Stochastic Structural Dynamics

Lecture-13

Random vibration of MDOF systems - 1

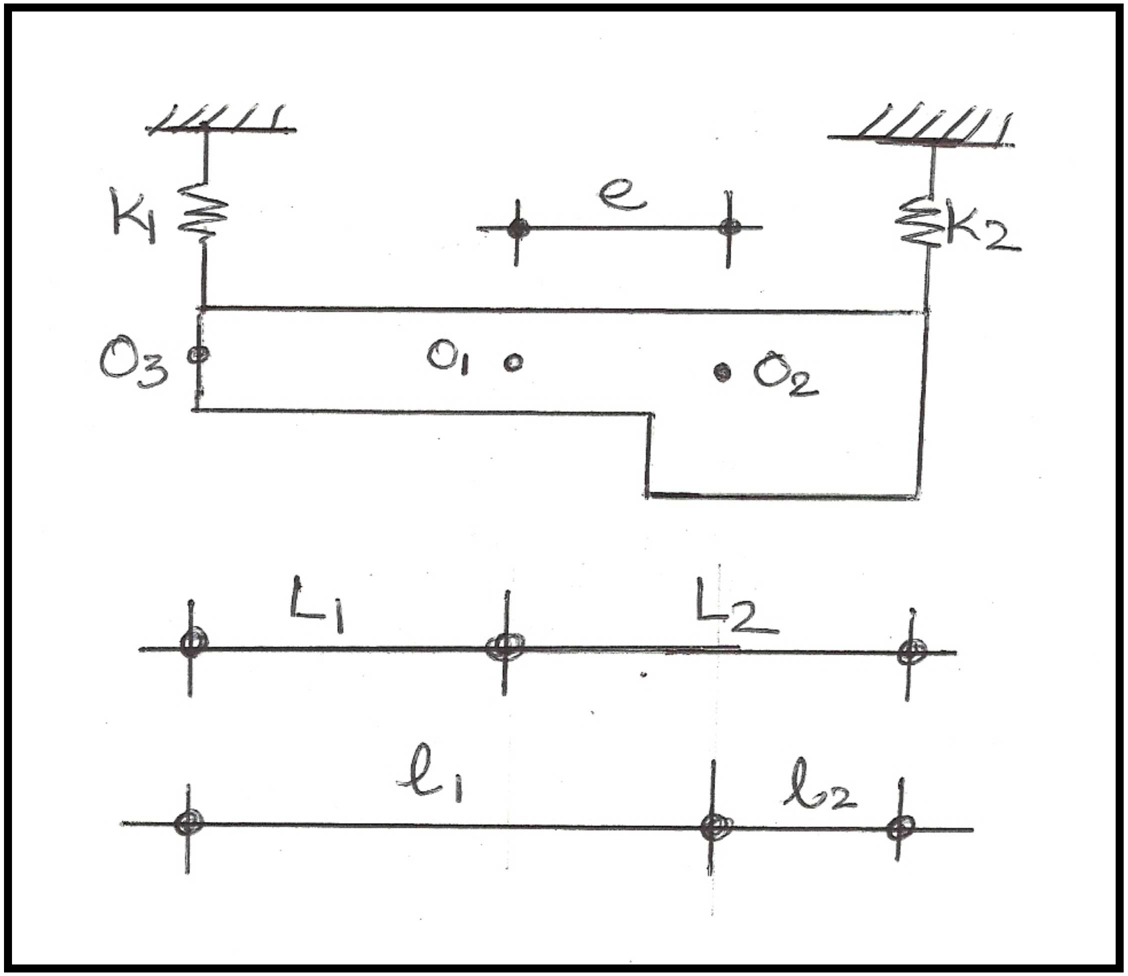
Dr C S Manohar
Department of Civil Engineering
Professor of Structural Engineering
Indian Institute of Science
Bangalore 560 012 India
manohar@civil.iisc.ernet.in

Preliminaries

Discrete MDOF systems under deterministic excitations

- **Nature of equations of motion**
- **Input - output relations in time domain**
- **Input - output relations in frequency domain**
- **Forced vibration analysis using modal expansion**

A rigid bar supported on two springs



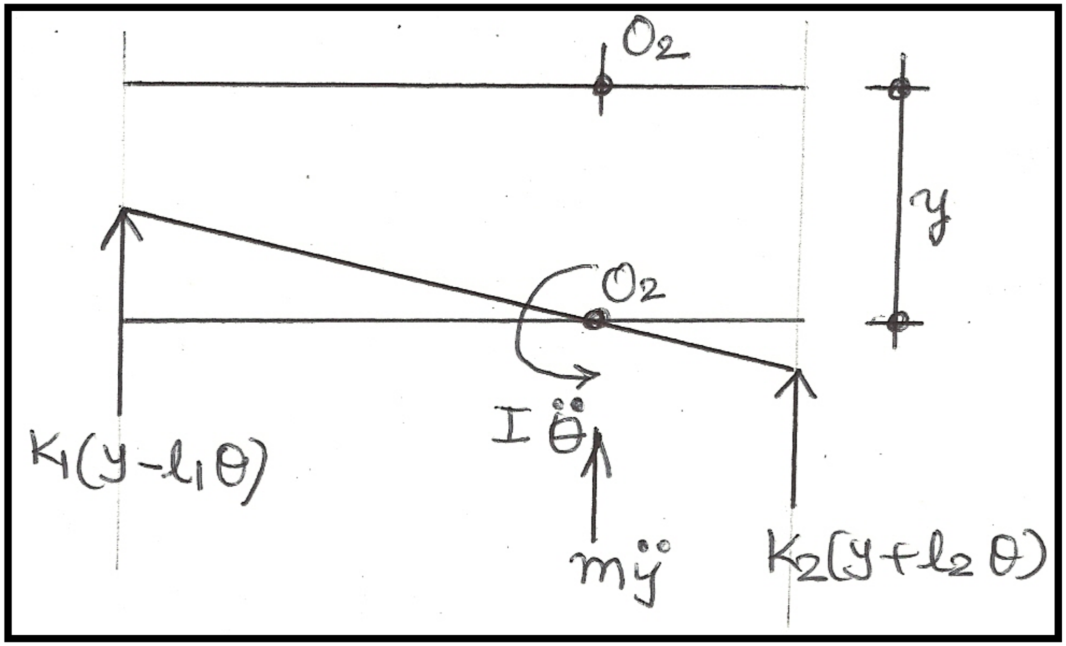
Two DOF-s

- 1 Translation
- 1 Rotation

O_1 : Elastic centre $\left(k_1 L_1 = k_2 L_2\right)$

O_2: Centre of gravity

$$l_1 + l_2 = L_1 + L_2 = L$$



$$m\ddot{y} + k_1\left(y - l_1\theta\right) + k_2\left(y + l_2\theta\right) = 0$$

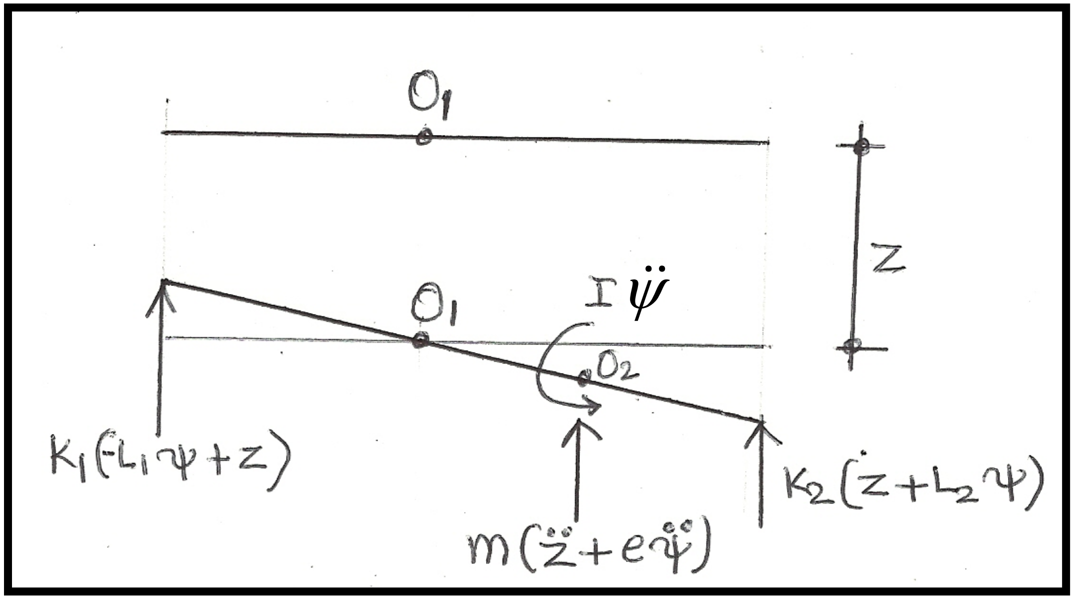
$$I\ddot{\theta} + k_2\left(y + l_2\theta\right)l_2 - k_1\left(y - l_1\theta\right)l_1 = 0$$

$$\Rightarrow$$

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1 l_1 + k_2 l_2 \\ -k_1 l_1 + k_2 l_2 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = 0$$

