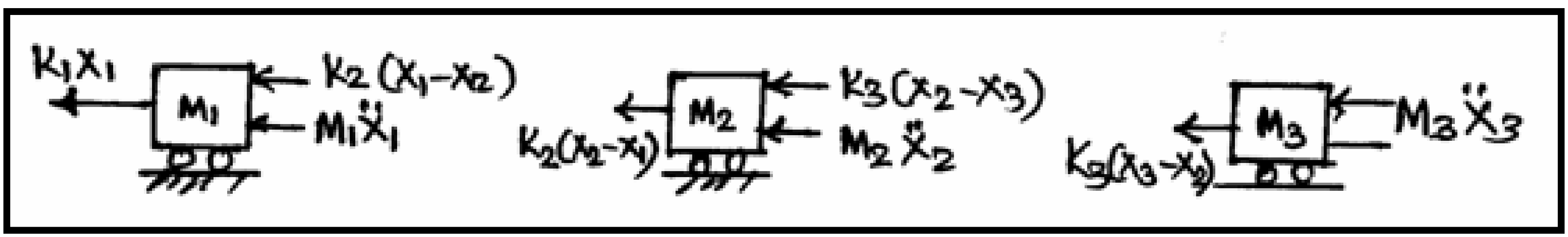
$$\left[H(\omega)\right] = \left[H_{rs}(\omega)\right]$$

Example



- $EI = 4.5 \times 10^6\ \text{Nm}^2$ for all columns
- η=0.03 for all modes





$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3(x_2 - x_3) = 0$$

$$m_3\ddot{x}_3 + k_3(x_3 - x_2) = 0$$

$$\begin{bmatrix} 5000 & 0 & 0 \\ 0 & 4000 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + 4\times 10^6 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

$$\begin{vmatrix} 8\times 10^6 - 5000\omega^2 & -4\times 10^6 & 0 \\ -4\times 10^6 & 8\times 10^6 - 4000\omega^2 & -4\times 10^6 \\ 0 & -4\times 10^6 & 4\times 10^6 - 3000\omega^2 \end{vmatrix} = 0$$

$$\omega^6 - 4933.3\omega^4 + 5.867\times 10^7\,\omega^2 - 1.066\times 10^9 = 0$$

$$\omega = \{14.86,\ 38.78,\ 56.64\} \quad \text{rad/s}$$

$$\Phi = \begin{bmatrix} 0.0058 & 0.0114 & -0.0061 \\ 0.0100 & 0.0014 & 0.0122 \\ 0.0120 & -0.0107 & -0.0087 \end{bmatrix}$$

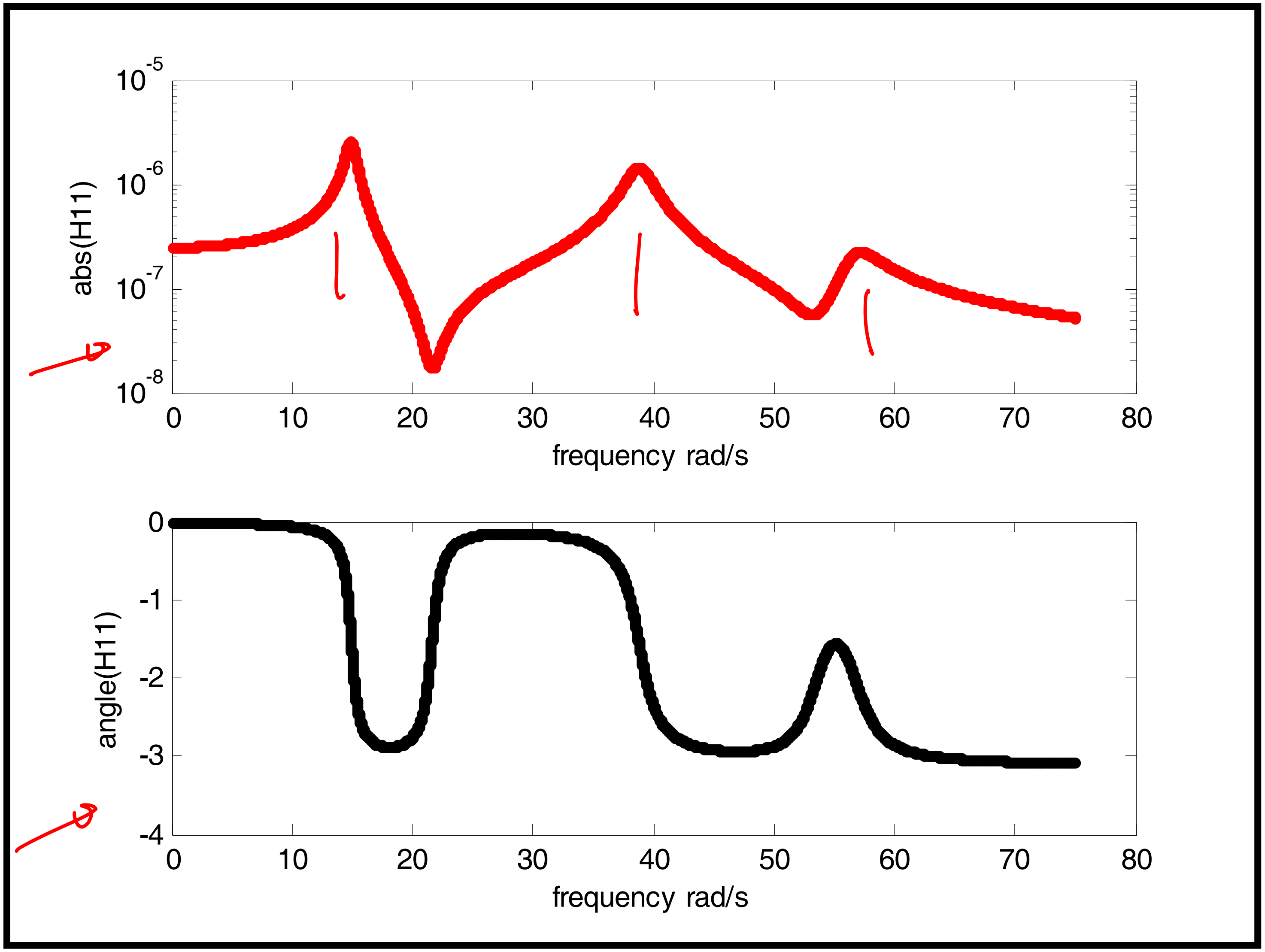
$$\Gamma_{nn} = 2\eta_n \omega_n$$

$$\phi^t C \phi = \Gamma$$

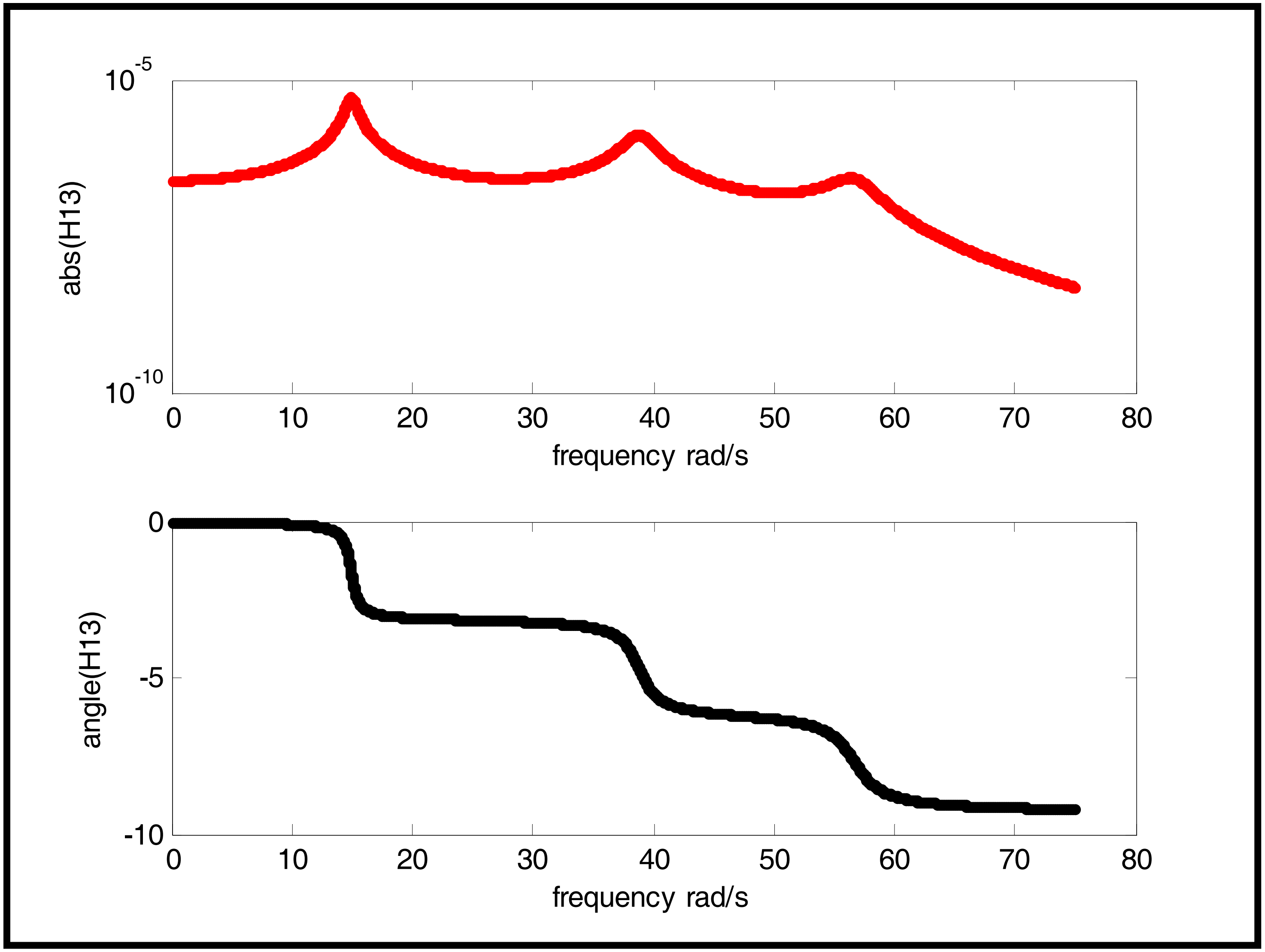
$$C = 1\times10^4 \begin{bmatrix} 1.1407 & -0.3244 & -0.0656 \\ -0.3244 & 0.9549 & -0.3419 \\ -0.0656 & -0.3419 & 0.5846 \end{bmatrix} \text{ Ns/m}$$

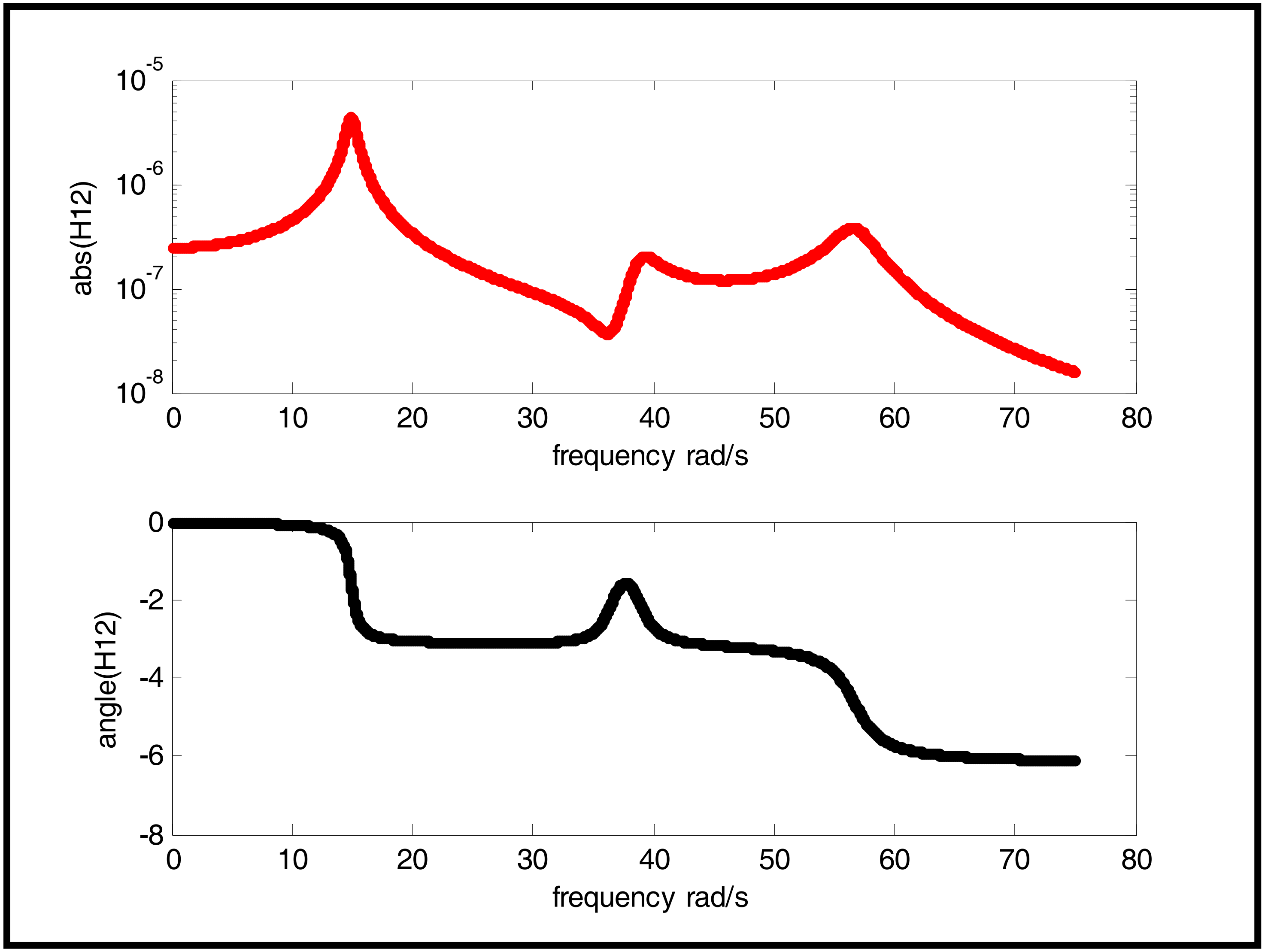
$$C = [\phi^t]^{-1} \Gamma \phi^{-1}$$

Clough & Penzien
Structural Dynamics



10^-5
10^-6
10^-7
10^-8
abs(H11)
0 10 20 30 40 50 60 70 80
frequency rad/s
0
-1
-2
-3
-4
angle(H11)
0 10 20 30 40 50 60 70 80
frequency rad/s


10^-5
10^-10
abs(H13)
0 10 20 30 40 50 60 70 80
frequency rad/s
0
-5
-10
angle(H13)
0 10 20 30 40 50 60 70 80
frequency rad/s



abs(H12)
10^-5
10^-6
10^-7
10^-8
0 10 20 30 40 50 60 70 80
frequency rad/s
angle(H12)
0
-2
-4
-6
-8
0 10 20 30 40 50 60 70 80
frequency rad/s

MDOF system with *s*-th dof driven by an unit impulse force

$$M\ddot{X} + C\dot{X} + KX = F\delta(t)$$

$$X(0) = 0; \dot{X}(0) = 0$$

$$F^t = \{0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0\}$$

$s -$ th entry

$X_{rs}(t) =$ response of the r-th coordinate due to unit impulse driving at s-th coordinate.

$$M\ddot{X} + C\dot{X} + KX = F\delta(t)$$

$$F^t = \{0 \quad 0 \quad \cdots \quad 1 \quad \cdots \quad 0 \quad 0\}$$

$$X(t) = \Phi Z(t)$$

$$\Phi^t M \Phi = I \;\&\; \Phi^t K \Phi = \Lambda$$

C is classical $\Rightarrow \Phi^t C \Phi = \Gamma$ (Diagonal) with $\Gamma_{nn} = 2\eta_n \omega_n$

$$M\Phi\ddot{Z}(t) + C\Phi\dot{Z}(t) + K\Phi Z(t) = F\delta(t)$$

$$\Phi^t M \Phi \ddot{Z}(t) + \Phi^t C \Phi \dot{Z}(t) + \Phi^t K \Phi Z(t) = \Phi^t F \delta(t)$$

$$I\ddot{Z} + \Gamma\dot{Z} + \Lambda Z = \Phi^t F \delta(t)$$

$$I\ddot{Z}+\Gamma\dot{Z}+\Lambda Z=\Phi^{t}F\delta(t)$$

$$\ddot{z}_n+2\eta_n\omega_n\dot{z}_n+\omega_n^2 z_n=\sum_{j=1}^{N}\Phi_{jn}F_j\delta(t)=\Phi_{ns}\delta(t)$$

sn

$$z_n(0)=0;\ \dot{z}_n(0)=0$$

$$\Rightarrow$$

$$z_n(t)=\Phi_{ns}h_n(t)=\frac{\Phi_{sn}}{\omega_{dn}}\exp(-\eta_n\omega_n t)\sin\omega_{dn}t$$

sn

$$X=\Phi Z\Rightarrow$$

$$X_r(t)=\sum_{n=1}^{N}\Phi_{rn}z_n(t)$$

$$h_{rs}(t)=\sum_{n=1}^{N}\Phi_{rn}\Phi_{sn}\frac{1}{\omega_{dn}}\exp(-\eta_n\omega_n t)\sin\omega_{dn}t$$

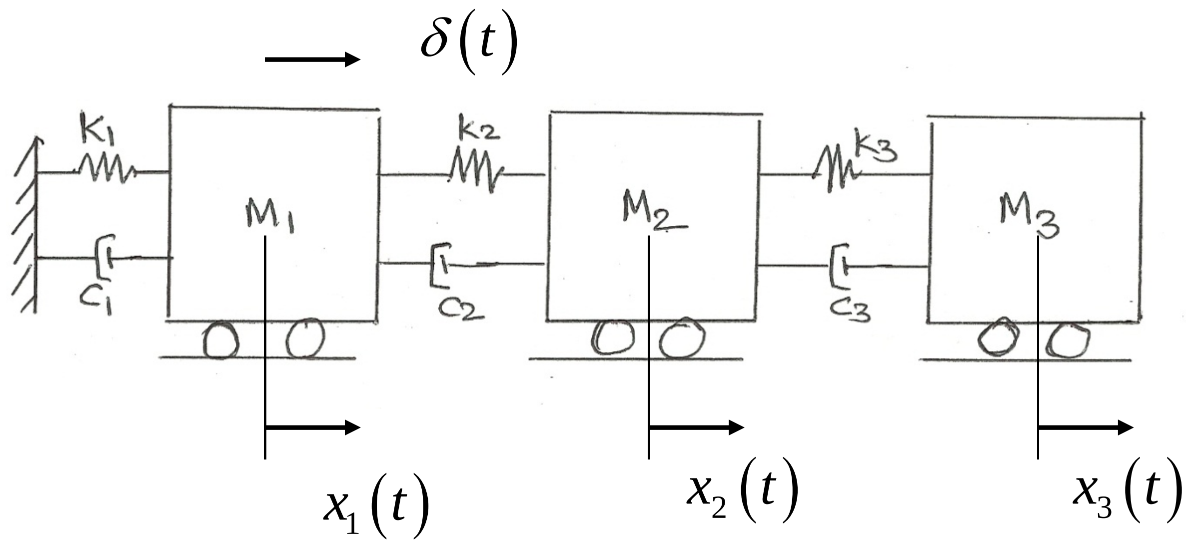
$$X_r(t) = h_{rs}(t) = \sum_{n=1}^{N} \Phi_{rn}\Phi_{sn}\frac{1}{\omega_{dn}}\exp(-\eta_n\omega_n t)\sin\omega_{dn}t$$