M is diagonal and K is non-diagonal

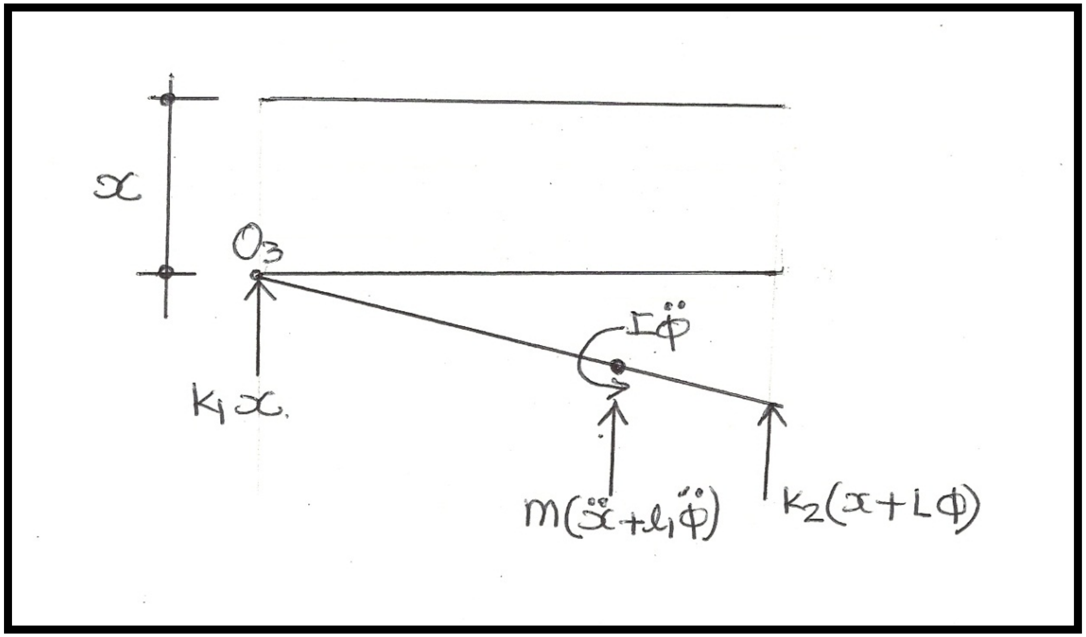
Static coupling



$$\begin{bmatrix} m & me \\ me & m \end{bmatrix} \begin{Bmatrix} \ddot{z} \\ \ddot{\psi} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & 0 \\ 0 & k_1 L_1^2 + k_2 L_2^2 \end{bmatrix} \begin{Bmatrix} z \\ \psi \end{Bmatrix} = 0$$

M is non-diagonal and K is diagonal

Inertial coupling



$$\begin{bmatrix} m & ml_1 \\ ml_1 & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix} \begin{Bmatrix} x \\ \phi \end{Bmatrix} = 0$$

M is non-diagonal and K is non-diagonal

Static and inertial coupling

Remarks

- Equations of motion for MDOF systems are generally coupled

- Coupling between co-ordinates is manifest in the form of structural matrices being nondiagonal

- Coupling is not an intrinsic property of a vibrating system. It is dependent upon the choice of the coordinate system. This choice itself is arbitrary.

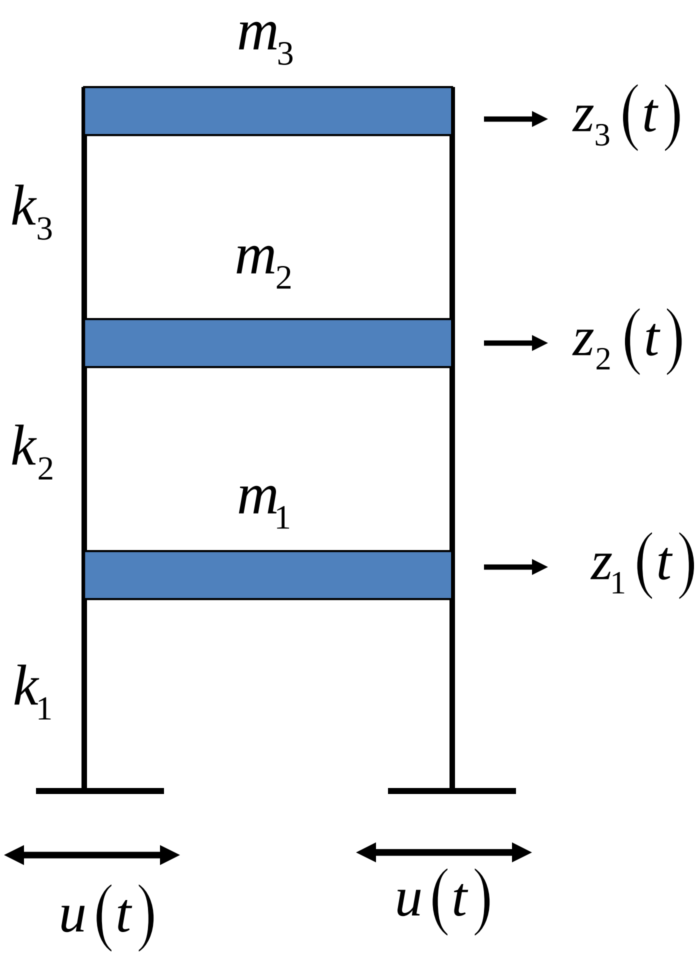
- Equations of motion are not unique.
They depend upon the choice of coordinate system.

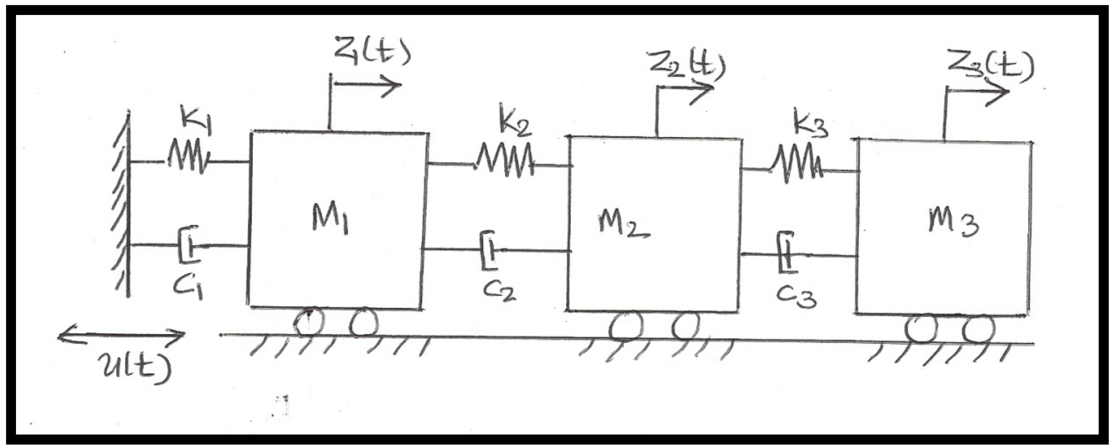
Remarks (continued)

- **The best choice of coordinate system is the one in which the coupling is absent. That is, the structural matrices are all diagonal.**

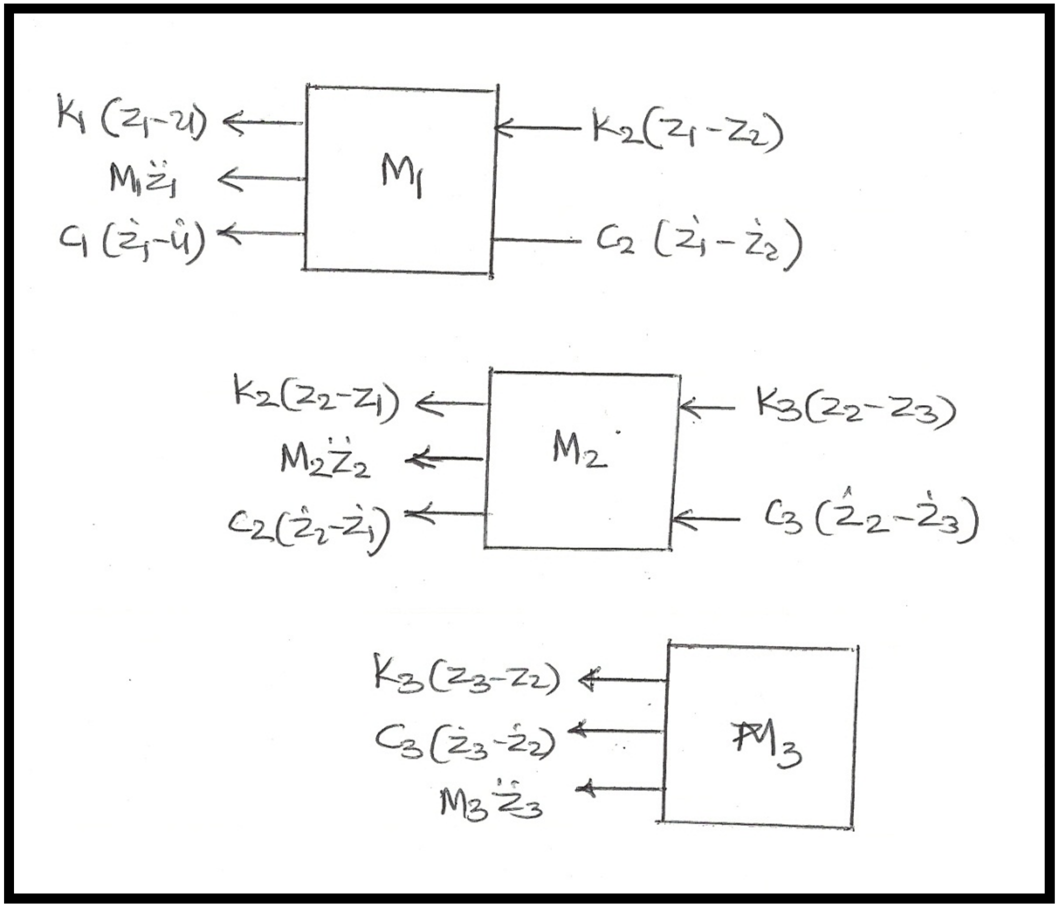
- **These coordinates are called the natural coordinates for the system. Determination of these coordinates for a given system constitutes a major theme in structural dynamics. Theory of ODEs and linear algebra help us.**

A building frame under support motion





$$K_1 = \underline{\underline{2}} \times \frac{12EI}{l^3}$$



$$m_1\ddot{z}_1 + c_1\left(\dot{z}_1 - \dot{u}\right) + c_2\left(\dot{z}_1 - \dot{z}_2\right) + k_1\left(z_1 - u\right) + k_2\left(z_1 - z_2\right) = 0$$

$$m_2\ddot{z}_2 + c_2\left(\dot{z}_2 - \dot{z}_1\right) + c_3\left(\dot{z}_2 - \dot{z}_3\right) + k_2\left(z_2 - z_1\right) + k_3\left(z_2 - z_3\right) = 0$$

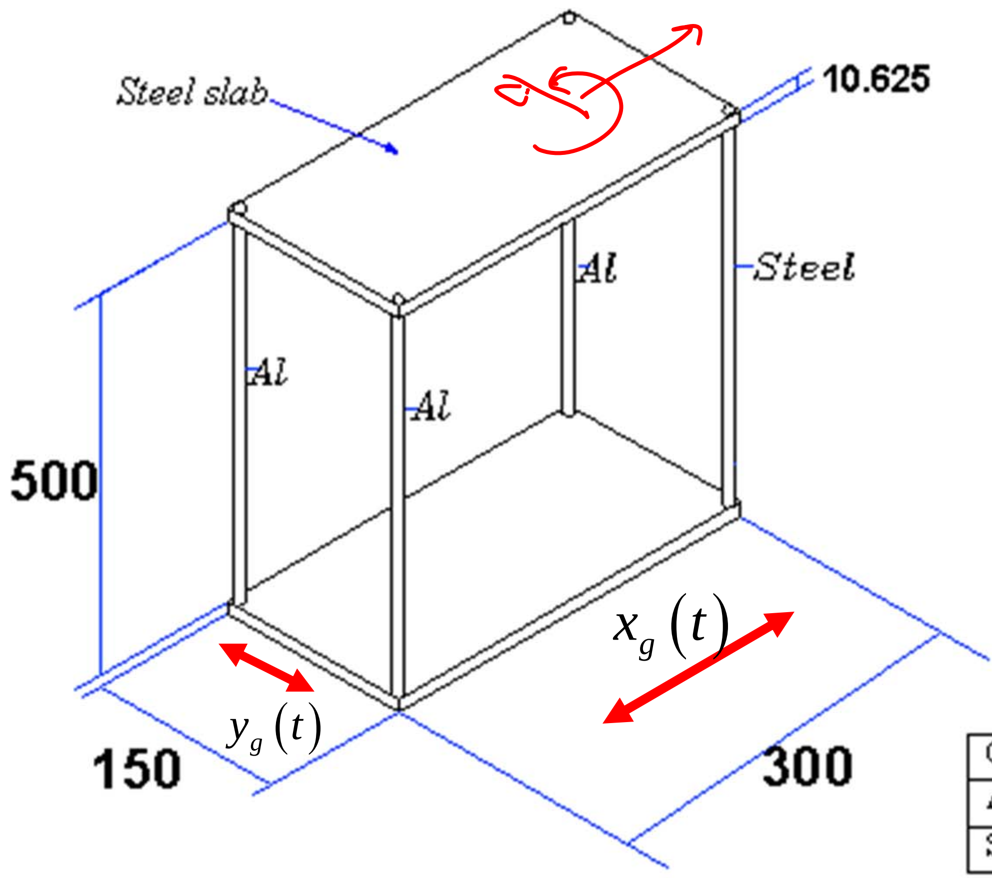
$$m_3\ddot{z}_3 + c_3\left(\dot{z}_3 - \dot{z}_2\right) + k_3\left(z_3 - z_2\right) = 0$$

$$x_1 = z_1 - u$$
$$x_2 = z_2 - u$$
$$x_3 = z_3 - u$$

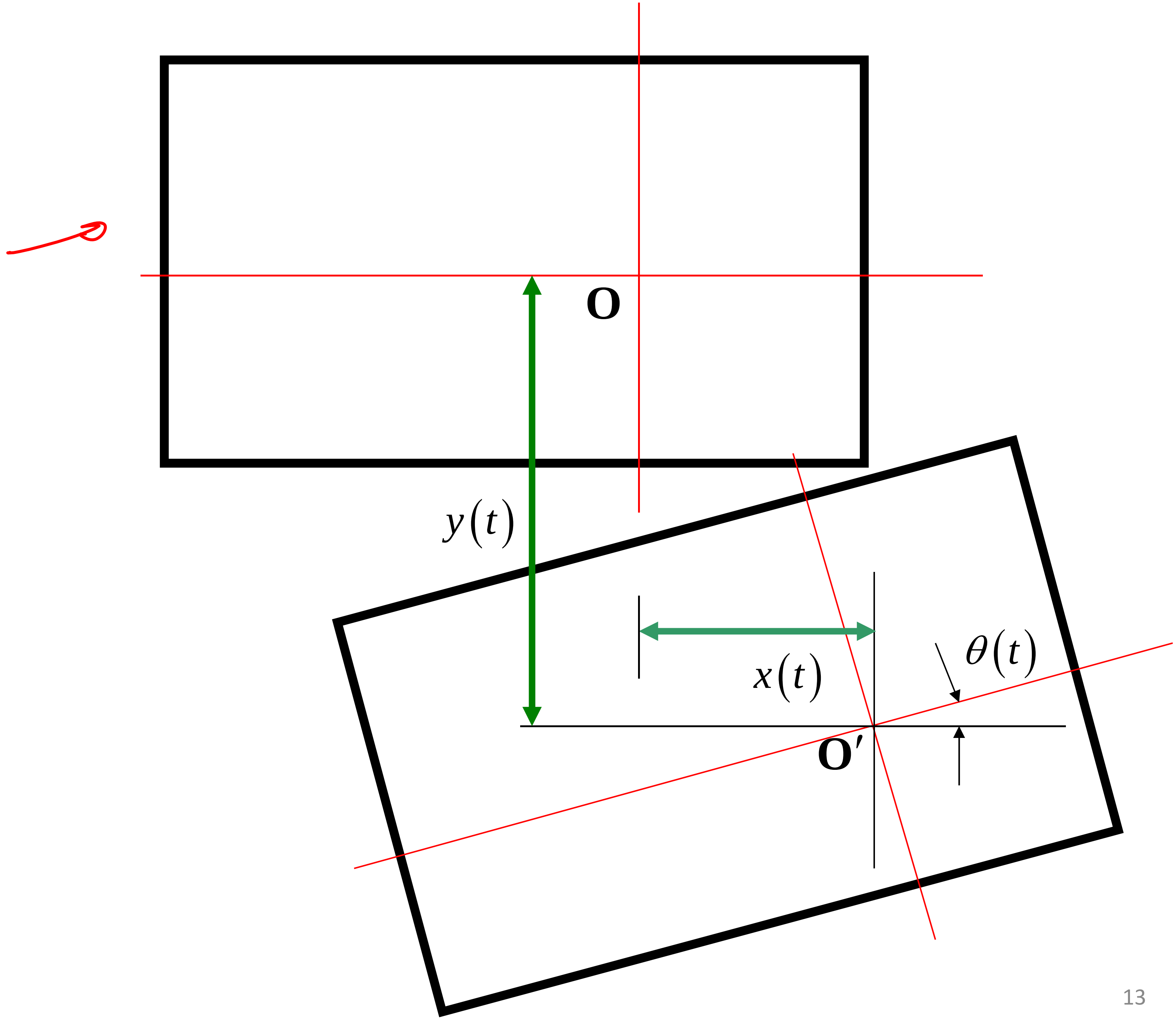
$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2\left(\dot{x}_1 - \dot{x}_2\right) + k_1 x_1 + k_2\left(x_1 - x_2\right) = -m_1\ddot{u}$$
$$m_2\ddot{x}_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) + c_3\left(\dot{x}_2 - \dot{x}_3\right) + k_2\left(x_2 - x_1\right) + k_3\left(x_2 - x_3\right) = -m_2\ddot{u}$$
$$m_3\ddot{x}_3 + c_3\left(\dot{x}_3 - \dot{x}_2\right) + k_3\left(x_3 - x_2\right) = -m_3\ddot{u}$$

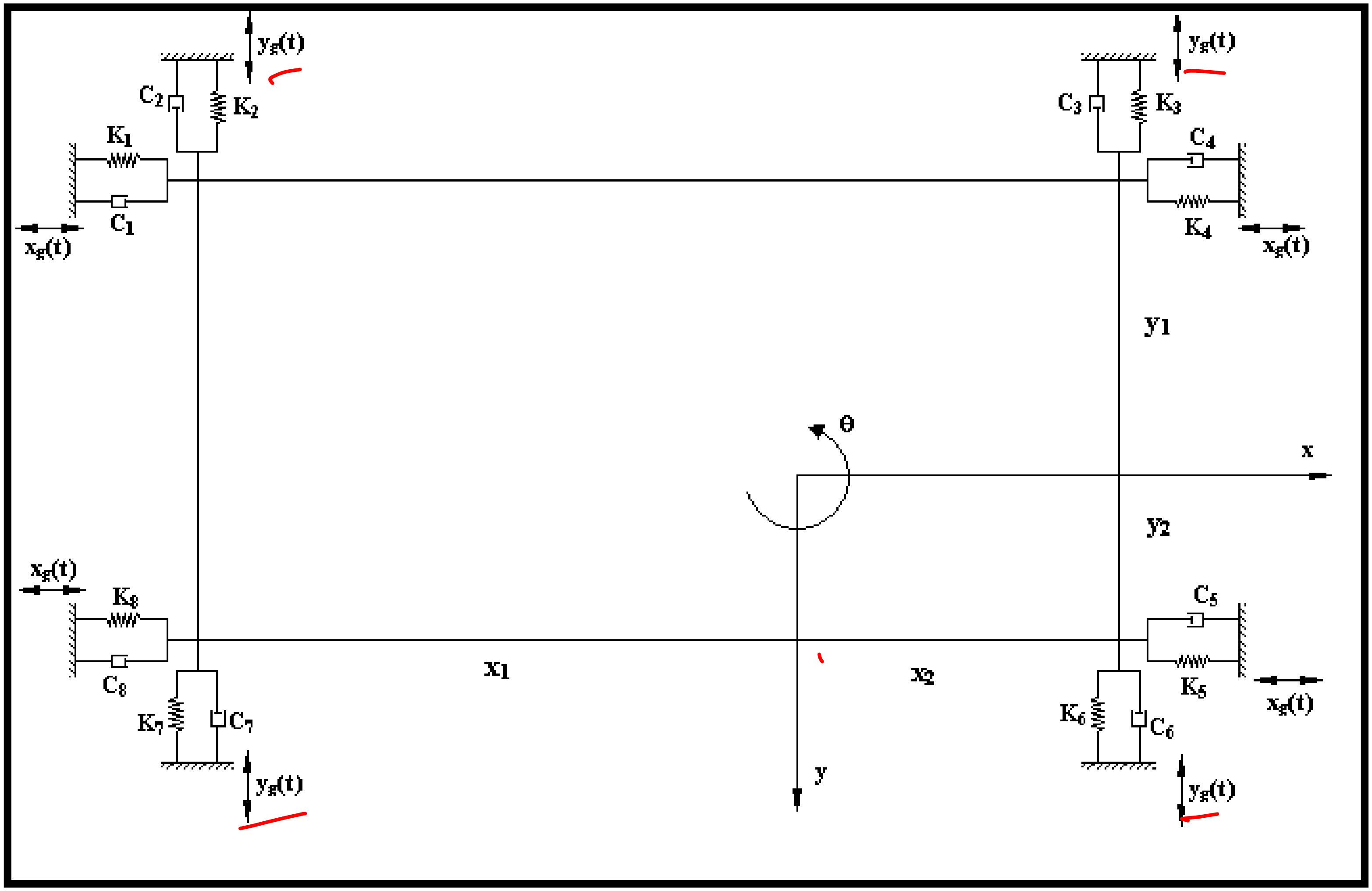
$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = -\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}\ddot{u}$$