Remarks

- $h_{rs}(t) = h_{sr}(t)$
- $\left[h(t)\right] = \left[h_{rs}(t)\right] =$ Matrix of impulse response functions
- $\left[h(t)\right] = \left[h(t)\right]^t$
- Not all modes need to be included in the summation
- If an arbitrary load $f_s(\tau)$ is applied at the s-th dof (instead of unit impulse excitation)

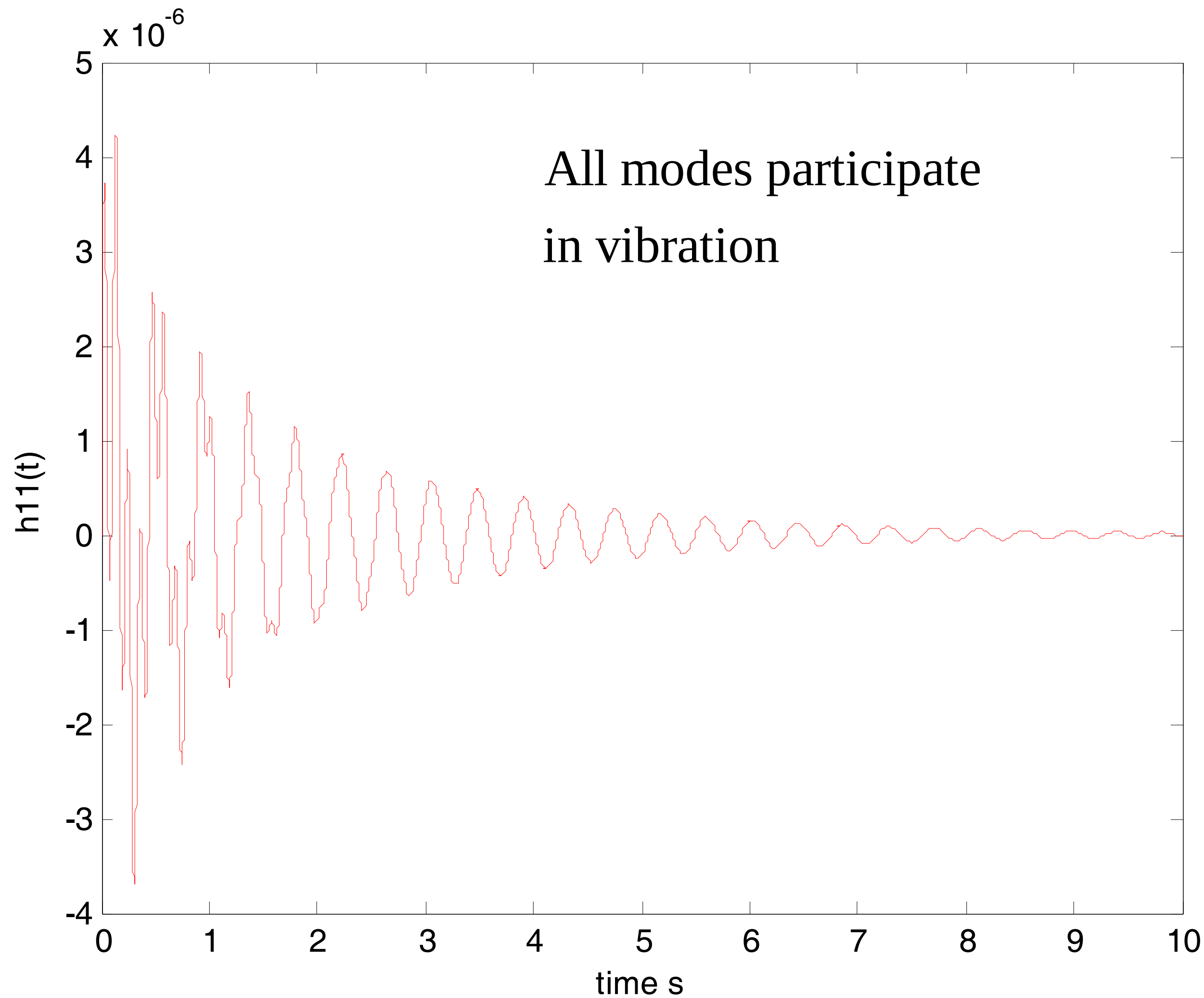
$$X_{rs}(t) = \int_0^t h_{rs}(t-\tau) f_s(\tau)\,d\tau$$

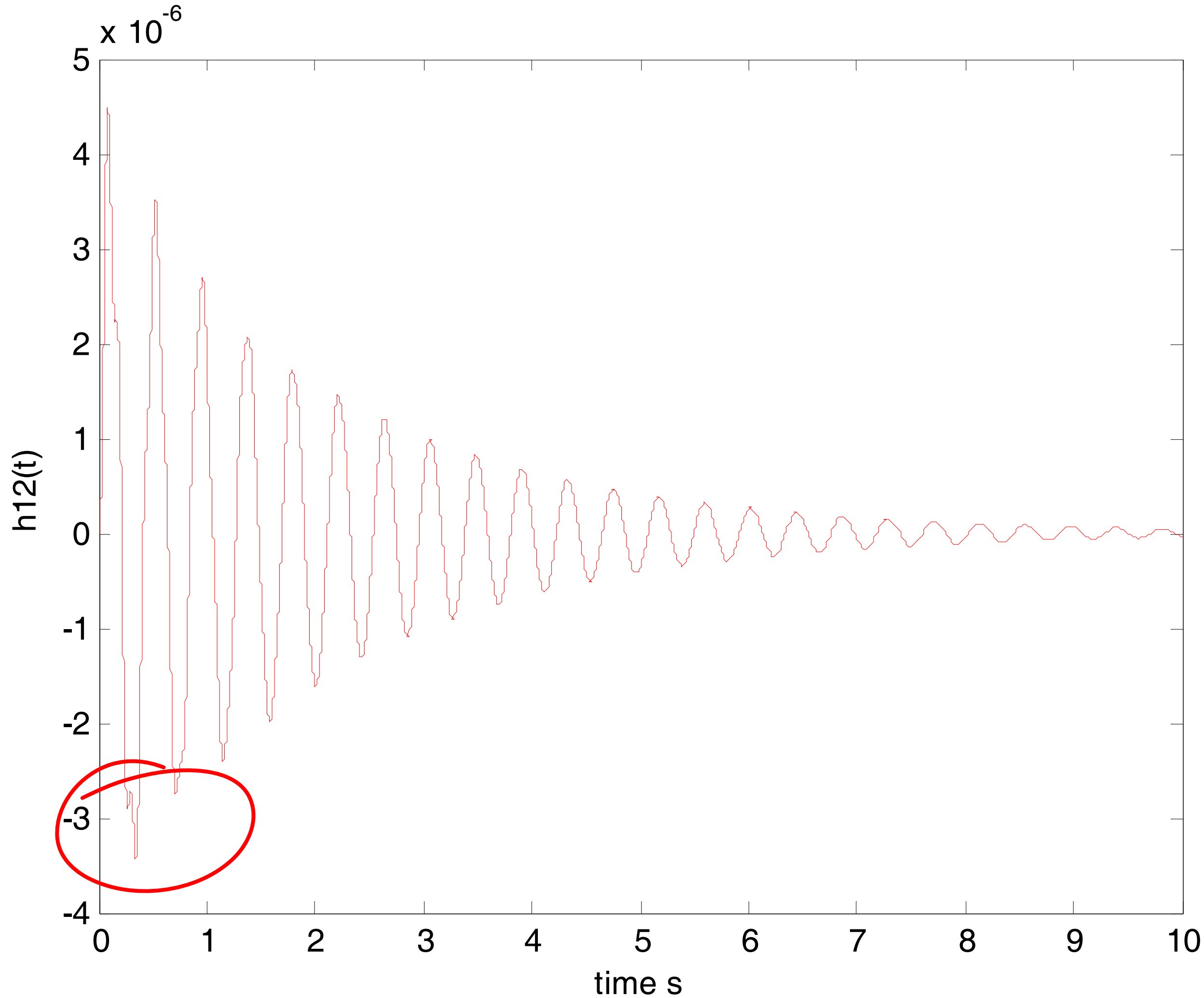
$$= \int_0^t f_s(\tau)\left\{\sum_{n=1}^{N} \Phi_{rn}\Phi_{sn}\frac{1}{\omega_{dn}}\exp\left[-\eta_n\omega_n(t-\tau)\right]\sin\omega_{dn}(t-\tau)\right\}d\tau$$