A frame with asymmetric plan under multi-component support motions

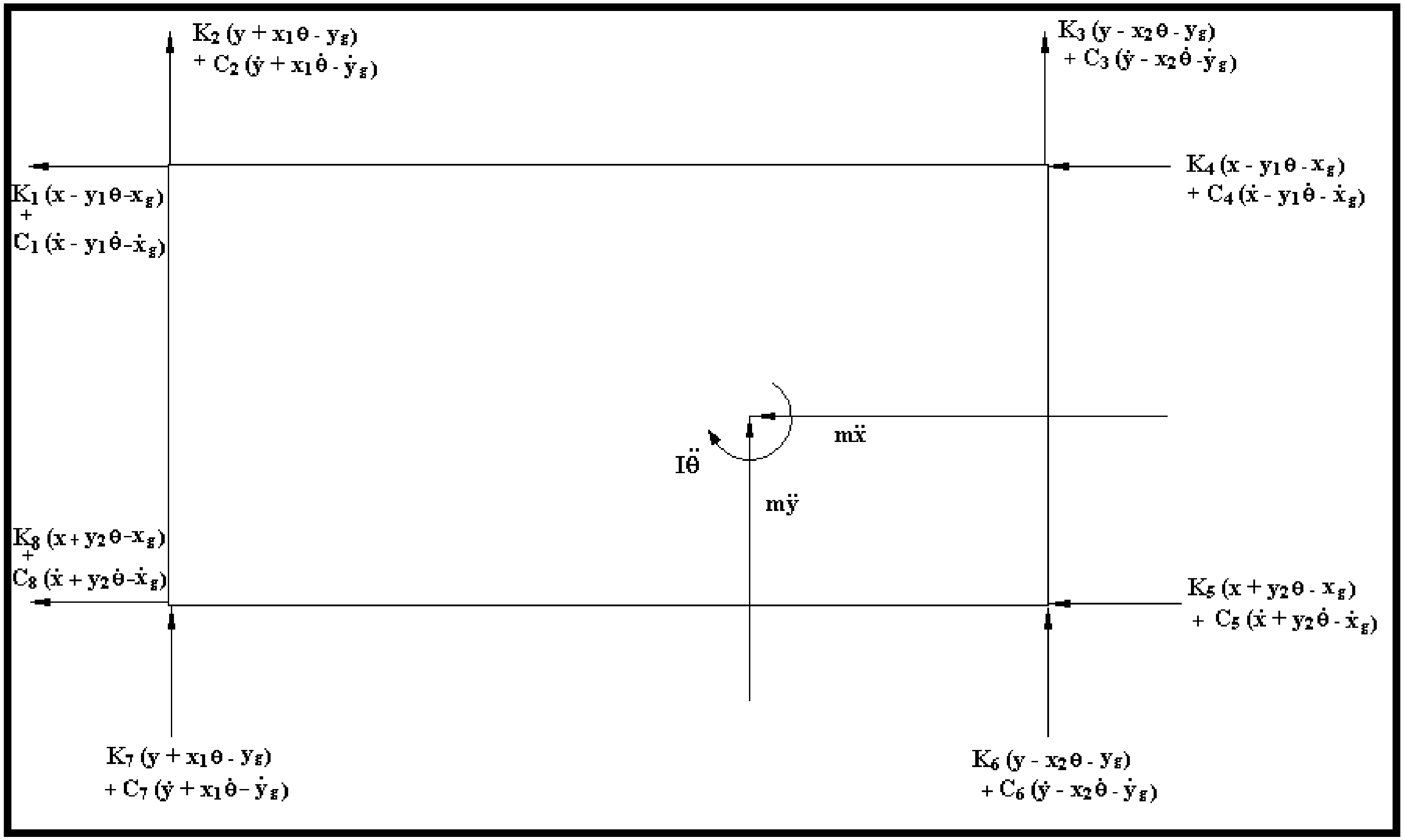


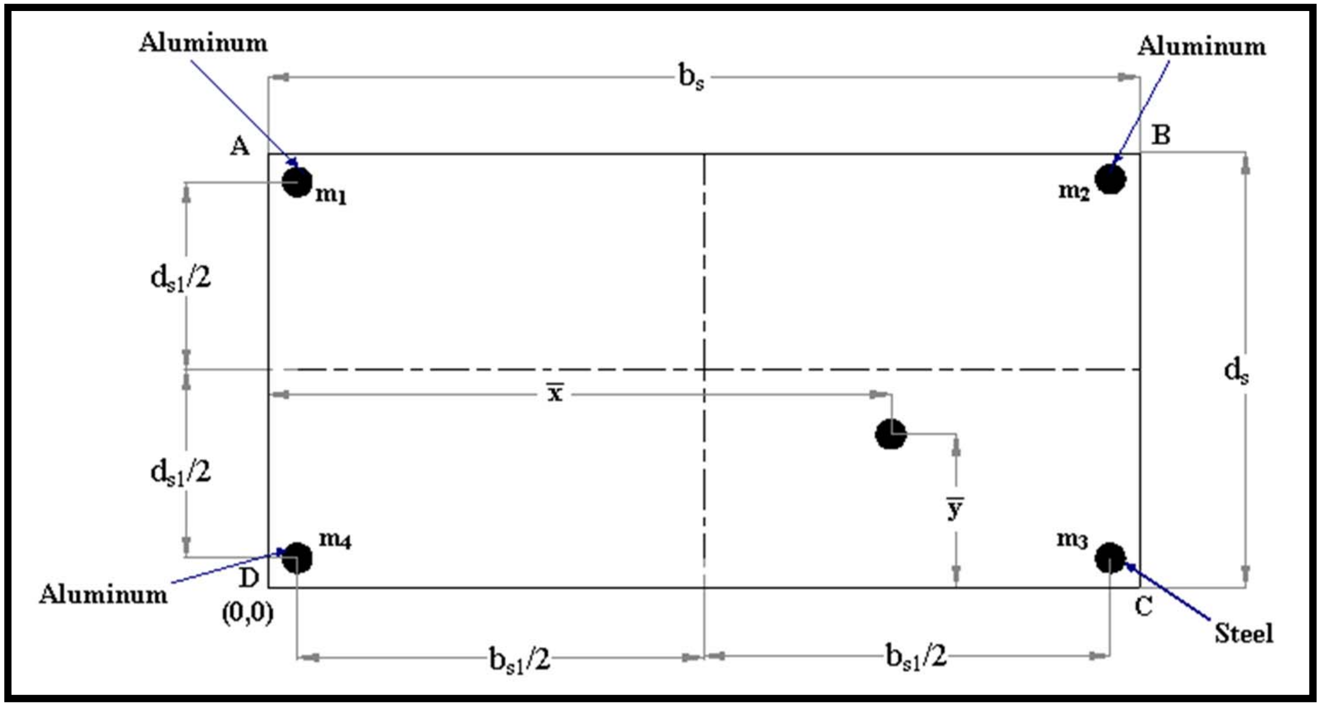
Columns	Diameter (mm)
Aluminum	10.13
Steel	9.958


O
y(t)
x(t)
θ(t)
O′



yg(t)
C2
K2
K1
C1
xg(t)
yg(t)
C3
K3
C4
K4
xg(t)
y1
θ
x
y2
xg(t)
K8
C8
K7
C7
yg(t)
x1
x2
y
C5
K5
xg(t)
K6
C6
yg(t)


K2 (y + x1θ - yg)
+ C2 (ẏ + x1θ̇ - ẏg)
K3 (y - x2θ - yg)
+ C3 (ẏ - x2θ̇ -ẏg)
K1 (x - y1θ-xg)
+
C1 (ẋ - y1θ̇-ẋg)
K4 (x - y1θ - xg)
+ C4 (ẋ - y1θ̇ - ẋg)
mẍ
Iθ̈
mÿ
K8 (x + y2θ-xg)
+
C8 (ẋ + y2θ̇-ẋg)
K5 (x + y2θ - xg)
+ C5 (ẋ + y2θ̇ - ẋg)
K7 (y + x1θ - yg)
+ C7 (ẏ + x1θ̇- ẏg)
K6 (y - x2θ - yg)
+ C6 (ẏ - x2θ̇ - ẏg)



$$\bar{x} = \frac{m_S \dfrac{b_s}{2} + (m_1 + m_4)\left(\dfrac{b_s - b_{s1}}{2}\right) + (m_2 + m_3)\left(\dfrac{b_s + b_{s1}}{2}\right)}{\rho_s t b_s d_s + m_1 + m_2 + m_3 + m_4}$$

$$\bar{y} = \frac{m_S \dfrac{d_s}{2} + (m_4 + m_3)\left(\dfrac{d_s - d_{s1}}{2}\right) + (m_1 + m_2)\left(\dfrac{d_s + d_{s1}}{2}\right)}{\rho_s t b_s d_s + m_1 + m_2 + m_3 + m_4}$$

$$m_1 = m_2 = m_3 = A_a \rho_a (h/2)$$

$$m_4 = A_s \rho_s (h/2), \quad m_S = \rho_s t b_s d_s$$

$$k_1 = k_2 = k_3 = \frac{12 E_a I_a}{h^3}; \; k_4 = \frac{12 E_s I_s}{h^3}$$

$$m = m_{slab} + m_{columns} = \rho_s t b_s d_s + 3\frac{\rho_a A_a h}{2} + \frac{\rho_s A_s h}{2}$$

$$I = \frac{\rho_s t b_s d_s}{12}\left(b_s^2 + d_s^2\right) + \rho_s t b_s d_s \left[\left(\bar{x} - \frac{b_s}{2}\right)^2 + \left(\bar{y} - \frac{d_s}{2}\right)^2\right] +$$

$$\frac{\rho_a A_a h}{2}(x_1^2 + y_1^2 + x_2^2 + y_1^2 + x_1^2 + y_2^2) + \frac{\rho_s A_s h}{2}\left(x_2^2 + y_2^2\right) + 3\frac{\rho_a A_a h}{2}\left(\frac{r_{al}}{2}\right)^2 + \frac{\rho_s A_s h}{2}\left(\frac{r_s}{2}\right)^2$$

$$m\ddot{x} + k_1(x - y_1\theta - x_g) + c_1(\dot{x} - y_1\dot{\theta} - \dot{x}_g) + k_4(x - y_1\theta - x_g) + c_4(\dot{x} - y_1\dot{\theta} - \dot{x}_g) +$$
$$k_8(x + y_2\theta - x_g) + c_8(\dot{x} + y_2\dot{\theta} - \dot{x}_g) + k_5(x + y_2\theta - x_g) + c_5(\dot{x} + y_2\dot{\theta} - \dot{x}_g) = 0$$

$$m\ddot{y} + k_2(y + x_1\theta - y_g) + c_2(\dot{y} + x_1\dot{\theta} - \dot{y}_g) + k_7(y + x_1\theta - y_g) + c_7(\dot{y} + x_1\dot{\theta} - \dot{y}_g) +$$
$$k_3(y - x_2\theta - y_g) + c_3(\dot{y} - x_2\dot{\theta} - \dot{y}_g) + k_6(y - x_2\theta - y_g) + c_6(\dot{y} - x_2\dot{\theta} - \dot{y}_g) = 0$$

$$I\ddot{\theta} + x_1\left[k_2(y + x_1\theta - y_g) + c_2(\dot{y} + x_1\dot{\theta} - \dot{y}_g) + k_7(y + x_1\theta - y_g) + c_7(\dot{y} + x_1\dot{\theta} - \dot{y}_g)\right] -$$
$$x_2\left[k_3(y - x_2\theta - y_g) + c_3(\dot{y} - x_2\dot{\theta} - \dot{y}_g) + k_6(y - x_2\theta - y_g) + c_6(\dot{y} - x_2\dot{\theta} - \dot{y}_g)\right] +$$
$$y_2\left[k_8(x + y_2\theta - x_g) + c_8(\dot{x} + y_2\dot{\theta} - \dot{x}_g) + k_5(x + y_2\theta - x_g) + c_5(\dot{x} + y_2\dot{\theta} - \dot{x}_g)\right] -$$
$$y_1\left[k_1(x - y_1\theta - x_g) + c_1(\dot{x} - y_1\dot{\theta} - \dot{x}_g) + k_4(x - y_1\theta - x_g) + c_4(\dot{x} - y_1\dot{\theta} - \dot{x}_g)\right] = 0$$

$$u(t) = \begin{Bmatrix} x(t) \\ y(t) \\ \theta(t) \end{Bmatrix}$$

$$M\ddot{u} + C\dot{u} + Ku = f(t)$$

How to uncouple equations of motion?

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

$$X(0) = X_0;\ \dot{X}(0) = \dot{X}_0$$

- M, C and K, in general, are non-diagonal
- Equations are coupled

Suppose we introduce a new set of dependent variables $Z(t)$ using the transformation

$$X(t) = TZ(t)$$

where T is a $n \times n$ transformaiton matrix, to be selected.

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

$$X(0) = X_0; \dot{X}(0) = X_0$$

$$X(t) = TZ(t)$$

$$\Rightarrow MT\ddot{Z}(t) + CT\dot{Z}(t) + KTZ(t) = F(t)$$

$$\Rightarrow T^t MT\ddot{Z}(t) + T^t CT\dot{Z}(t) + T^t KTZ(t) = T^t F(t)$$

$$\Rightarrow \bar{M}\ddot{Z}(t) + \bar{C}\dot{Z}(t) + \bar{K}Z(t) = \bar{F}(t)$$

$\bar{M}, \bar{C}, \& \bar{K} =$ structural matrices in the new coordinate system.

$\bar{F}(t) =$ force vector in the new coordinate system

Question

Can we select T such that $\bar{M}, \bar{C}, \& \bar{K}$ are all **DIAGONAL**?

If yes, equation for $Z(t)$ would then represent a set of uncoupled equations and hence can be solved easily.

How to select T to achieve this?

Consider the seemingly unrelated problem of undamped free vibration analysis

$$M\ddot{X} + KX = 0$$

Seek a special solution to this set of equations in which all points on the sturcutre oscillate harmonically at the same frequency.

$$\frac{x_1(t) = r_1 \exp(i\omega t)}{x_2(t) = r_2 \exp(i\omega t)} = \frac{r_1}{r_2}$$

That is

$$x_k(t) = r_k \exp(i\omega t); k = 1, 2, \cdots, n$$

or, $X(t) = R\exp(i\omega t)$ where R is a $n \times 1$ vector.

$$\Rightarrow \dot{X}(t) = i\omega R \exp(i\omega t) \;\&\; \ddot{X}(t) = -\omega^2 R \exp(i\omega t)$$

$$\Rightarrow \left[-\omega^2 MR + KR\right]\exp(i\omega t) = 0$$

$$\left[-\omega^2 MR + KR\right]\exp\left(i\omega t\right) = 0$$

$$\Rightarrow \left[-\omega^2 RM + KR\right] = 0$$

$$\Rightarrow KR = \omega^2 MR$$

This is a algebraic eigenvalue problem.

Note

- $K = K^t; M = M^t$
- K is positive semi-definite
- M is positive definite

$\Rightarrow$

Eigensolutions would be real valued and eigenvalues would be non-negative.

$$KR = \omega^2 MR$$

$$\left[K - \omega^2 M\right] R = 0$$

Let $\left[K - \omega^2 M\right]^{-1}$ exist.

$$\Rightarrow \left[K - \omega^2 M\right]^{-1} \left[K - \omega^2 M\right] R = 0$$

$$\Rightarrow IR = 0 \Rightarrow R = 0$$

$\Rightarrow$ If $\left[K - \omega^2 M\right]^{-1}$ exists, R=0 is the solution.

Condition for existence of nontrivial solution is that

$\left[K - \omega^2 M\right]^{-1}$ should not exist.

$$\Rightarrow \left|K - \omega^2 M\right| = 0$$

This is called the characteristic equation.

This leads to the characteristic values

$\omega_1^2 \leq \omega_2^2 \leq \cdots \leq \omega_n^2$ and associated eigenvectors

$R_1, R_2, \cdots, R_n$.

Orthogonality property of eigenvectors

Consider r-th and s-th eigenpairs. $\Rightarrow$

$$KR_r = \omega_r^2 MR_r \qquad (1)$$

$$KR_s = \omega_s^2 MR_s \qquad (2)$$

$(1)\times R_s^t \Rightarrow$

$$R_s^t KR_r = \omega_r^2 R_s^t MR_r \qquad (3)$$

$(2)\times R_r^t \Rightarrow$

$$R_r^t KR_s = \omega_s^2 R_r^t MR_s \qquad (4)$$

Transpose both sides of equation (4) $\Rightarrow$

$$R_s^t K^t R_r = \omega_s^2 R_s^t M^t R_r$$

Since $K^t = K$ & $M^t = M$, we get

$$R_s^t KR_r = \omega_s^2 R_s^t MR_r \qquad (5)$$

Substract (3) and (5) $\Rightarrow$

$$\left(\omega_r^2 - \omega_s^2\right) R_s^t MR_r = 0$$

$(AB)^t = B^t A^t$

$$R_s^t MR_r = 0 \quad r \neq s$$

$$R_s^t KR_r = 0 \quad r \neq s$$

Normalization

$$R_s^t MR_s = 1$$

$$R_s^t KR_s = \omega_s^2$$

Introduce

$$\Phi = \begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}_{(n\times n)}$$

$$\Lambda = \text{Diag}\begin{bmatrix} \omega_1^2 & \omega_2^2 & \cdots & \omega_n^2 \end{bmatrix}$$