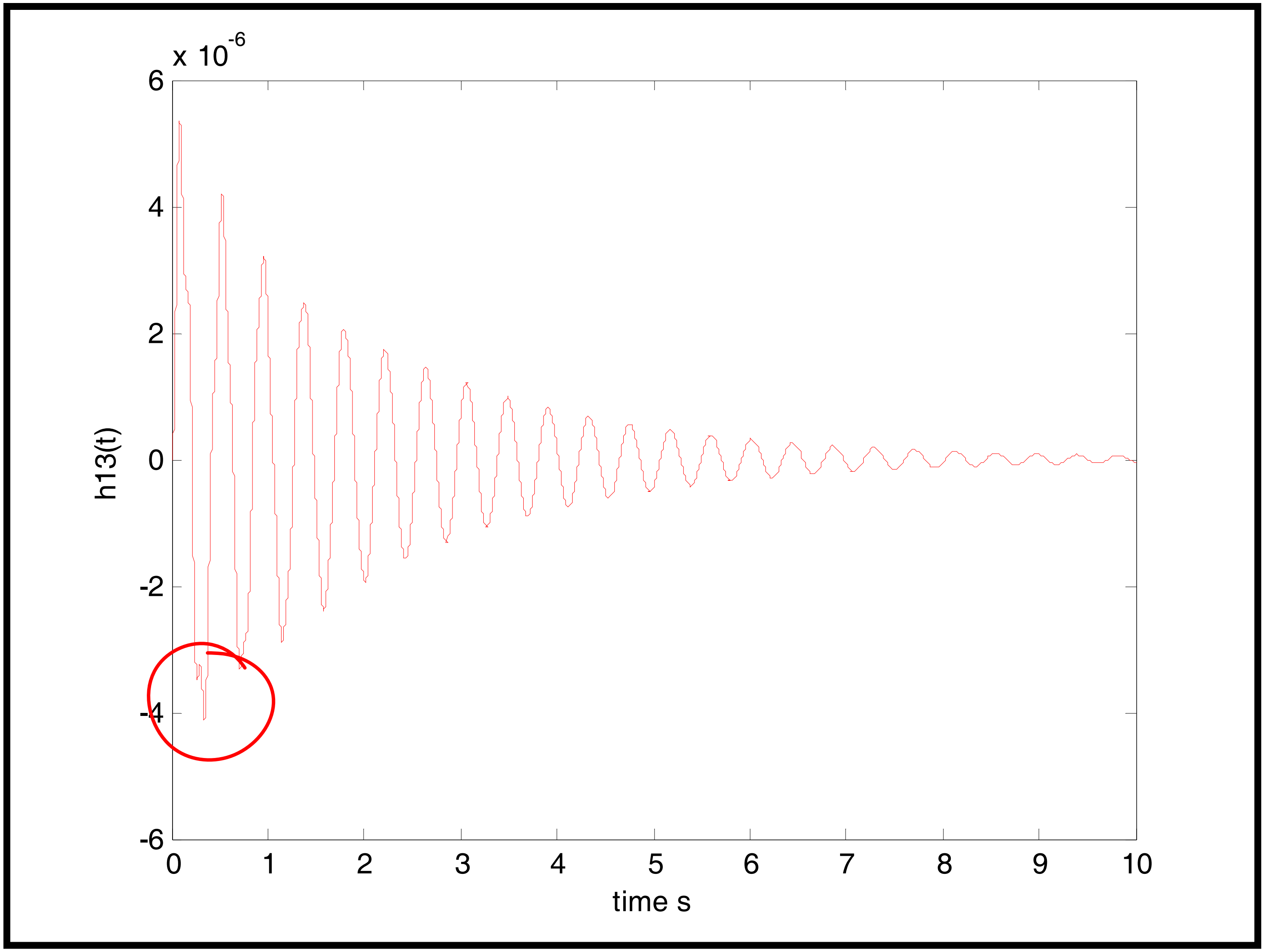
$$x_1(t) = X_{11}(t) = h_{11}(t)$$
$$x_2(t) = X_{12}(t) = h_{12}(t)$$
$$x_3(t) = X_{13}(t) = h_{13}(t)$$

$$\left[h(t)\right] = \left[h_{rs}(t)\right]$$


x 10^-6
5
4
3
2
1
0
-1
-2
-3
-4
h11(t)
All modes participate
in vibration
0
1
2
3
4
5
6
7
8
9
10
time s



x 10^-6
5
4
3
2
1
0
-1
-2
-3
-4
h12(t)
0
1
2
3
4
5
6
7
8
9
10
time s


x 10^-6
6
4
2
0
-2
-4
-6
h13(t)
0
1
2
3
4
5
6
7
8
9
10
time s

Recall

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(i\omega t)\,d\tau$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(-i\omega t)\,d\omega$$

Exercise

Show that

$$H_{ij}(\omega) = \int_{-\infty}^{\infty} h_{ij}(t)\exp(i\omega t)\,d\tau$$

$$h_{ij}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{ij}(\omega)\exp(-i\omega t)\,d\omega$$

MDOF system under stationary vector random excitation

Response analysis in frequency domain

$$M\ddot{X} + C\dot{X} + KX = F(t); X(0) = 0; \dot{X}(0) = 0$$

$$X(t) \rightarrow N \times 1$$

$$\langle F(t) \rangle = 0 \quad \langle F(t) F^t(t+\tau) \rangle = R_{FF}(\tau);$$

(N×1) (1×N) (N×N)

$R_{FF}(\tau)$ is a $N \times N$ matrix

We are interested in characterizing the response in the steady state.

$$X_T(\omega) = H(\omega) F_T(\omega)$$

(N×1)

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{T} \langle X_T(\omega) X_T^{*t}(\omega) \rangle$$

(N×N) (N×1) (1×N) N×N

$$= \lim_{T\to\infty} \frac{1}{T} \langle H(\omega) F_T(\omega) F_T^{*t}(\omega) H^{*t}(\omega) \rangle$$

$$= H(\omega) \lim_{T\to\infty} \frac{1}{T} \langle F_T(\omega) F_T^{*t}(\omega) \rangle H^{*t}(\omega)$$

$$= H(\omega) S_{FF}(\omega) H^{*t}(\omega)$$ ← MDOF

SDOFS: $S_{XX}(\omega) = |H(\omega)|^2 S_{FF}(\omega)$

MDOF system under stationary vector random excitation

Response analysis in time domain

$$M\ddot{X} + C\dot{X} + KX = F(t); X(0) = 0; \dot{X}\left(0\right) = 0$$

$$X\left(t\right) \rightarrow N \times 1$$

$$\langle F(t) \rangle = 0 \quad \left\langle F\left(t\right) F^{t}\left(t+\tau\right)\right\rangle = R_{FF}\left(\tau\right);$$

$R_{FF}\left(\tau\right)$ is a $N \times N$ matrix

$$X\left(t\right) = \int_0^t \left[h\left(t-\tau\right)\right] F\left(\tau\right) d\tau$$

N×1 N×N N×1

$$\left\langle X\left(t\right)\right\rangle = \int_0^t \left[h\left(t-\tau\right)\right] \left\langle F\left(\tau\right)\right\rangle d\tau = 0$$

$$X(t_1) = \int_0^{t_1} \left[h(t_1 - \tau_1)\right] F(\tau_1)\, d\tau_1$$

$$X(t_2) = \int_0^{t_2} \left[h(t_2 - \tau_2)\right] F(\tau_2)\, d\tau_1$$

$$X^t(t_2) = \int_0^{t_2} F^t(\tau_2)\left[h(t_2 - \tau_2)\right]^t d\tau_2$$

$$\left\langle X(t_1) X^t(t_2) \right\rangle = \int_0^{t_1}\int_0^{t_2} \left[h(t_1 - \tau_1)\right] \left\langle F(\tau_2) F^t(\tau_2) \right\rangle \left[h(t_2 - \tau_2)\right]^t d\tau_1 d\tau_2$$

$$R_{XX}(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} \left[h(t_1 - \tau_1)\right] R_{FF}(\tau_1, \tau_2) \left[h(t_2 - \tau_2)\right]^t d\tau_1 d\tau_2$$

$$R_{XX}(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} \left[h(t_1 - \tau_1)\right] R_{FF}(\tau_2 - \tau_1) \left[h(t_2 - \tau_2)\right]^t d\tau_1 d\tau_2$$

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

System starts from rest.

$$X(t) = \Phi Z(t) \Rightarrow X_k(t) = \sum_{n=1}^{N} \Phi_{kn} Z_n(t)$$

$$\ddot{Z}_n + 2\eta_n \omega_n \dot{Z}_n + \omega_n^2 Z_n = p_n(t)$$

$$p(t) = \Phi^t F(t)$$

$$p_n(t) = \sum_{s=1}^{N} \Phi^t_{ns} F_s(t) = \sum_{s=1}^{N} \Phi_{sn} F_s(t)$$

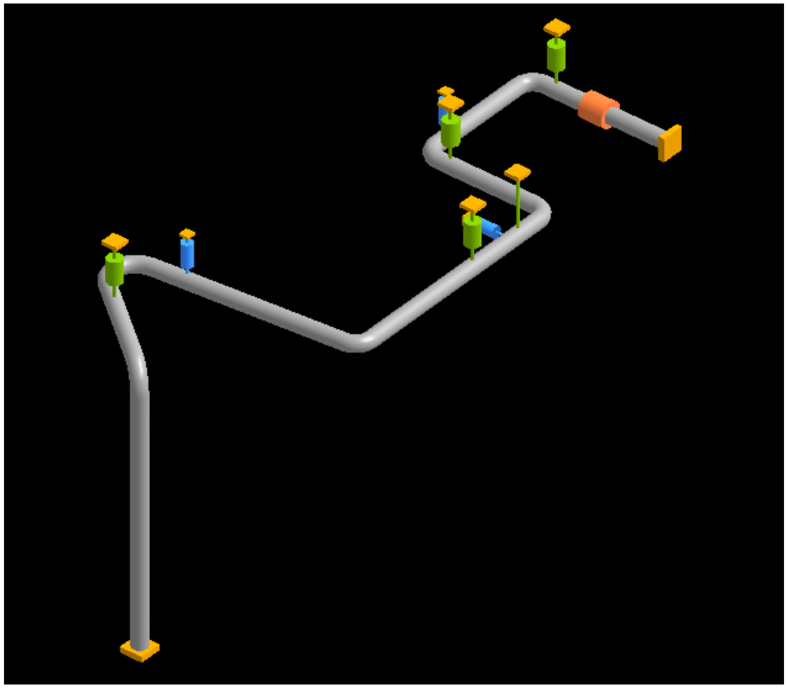
$$Z_n(t) = \int_0^t h_n(t-\tau) \left[\sum_{s=1}^{N} \Phi_{sn} F_s(\tau) \right] d\tau$$

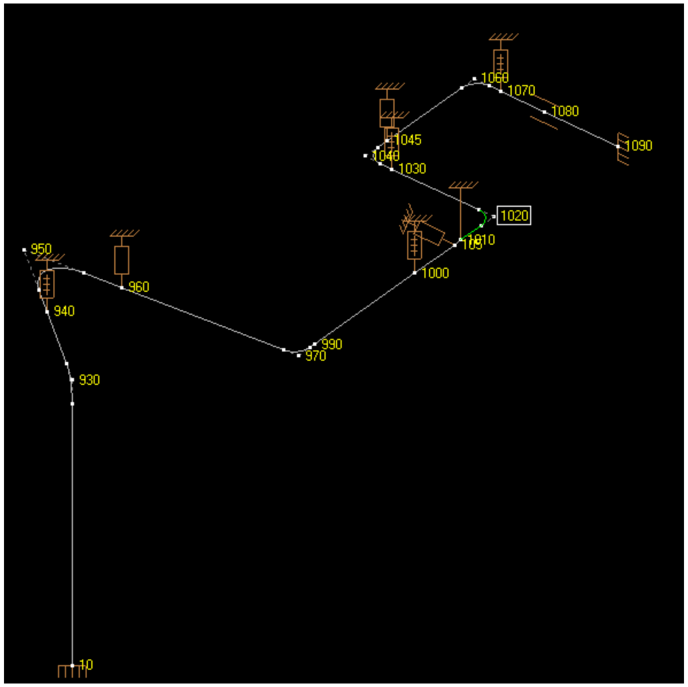
$$X_k(t) = \sum_{n=1}^{N} \Phi_{kn} \int_0^t h_n(t-\tau) \left[\sum_{s=1}^{N} F_s(\tau) \right] d\tau = \sum_{n=1}^{N} \sum_{s=1}^{N} \Phi_{kn} \Phi_{sn} \int_0^t h_n(t-\tau) F_s(\tau) d\tau$$

Based on this expression, we can evaluate mean, covariance and other moments of $X(t)$.