Orthogonality relations

$$\Phi^t M \Phi = I$$

$$\Phi^t K \Phi = \Lambda$$

Select $T = \Phi$

Consider Undamped Forced Vibration Analysis

$$M\ddot{X} + KX = F(t)$$

$$X(0) = X_0; \dot{X}(0) = X_0$$

$$X(t) = \Phi Z(t)$$

$$\Rightarrow M\Phi\ddot{Z}(t) + K\Phi Z(t) = F(t)$$

$$\Rightarrow \Phi^t M\Phi\ddot{Z}(t) + \Phi^t K\Phi Z(t) = \Phi^t F(t)$$

$$\Rightarrow I\ddot{Z}(t) + \Lambda Z(t) = \bar{F}(t)$$

$$\Rightarrow \ddot{z}_r + \omega_r^2 z_r = f_r(t); r = 1, 2, \cdots, n$$

How about initial conditions?

$$X(0) = \Phi Z(0)$$

$$\Phi^t MX(0) = \Phi^t M\Phi Z(0) = Z(0)$$

$$Z(0) = \Phi^t MX(0) \,\&\, \dot{Z}(0) = \Phi^t M\dot{X}(0)$$

$$z_r(t) = z_r(0)\cos\omega_r t + \frac{\dot{z}_r(0)}{\omega_r}\sin\omega_r t + \int_0^t \frac{1}{\omega_r}\sin\omega_r(t-\tau)\, f_r(\tau)d\tau$$

$$X(t) = \Phi Z(t)$$

$$x_k(t) = \sum_{r=1}^{n} \Phi_{kr} z_r(t)$$

$k=1,2,\cdots,n$

$$= \sum_{r=1}^{n} \Phi_{kr}\left\{ z_r(0)\cos\omega_r t + \frac{\dot{z}_r(0)}{\omega_r}\sin\omega_r t + \int_0^t \frac{1}{\omega_r}\sin\omega_r(t-\tau)\, f_r(\tau)d\tau \right\}$$

How about damped forced response analysis?

$$M\ddot{X} + C\dot{X} + KX = F(t)$$

$$X(0) = X_0; \dot{X}(0) = X_0$$

$$X(t) = \Phi Z(t)$$

$$\Rightarrow M\Phi\ddot{Z}(t) + C\Phi\dot{Z}(t) + K\Phi Z(t) = F(t)$$

$$\Rightarrow \Phi^t M\Phi\ddot{Z}(t) + \Phi^t C\Phi\dot{Z}(t) + \Phi^t K\Phi Z(t) = \Phi^t F(t)$$

$$\Rightarrow I\ddot{Z}(t) + \Phi^t C\Phi\dot{Z}(t) + \Lambda Z(t) = \bar{F}(t)$$

If $\Phi^t C\Phi$ is not a diagonal matrix, the equations of motion would still remain coupled.

Classical damping models

If the damping matrix C is such that

$\Phi^t C \Phi$ is a diagonal matrix, then equations would get uncoupled.

Such C matrices are called classical damping matrices.

Example

Rayleigh's proportional damping matrix

$$C = \alpha M + \beta K$$

$$\Rightarrow$$

$$\Phi^t C \Phi = \alpha \Phi^t M \Phi + \beta \Phi^t K \Phi$$

$$= \alpha I + \beta \Lambda$$

$$C = \alpha M + \beta K$$

$$\Rightarrow \Phi^T C \Phi = \Phi^T [\alpha M + \beta K] \Phi$$

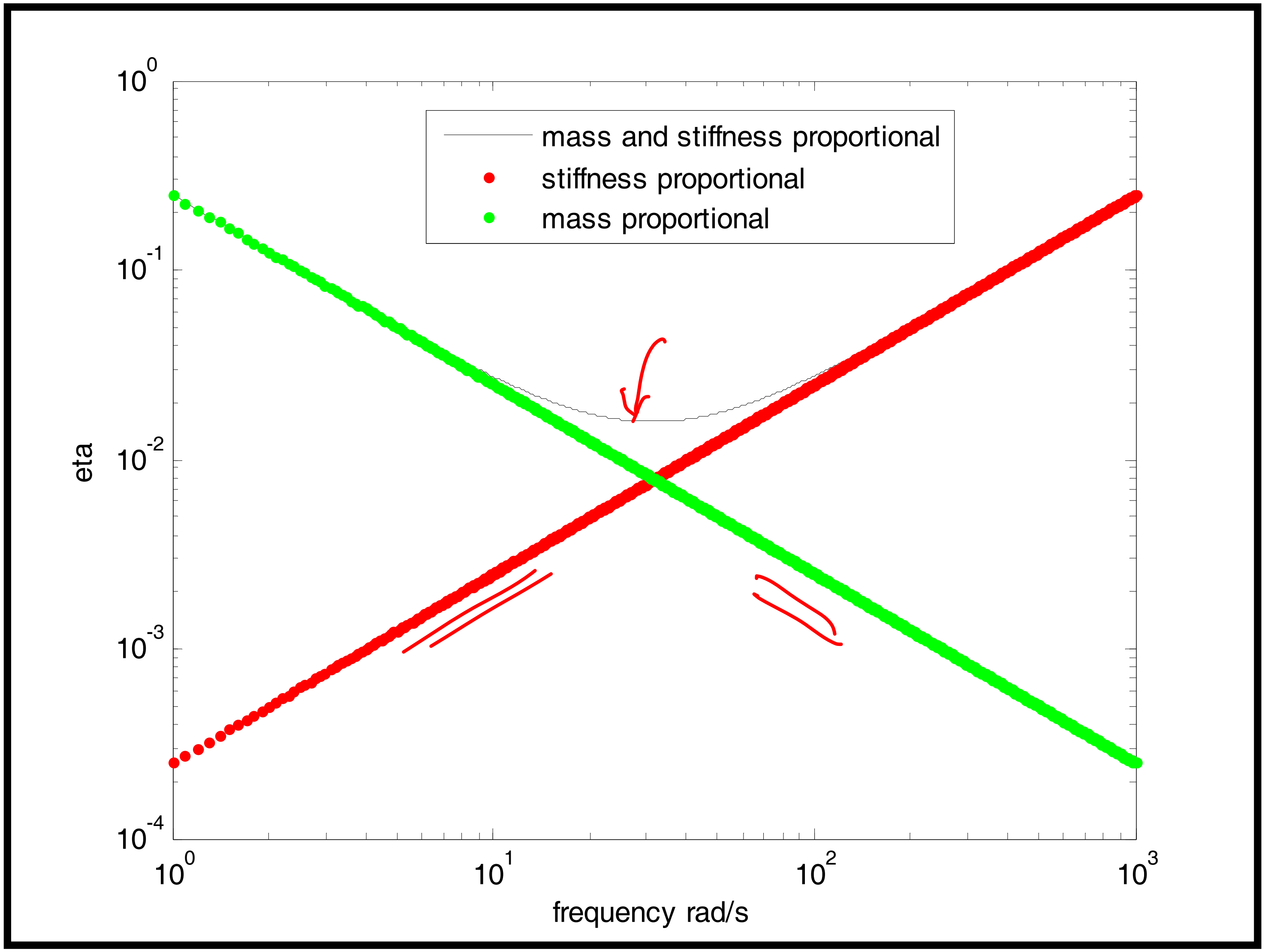
$$= \alpha \Phi^T I \Phi + \beta \Phi^T K \Phi$$

$$= \alpha [I] + \beta \quad Diag[\omega_i^2]$$

$$\Rightarrow c_n = \alpha + \beta \omega_n^2$$

$$\eta_n = \frac{\alpha}{2\omega_n} + \frac{\beta \omega_n}{2}$$

$c_n = 2\eta_n \omega_n m_n$



mass and stiffness proportional
stiffness proportional
mass proportional
eta
frequency rad/s
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{0}
10^{1}
10^{2}
10^{3}

$$I\ddot{Z}(t) + \Phi^t C \Phi \dot{Z}(t) + \Lambda Z(t) = \bar{F}(t)$$

$$Z(0) = \Phi^t M X(0) \;\&\; \dot{Z}(0) = \Phi^t M \dot{X}(0)$$

$$\Rightarrow$$

$$\ddot{z}_r + 2\eta_r\omega_r\dot{z}_r + \omega_r^2 z_r = f_r(t); r = 1, 2, \cdots, n$$

with $z_r(0) \;\&\; \dot{z}_r(0)$ specified.

$$\Rightarrow$$

$$z_r(t) = \exp(-\eta_r\omega_r t)\left[a_r \cos\omega_{dr} t + b_r \sin\omega_{dr} t\right] +$$

$$\int_0^t \frac{1}{\omega_{dr}} \exp\left[-\eta_r\omega_r(t-\tau)\right] f_r(\tau)\, d\tau$$

$$z_r(t) = \exp(-\eta_r \omega_r t)\left[a_r \cos \omega_{dr} t + b_r \sin \omega_{dr} t\right] + \int_0^t \frac{1}{\omega_{dr}} \exp\left[-\eta_r \omega_r (t-\tau)\right] f_r(\tau) d\tau$$