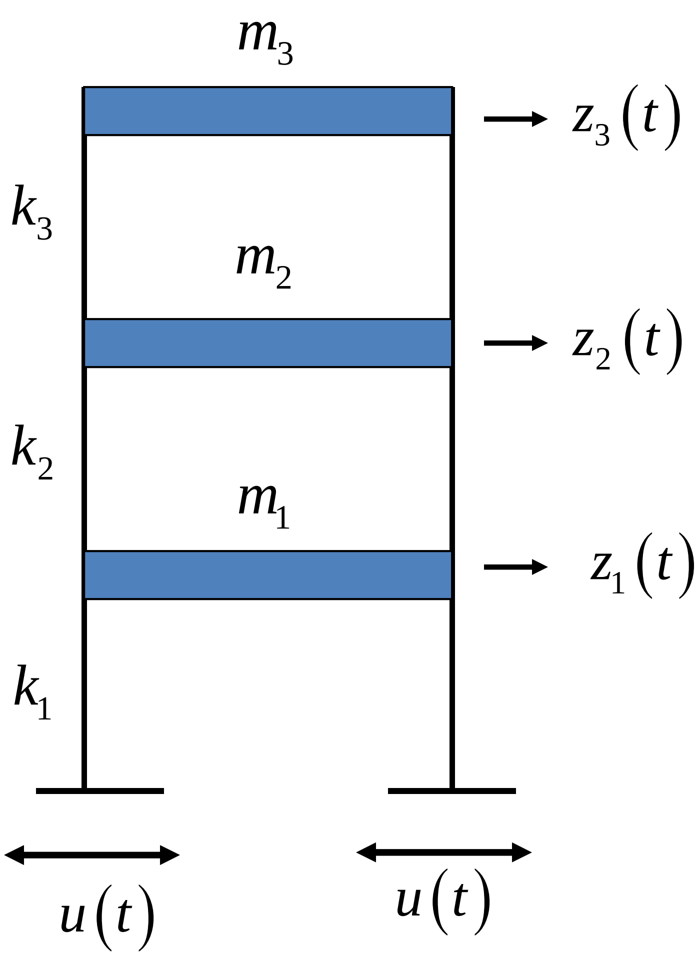
MDOF systems under random support motions

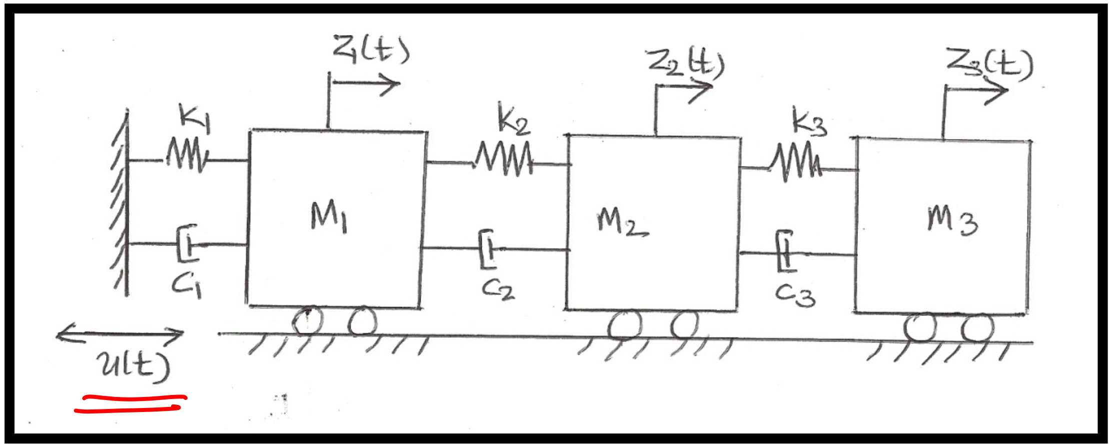
Case of uniform support motions





A building frame under random support motion





$$\langle u(t) \rangle = 0$$

$$\langle u(t)u(t+\tau) \rangle = R_{uu}(\tau)$$

$$m_1\ddot{z}_1 + c_1(\dot{z}_1 - \dot{u}) + c_2(\dot{z}_1 - \dot{z}_2) + k_1(z_1 - u) + k_2(z_1 - z_2) = 0$$

$$m_2\ddot{z}_2 + c_2(\dot{z}_2 - \dot{z}_1) + c_3(\dot{z}_2 - \dot{z}_3) + k_2(z_2 - z_1) + k_3(z_2 - z_3) = 0$$

$$m_3\ddot{z}_3 + c_3(\dot{z}_3 - \dot{z}_2) + k_3(z_3 - z_2) = 0$$

$$x_1 = z_1 - u$$

$$x_2 = z_2 - u$$

$$x_3 = z_3 - u$$

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = -m_1\ddot{u}$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + c_3(\dot{x}_2 - \dot{x}_3) + k_2(x_2 - x_1) + k_3(x_2 - x_3) = -m_2\ddot{u}$$

$$m_3\ddot{x}_3 + c_3(\dot{x}_3 - \dot{x}_2) + k_3(x_3 - x_2) = -m_3\ddot{u}$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = -\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \ddot{u}$$

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

$$F(t) = -M\{\mathbf{1}\}\ddot{u}(t)$$

$$F_T(\omega) = -M\{\mathbf{1}\}\ddot{U}_T(\omega)$$

$$S_{FF}(\omega) = M\{\mathbf{1}\}\left\langle \ddot{U}_T(\omega)\ddot{U}_T^*(\omega)\right\rangle\{\mathbf{1}\}^t M = M\{\mathbf{1}\}S_{\ddot{u}\ddot{u}}(\omega)\{\mathbf{1}\}^t M$$

$$S_{XX}(\omega) = H(\omega)S_{FF}(\omega)H^{*t}(\omega)$$

$$S_{XX}(\omega) = H(\omega)M\{\mathbf{1}\}S_{\ddot{u}\ddot{u}}(\omega)\{\mathbf{1}\}^t MH^{*t}(\omega)$$

$$H(\omega) = \left[-\omega^2 M + i\omega C + K\right]^{-1}$$

$$S_{FF}(\omega) = \lim_{T\to\infty}\frac{1}{T}\left\langle F_T(\omega)\,F_T^*(\omega)\right\rangle$$

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

$$F(t) = -M\{\mathbf{1}\}\ddot{u}(t)$$

$$\langle F(t)\rangle = -M\{\mathbf{1}\}\langle \ddot{u}(t)\rangle = 0$$

$$R_{FF}(t_1,t_2) = \langle F(t_1)F^t(t_2)\rangle \quad (N\times N)$$

$$= M\{\mathbf{1}\}\langle \ddot{u}(t_1)\ddot{u}(t_2)\rangle\{\mathbf{1}\}^t M \quad = M\{1\}R_{\ddot{u}\ddot{u}}(t_2,t_1)\{1\}^t M$$

$$\Rightarrow X(t) = \int_0^t [h(t-\tau)]F(\tau)\,d\tau = -\int_0^t [h(t-\tau)]M\{\mathbf{1}\}\ddot{u}(\tau)\,d\tau$$

$$\langle X(t)\rangle = -\int_0^t [h(t-\tau)]M\{\mathbf{1}\}\langle \ddot{u}(\tau)\rangle\,d\tau = 0$$

$$R_{XX}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2}[h(t_1-\tau_1)]M\{\mathbf{1}\}R_{\ddot{u}\ddot{u}}(\tau_1,\tau_2)\{\mathbf{1}\}^t M[h(t_1-\tau_1)]^t\,d\tau_1 d\tau_2$$

$$x(t) = \Phi y(t)$$

$$x_k(t) = \sum_{n=1}^{N} \Phi_{kn} y_n(t)$$

$$\ddot{y}_n + 2\eta_n \omega_n \dot{y}_n + \omega_n^2 y_n = p_n(t)$$

$$\{p_n(t)\} = -\Phi^t M\{\mathbf{1}\}\ddot{u}(t) = \{\gamma\}\ddot{u}(t)$$

$$\{\gamma\} = -\Phi^t M\{\mathbf{1}\} \text{ [Modal participation factor]}$$

$$\ddot{y}_n + 2\eta_n \omega_n \dot{y}_n + \omega_n^2 y_n = \gamma_n \ddot{u}(t)$$

$$x(t) = \Phi y(t)$$

$$x_k(t) = \sum_{n=1}^{N} \Phi_{kn} y_n(t)$$

$$\ddot{y}_n + 2\eta_n \omega_n \dot{y}_n + \omega_n^2 y_n = \gamma_n \ddot{u}(t)$$

$$y_n(t) = \int_0^t h_n(t-\tau)\gamma_n \ddot{u}(\tau)\, d\tau$$

$$x_k(t) = \sum_{n=1}^{N} \Phi_{kn} \int_0^t h_n(t-\tau)\gamma_n \ddot{u}(\tau)\, d\tau$$