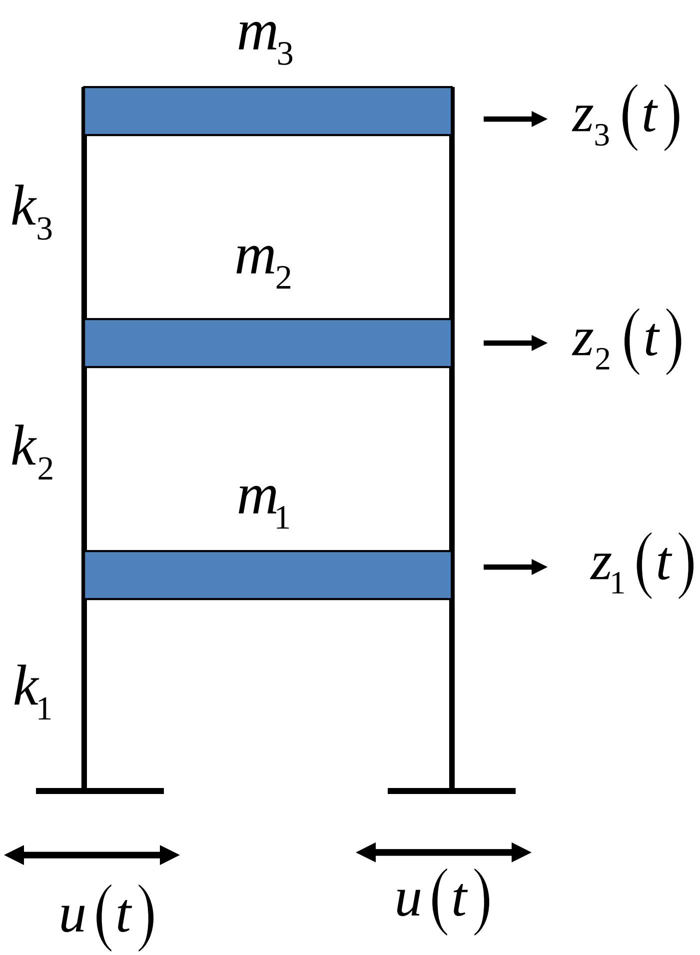
$$X(t) = \Phi Z(t)$$

$$x_k(t) = \sum_{r=1}^{n} \Phi_{kr} z_r(t)$$

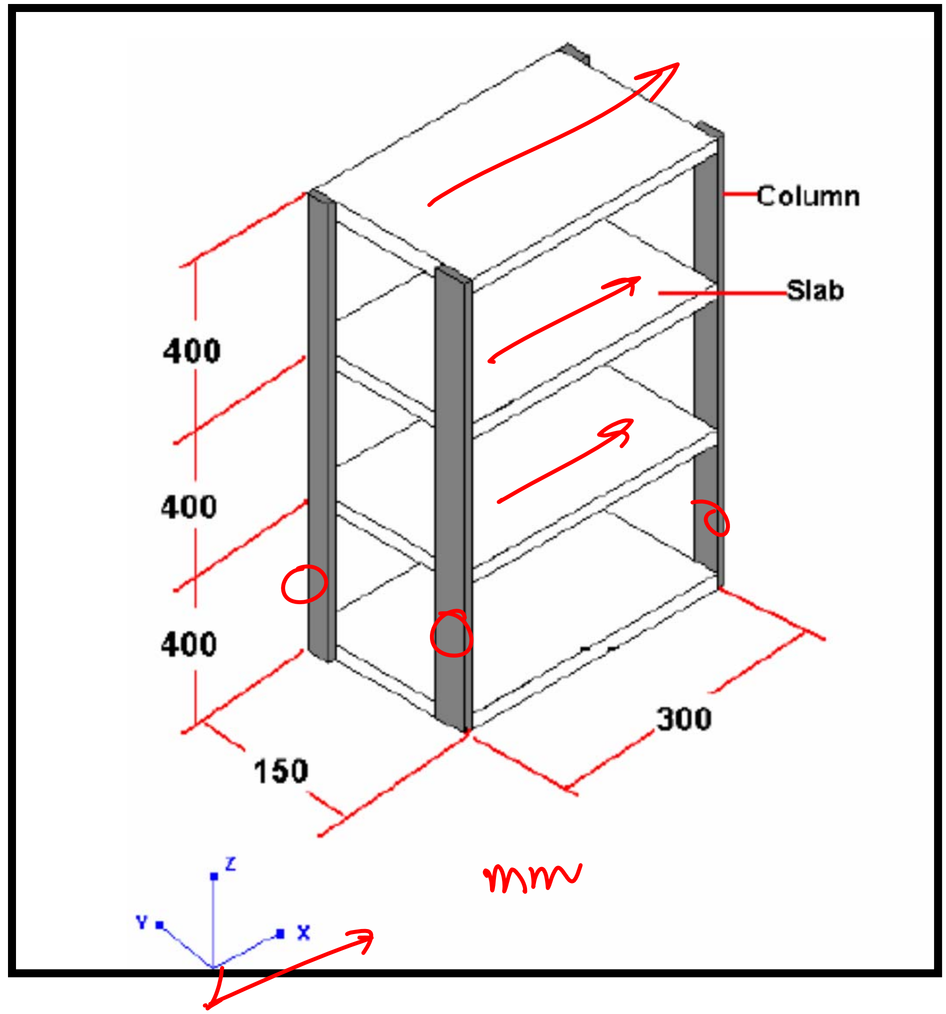
$$= \sum_{r=1}^{n} \Phi_{kr} \left\{ \exp(-\eta_r \omega_r t)\left[a_r \cos \omega_{dr} t + b_r \sin \omega_{dr} t\right] + \int_0^t \frac{1}{\omega_{dr}} \exp\left[-\eta_r \omega_r (t-\tau)\right] f_r(\tau) d\tau \right\}$$

$$k = 1, 2, \cdots, n$$

Example 1



m_3
$z_3(t)$
k_3
m_2
$z_2(t)$
k_2
m_1
$z_1(t)$
k_1
$u(t)$
$u(t)$



Column
Slab
400
400
400
150
300
mm
Z
Y
X

Part	Dimensions in mm		
	Depth (D)	Width (B)	Length (L)
Column	D_A =3.00	B_A =25.11	L_A = **400.00**
Slab	D_B =12.70	B_B =150.00	L_B =300.00

Sl. No.	Part	Material	Mass kg	Material Properties	
				Young's Modulus (E) N/m^2	Mass density (ρ) kg/m^3
1	Column	Aluminum	M_c **=0.0814**	69.0E+009	2700
2	Slab	Aluminum	M_s= 1.5430	69.0E+009	2700
3	Allen screw, M8	Steel	M_{sc}=0.0035	-	-

Mass matrix (kg)

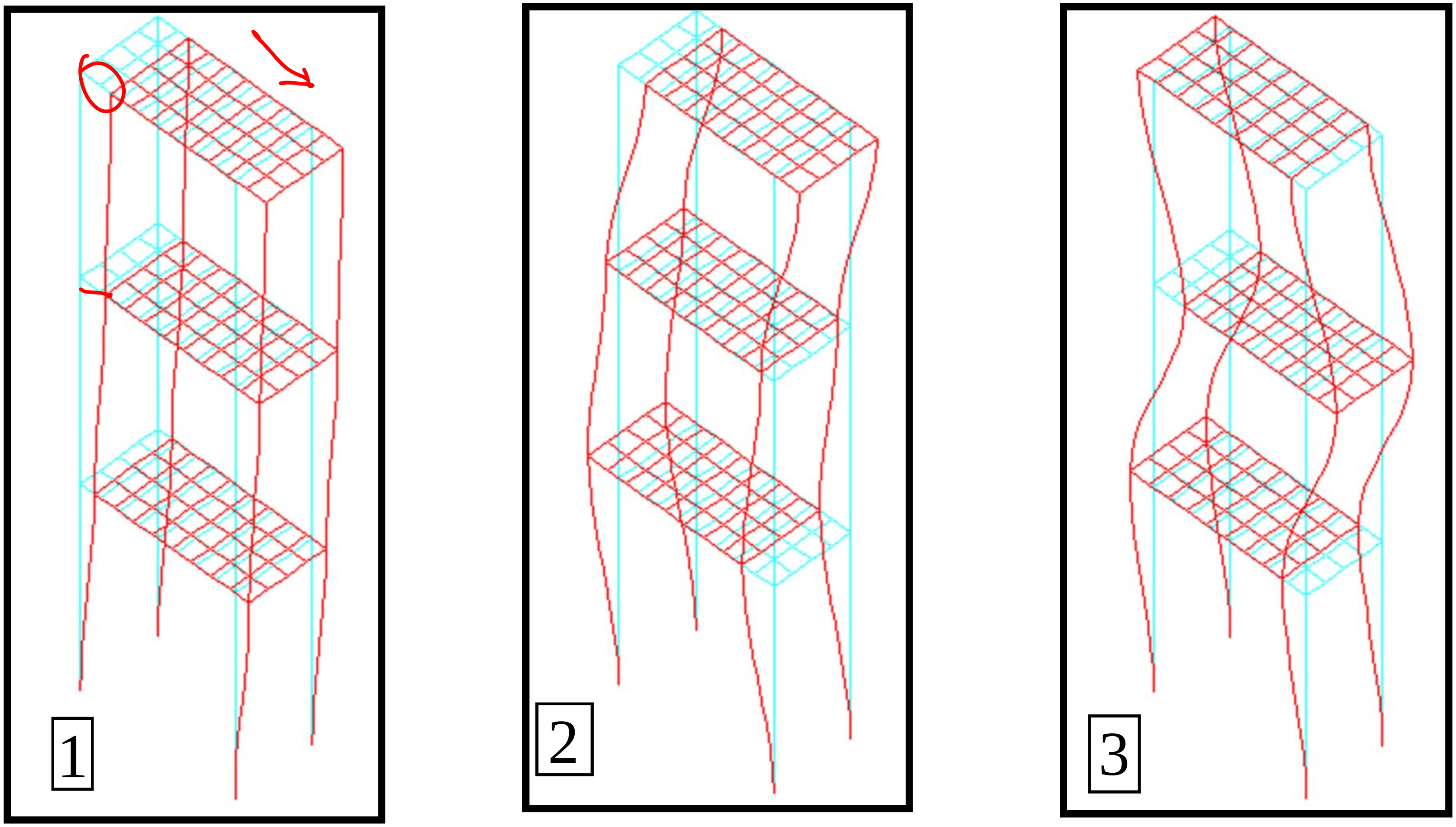
$$M = \begin{bmatrix} 1.8965 & 0 & 0 \\ 0 & 1.8965 & 0 \\ 0 & 0 & 1.7338 \end{bmatrix}$$

$$K\phi = \omega^2 M \phi$$

Stiffness matrix (N/m)

$$K = 1.0e{+}003 * \begin{bmatrix} 5.8475 & -2.9237 & 0.0000 \\ -2.9237 & 5.8475 & -2.9237 \\ 0.0000 & -2.9237 & 2.9237 \end{bmatrix}$$

$$\eta = \begin{bmatrix} 0.024 & 0.009 & 0.007 \end{bmatrix}^t$$



Mass normalized modal matrix

$$\Phi = \begin{bmatrix} -0.2464 & 0.5401 & -0.4181 \\ -0.4416 & 0.2132 & 0.5356 \\ -0.5451 & -0.4560 & -0.2679 \end{bmatrix}$$