$$\langle x_k(t) \rangle = \sum_{n=1}^{N} \Phi_{kn} \int_0^t h_n(t-\tau)\gamma_n \langle \ddot{u}(\tau) \rangle\, d\tau = 0$$

$$x_k(t) = \sum_{n=1}^{N} \Phi_{kn} \int_0^t h_n(t-\tau)\gamma_n \ddot{u}(\tau)\, d\tau$$

$$\langle x_k(t_1)\, x_k(t_2)\rangle = \sum_{n=1}^{N}\sum_{m=1}^{N} \Phi_{kn}\Phi_{km} \langle y_n(t_1)\, y_m(t_2)\rangle$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} \Phi_{kn}\Phi_{km} \int_0^{t_1}\int_0^{t_2} h_n(t_1-\tau_1)\, h_m(t_2-\tau_2)\,\gamma_n\gamma_m \langle \ddot{u}(\tau_1)\,\ddot{u}(\tau_2)\rangle d\tau_1 d\tau_2$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} \Phi_{kn}\Phi_{km} \int_0^{t_1}\int_0^{t_2} h_n(t_1-\tau_1)\, h_m(t_2-\tau_2)\,\gamma_n\gamma_m R_{\ddot{u}\ddot{u}}(\tau_1,\tau_2)\, d\tau_1 d\tau_2$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} \Phi_{kn}\Phi_{km} \int_0^{t_1}\int_0^{t_2} h_n(t_1-\tau_1)\, h_m(t_2-\tau_2)\,\gamma_n\gamma_m R_{\ddot{u}\ddot{u}}(\tau_1-\tau_2)\, d\tau_1 d\tau_2$$

$$\sigma_{x_k}^2(t) = \sum_{n=1}^{N}\sum_{m=1}^{N} \Phi_{kn}\Phi_{km} \int_0^{t}\int_0^{t} h_n(t-\tau_1)\, h_m(t-\tau_2)\,\gamma_n\gamma_m R_{\ddot{u}\ddot{u}}(\tau_1-\tau_2)\, d\tau_1 d\tau_2$$

$$x(t) = \Phi y(t)$$

$$x_k(t) = \sum_{n=1}^{N} \Phi_{kn} y_n(t)$$

$$X_{kT}(\omega) = \sum_{n=1}^{N} \Phi_{kn} Y_{nT}(\omega)$$

$$\ddot{y}_n + 2\eta_n \omega_n \dot{y}_n + \omega_n^2 y_n = \gamma_n \ddot{u}(t)$$

$$\Rightarrow$$

$$Y_{nT}(\omega) = \frac{\gamma_n \ddot{U}_T(\omega)}{\omega_n^2 - \omega^2 + i2\eta_n \omega_n \omega}$$

$$X_{kT}(\omega)=\sum_{n=1}^{N}\Phi_{kn}\frac{\gamma_n\ddot{U}_T(\omega)}{\omega_n^2-\omega^2+i2\eta_n\omega_n\omega}$$

$$\lim_{T\to\infty}\frac{1}{T}\left\langle X_{kT}(\omega)X_{kT}^{*}(\omega)\right\rangle$$

$$S_{x_kx_k}(\omega)=\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{\Phi_{kn}\Phi_{km}\gamma_n\gamma_m}{\left[\omega_n^2-\omega^2+i2\eta_n\omega_n\omega\right]\left[\omega_m^2-\omega^2-i2\eta_m\omega_m\omega\right]}S_{\ddot{U}\ddot{U}}(\omega)$$

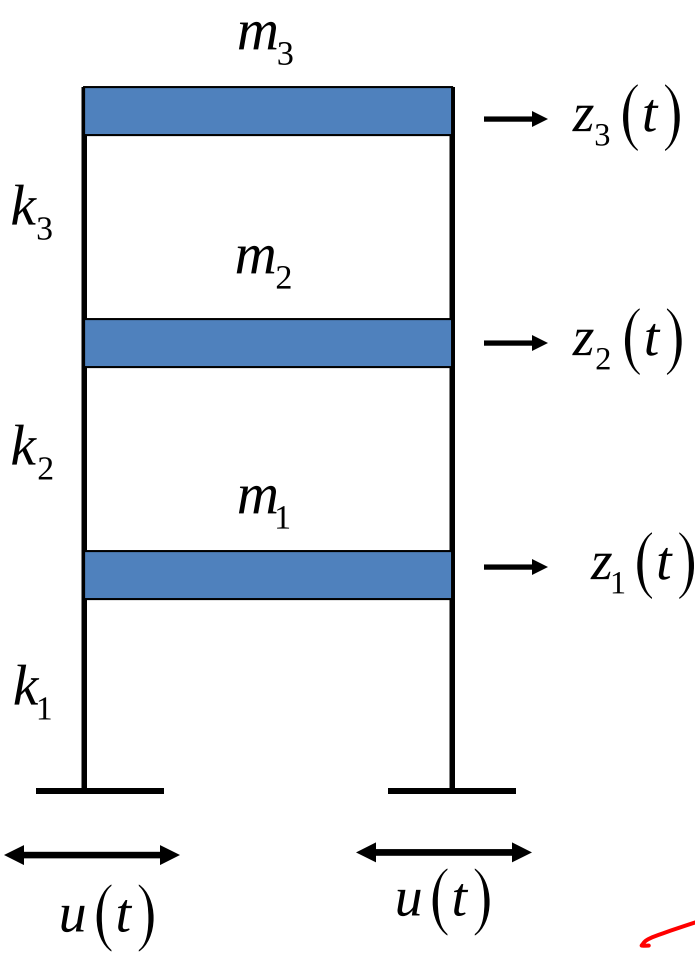
$$=\sum_{n=1}^{N}\frac{\Phi_{kn}^2\gamma_n^2S_{\ddot{U}\ddot{U}}(\omega)}{\left[\left(\omega_n^2-\omega^2\right)^2+\left(2\eta_n\omega_n\omega\right)^2\right]}$$

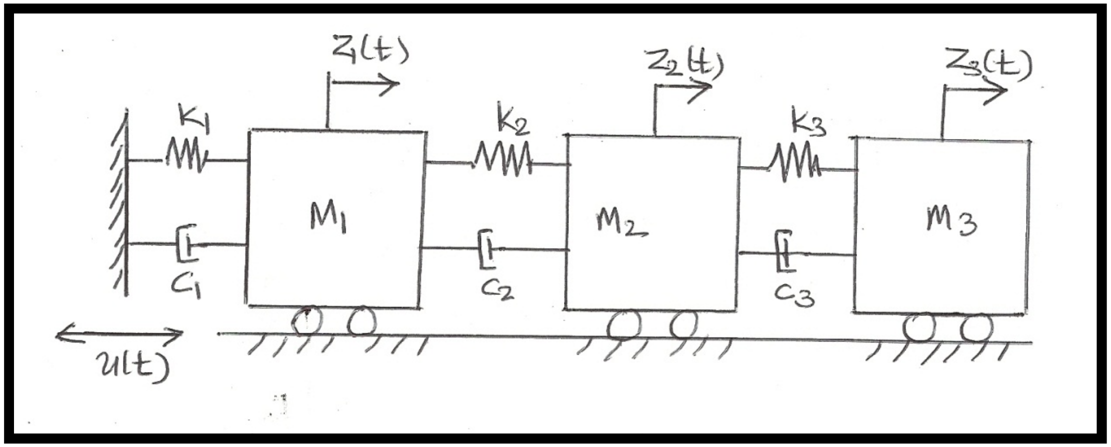
$$+\sum_{n=1}^{N}\sum_{\substack{m=1\\n\neq m}}^{N}\frac{\Phi_{kn}\Phi_{km}\gamma_n\gamma_m}{\left[\omega_n^2-\omega^2+i2\eta_n\omega_n\omega\right]\left[\omega_m^2-\omega^2-i2\eta_m\omega_m\omega\right]}S_{\ddot{U}\ddot{U}}(\omega)$$

$$\sigma_{x_k}^2=\int_0^\infty S_{x_kx_k}(\omega)\,d\omega$$

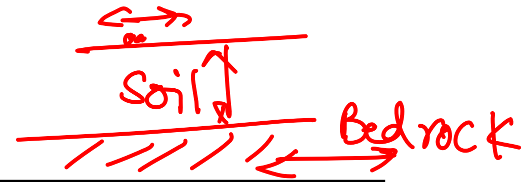
= Contribution ignoring modal interactions + correction due to modal interactions

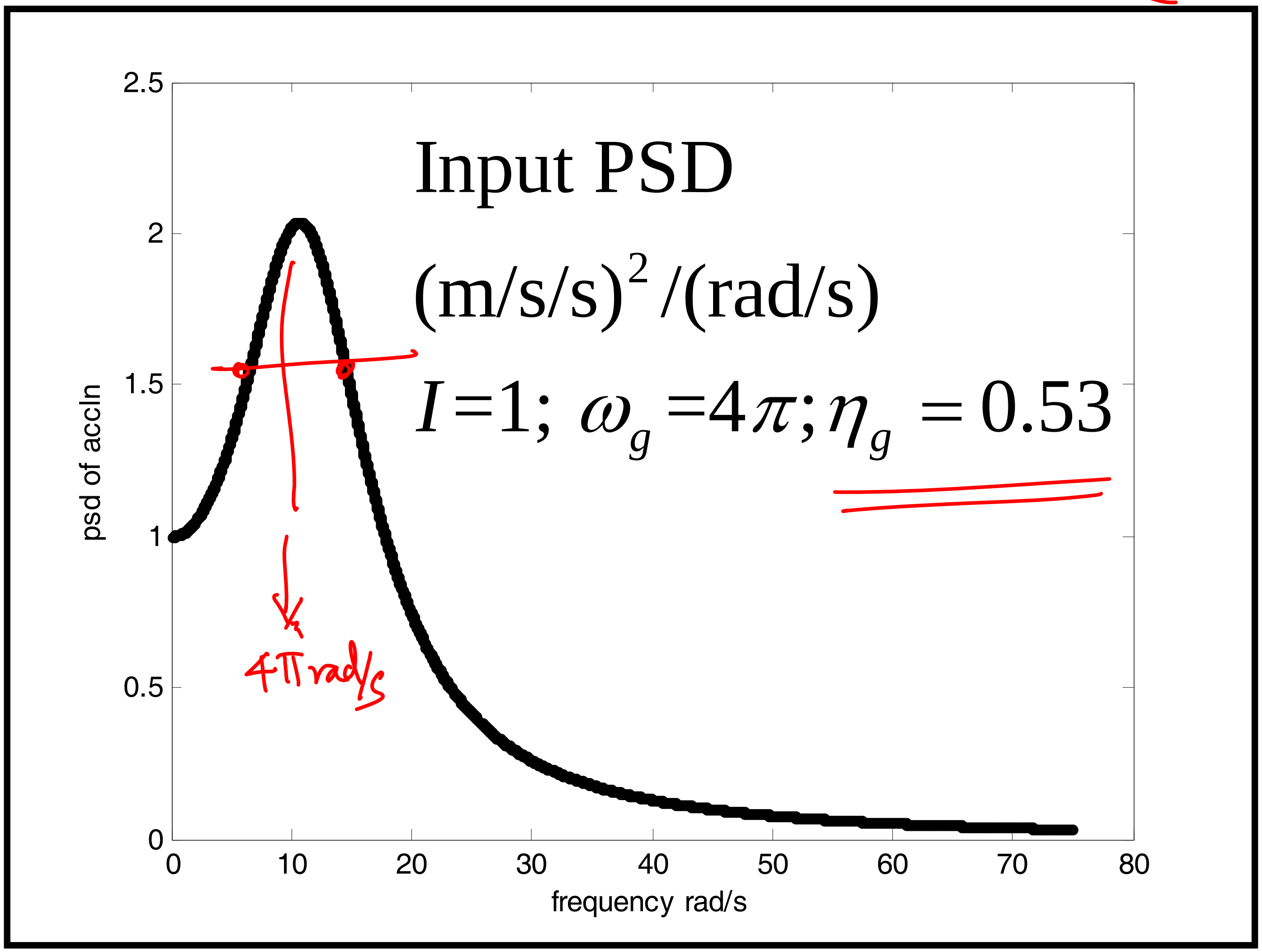
A building frame under random support motion

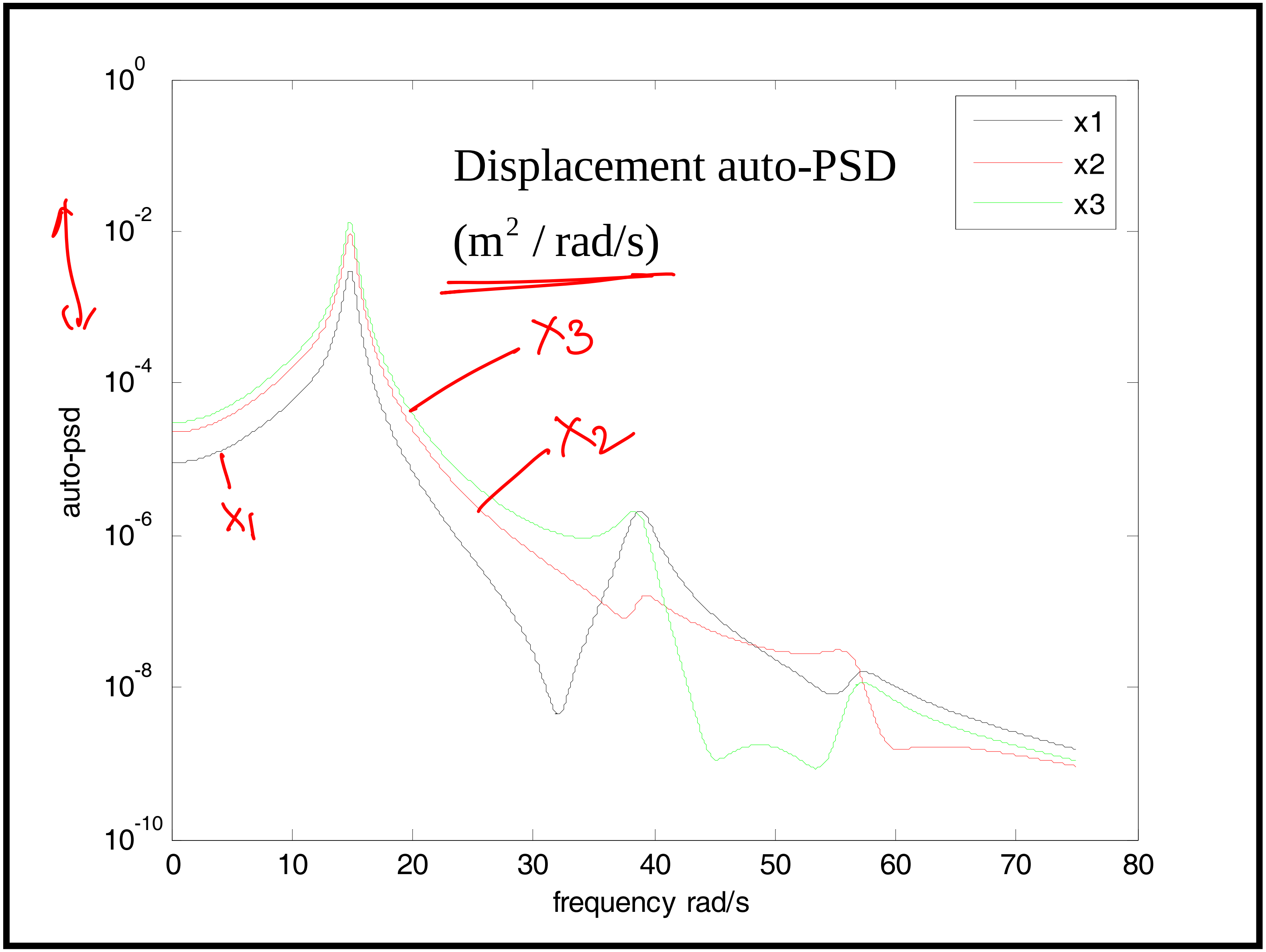


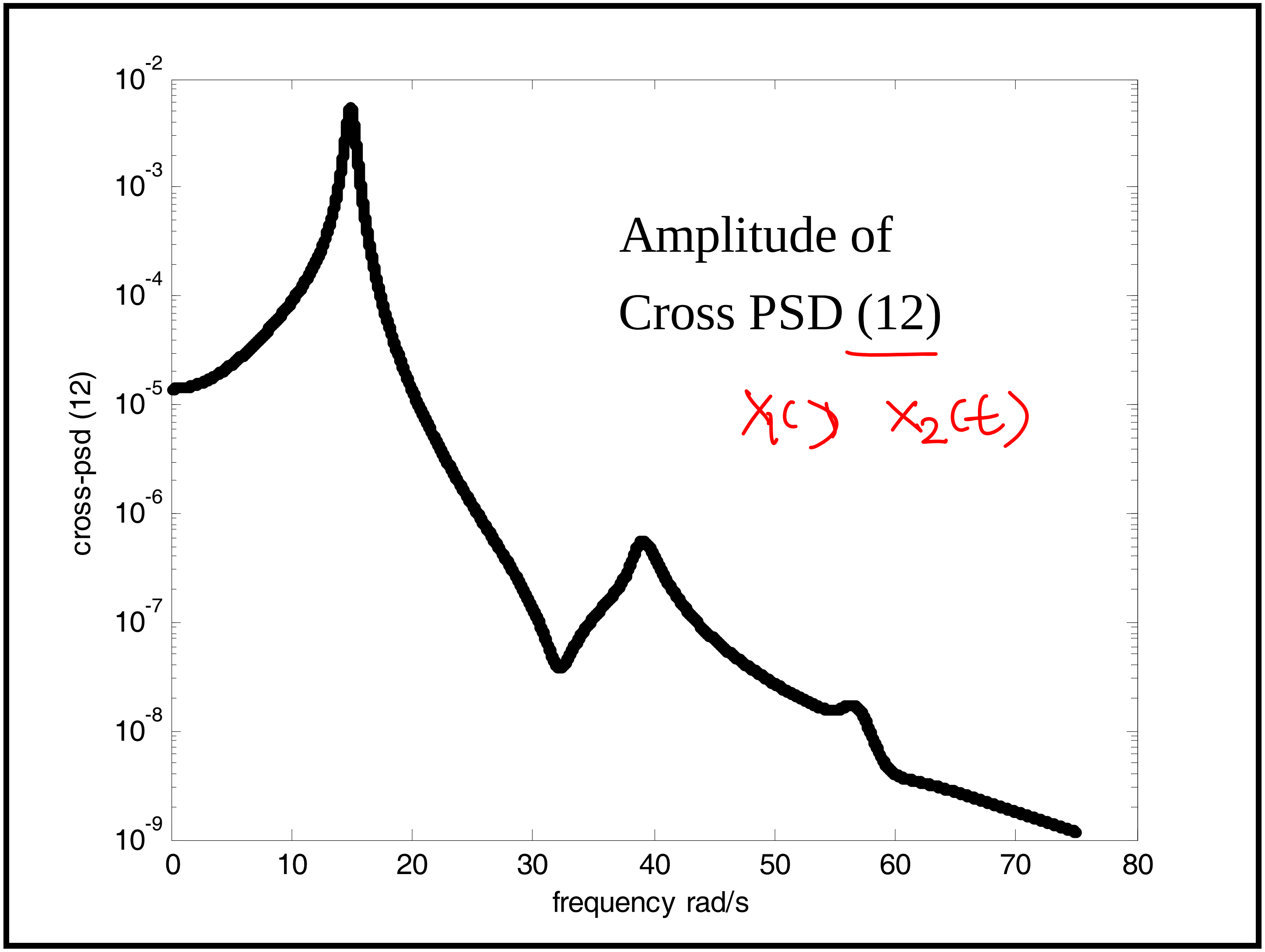


$$S_{\ddot{U}\ddot{U}}(\omega) = I \frac{\left(\omega_g^4 + 4\eta_g^2\omega_g^2\omega^2\right)}{\left(\omega^2 - \omega_g^2\right)^2 + 4\eta_g^2\omega_g^2\omega^2}$$

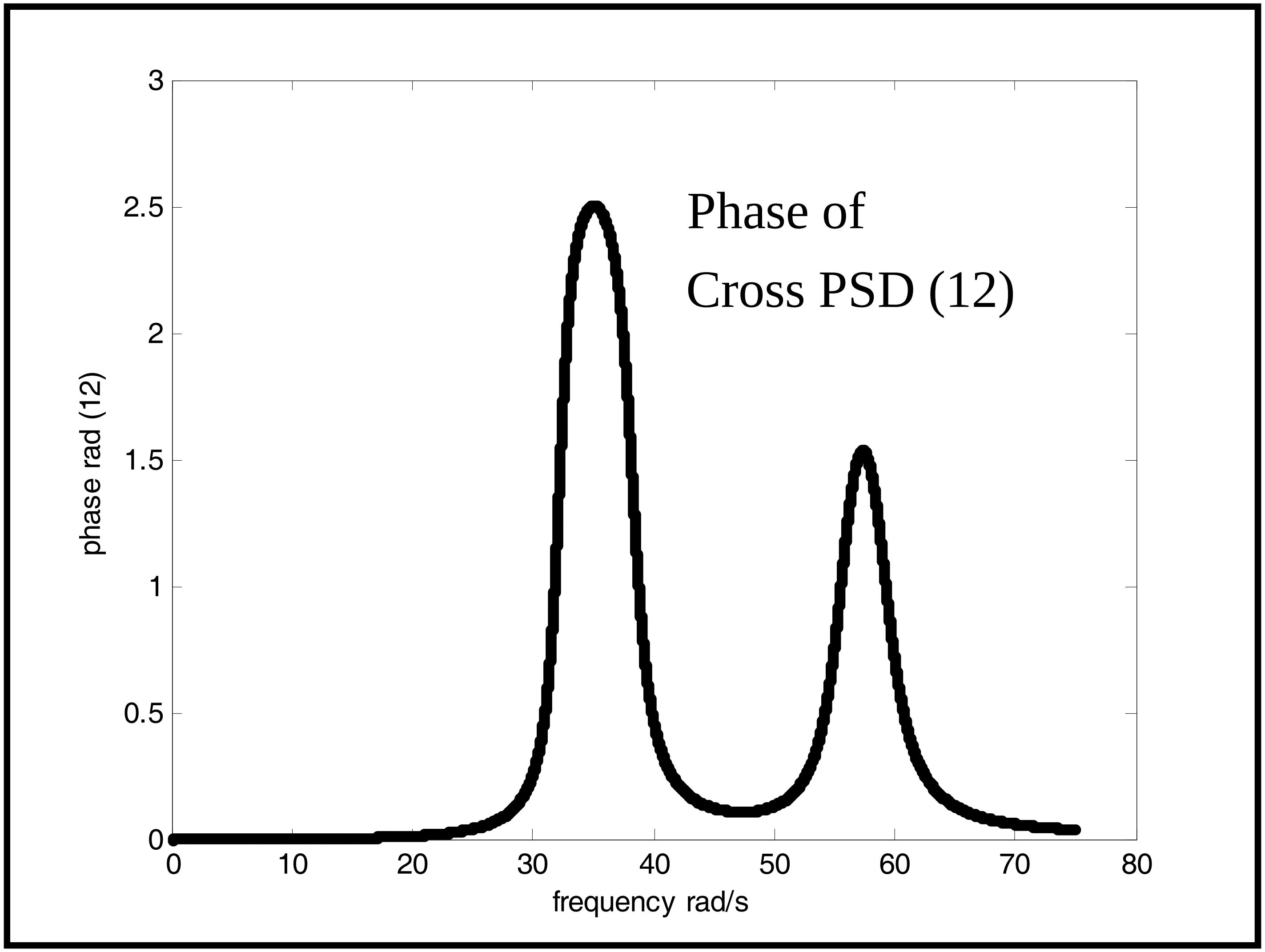

Soil
Bedrock


Input PSD
(m/s/s)^2 /(rad/s)
$I = 1;\ \omega_g = 4\pi;\ \eta_g = 0.53$
psd of accln
frequency rad/s
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80
4π rad/s

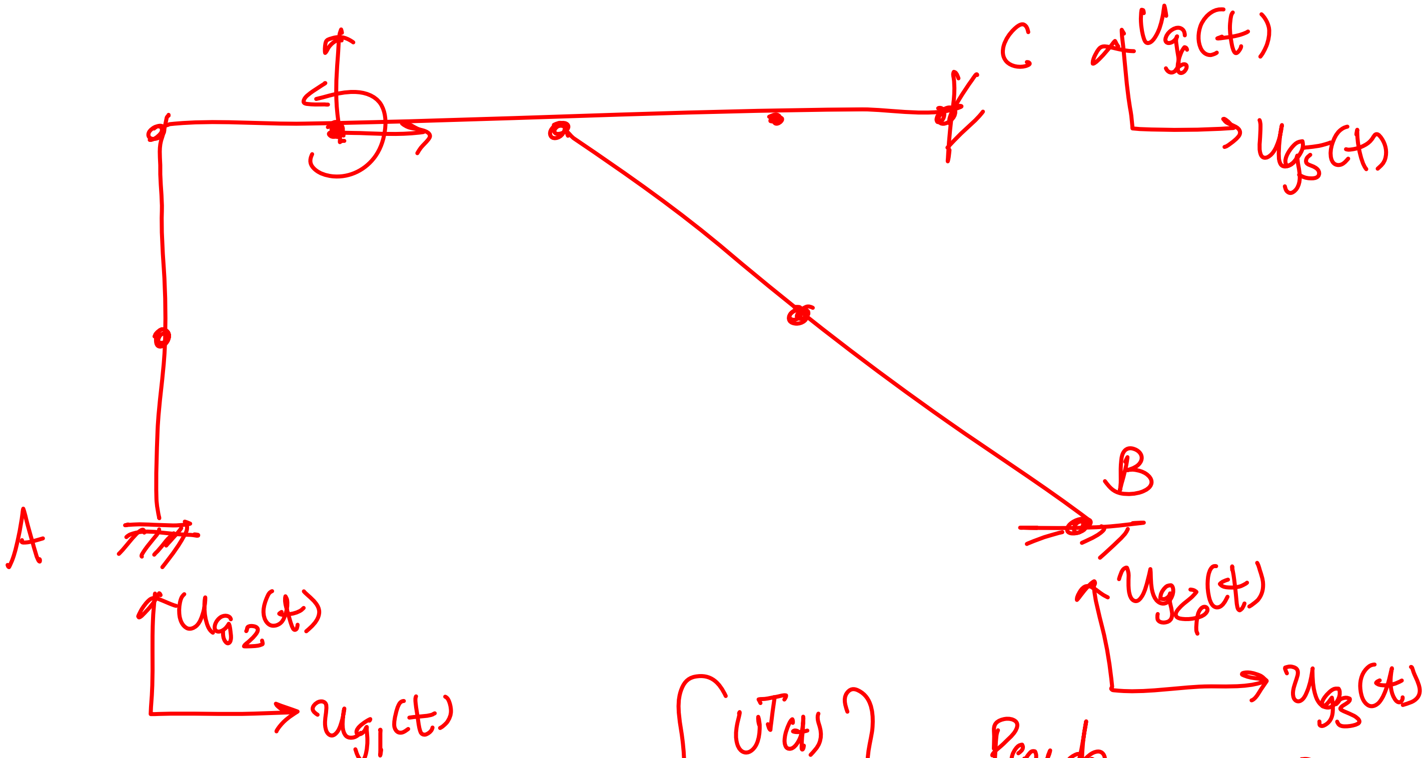

Displacement auto-PSD
(m^2 / rad/s)
x1
x2
x3
auto-psd
frequency rad/s
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
0
10
20
30
40
50
60
70
80



Amplitude of
Cross PSD (12)
$x_1(\cdot)$ $x_2(t)$
cross-psd (12)
frequency rad/s
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
0
10
20
30
40
50
60
70
80


Phase of
Cross PSD (12)
phase rad (12)
frequency rad/s
3
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80

Structures under differential support motions



$$U = \begin{Bmatrix} U^T(t) \\ u_g(t) \end{Bmatrix} = \underline{\text{Pseudo dynamic comp}} + \underline{\text{Dynamic comp}}$$

$$\begin{bmatrix} M & M_g \\ M_g^t & M_{gg} \end{bmatrix} \begin{Bmatrix} \ddot{u}^T \\ \ddot{u}_g \end{Bmatrix} + \begin{bmatrix} C & C_g \\ C_g^t & C_{gg} \end{bmatrix} \begin{Bmatrix} \dot{u}^T \\ \dot{u}_g \end{Bmatrix} + \begin{bmatrix} K & K_g \\ K_g^t & K_{gg} \end{bmatrix} \begin{Bmatrix} u^T \\ u_g \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_g(t) \end{Bmatrix}$$

$$\ddot{u}^T \sim N \times 1$$

$$\ddot{u}_g, p_g(t) \sim N_g \times 1; N_T = N + N_g$$

$$M, C, K \sim N \times N$$

$$M_g, C_g, K_g \sim N \times N_g$$

$$M_{gg}, C_{gg}, K_{gg} \sim N_g \times N_g$$

Pseudo-dynamic response

$$\begin{bmatrix} K & K_g \\ K_g^t & K_{gg} \end{bmatrix} \begin{Bmatrix} u^p \\ u_g \end{Bmatrix} = \begin{Bmatrix} 0 \\ p_g^p(t) \end{Bmatrix}$$

$$Ku^p + K_g u_g = 0 \Rightarrow u^p = -K^{-1} K_g u_g(t) = -\Gamma u_g(t)$$

$$\Gamma = K^{-1} K_g$$

$$p_g^p(t) = K_g^t u^p + K_{gg} u_g$$