Natural frequencies (rad/s)

$$\{\omega_n\} = \begin{bmatrix} 17.8939 \\ 49.7476 \\ 71.1199 \end{bmatrix}$$

Orthogonality checks:

$$\Phi^t M\Phi = [1.0000 \quad 0.0000 \quad 0.0000$$
$$0.0000 \quad 1.0000 \quad 0.0000$$
$$0.0000 \quad 0.0000 \quad 1.0000]$$

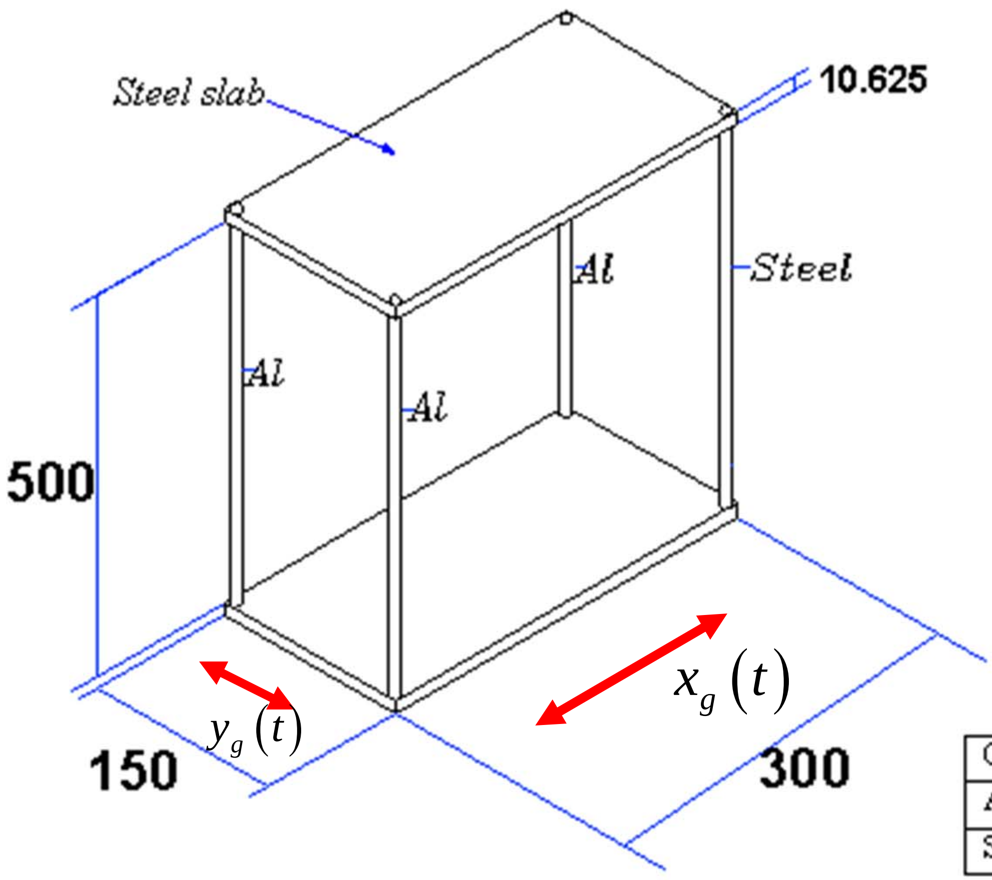
I

ω_1^2

$$\Phi^t K\Phi = 1.0e{+}003\ [0.3202 \quad -0.0000 \quad -0.0000$$
$$-0.0000 \quad 2.4748 \quad -0.0000$$
$$-0.0000 \quad -0.0000 \quad 5.0580]$$

ω_2^2 ω_3^2

Example 2



Columns	Diameter (mm)
Aluminum	10.13
Steel	9.958

All dimensions in mm

Physical properties of the frame members

Sl. No.	Part	Qty. Nos.	Material	Mass Kg	Material Properties		
					Mass density (ρ) kg/m^3	Modulus of elasticity (E) N/m^2	Poisson's ratio (μ)
1	Columns	3	Aluminum	$(m_1+m_2+m_3) = 0.3264$	2700	69.0E+009	0.3
2	Column	1	Steel	m_4=0.3037	7800	2.00E+011	0.3
3	Slab	1	Steel	M_s=3.7294	7800	2.00E+011	0.3

Location of mass center (m)

bs=0.30 , ds=0.15 , ts=10.625e-3
bs1=0.286 , ds1=0.136

$\bar{x} = 0.1436$, $\bar{y} = 0.0720$

Mass matrix

$$M = \begin{bmatrix} 4.0444\,(kg) & 0 & 0 \\ 0 & 4.0444\,(kg) & 0 \\ 0 & 0 & 0.0431\,(kgm^2) \end{bmatrix};$$

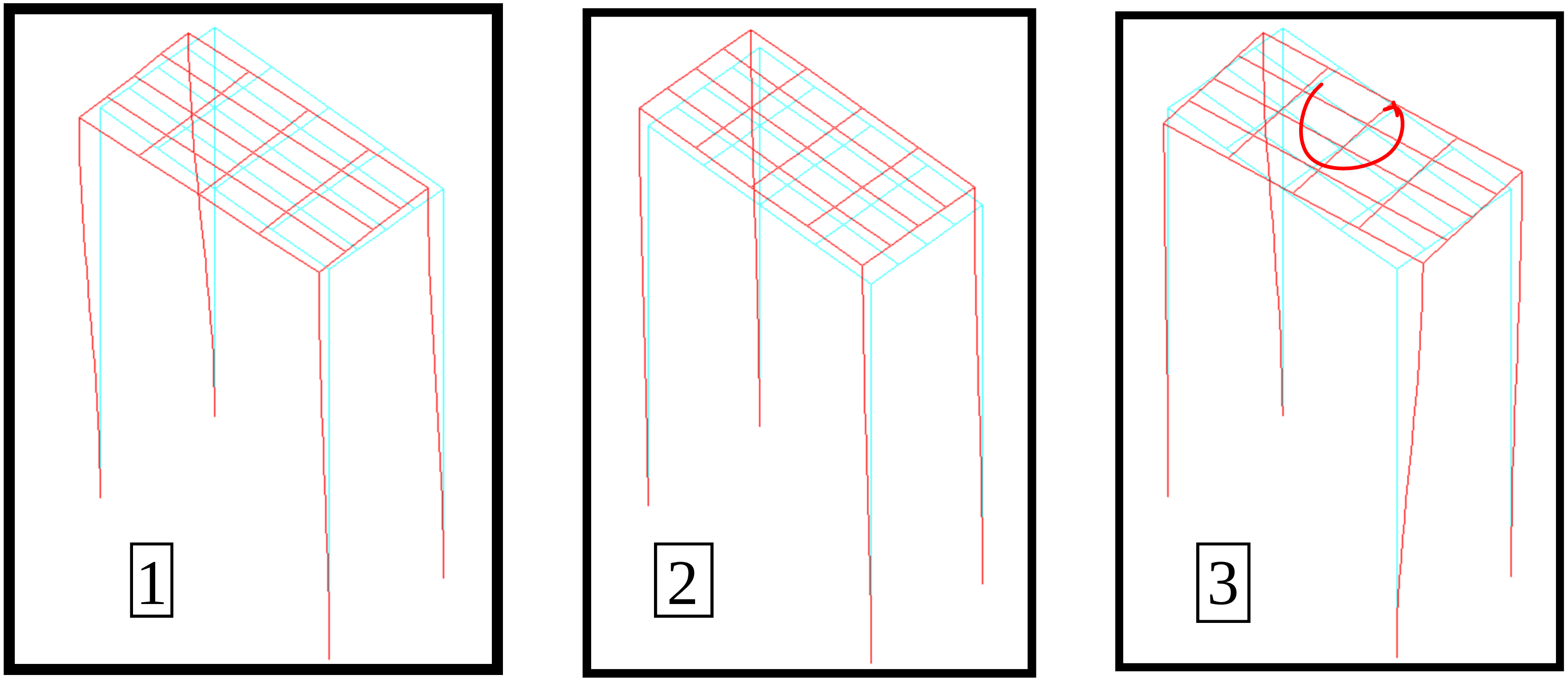
Stiffness matrix (force in N, distance in m and angle in rad).

k1=k2=k3=k4=k7=k8= 3.4240e+003 N/m
k5=k6=9.2674e+003 N/m
k*=313.1298 Nm

$$K = 1.0e+004 * \begin{bmatrix} 1.9539 & 0 & 0.0475 \\ 0 & 1.9539 & -0.0824 \\ 0.0475 & -0.0824 & 0.0805 \end{bmatrix};$$

$$\eta = \{0.02 \quad 0.02 \quad 0.01\}^t$$

Mode shapes



Mass normalized modal matrix

$$\Phi = [\begin{matrix} -0.2452 & 0.4308 & 0.0392 \\ 0.4254 & 0.2483 & -0.0681 \\ 0.7615 & 0 & 4.7567 \end{matrix}]$$

Natural frequencies (rad/s)

$$\{\omega_n\} = [\begin{matrix} 66.8321 \\ 69.5064 \\ 138.0460 \end{matrix}];$$

Orthogonality checks:

$$\Phi^t M\Phi = [1.0000 \quad 0.0000 \quad 0.0000$$
$$0.0000 \quad 1.0000 \quad 0.0000$$
$$0.0000 \quad 0.0000 \quad 1.0000];$$

I

$$\Phi^t K\Phi = 1.0e{+}004 *[\ 0.4467 \quad -0.0000 \quad 0.0000$$
$$-0.0000 \quad 0.4831 \quad 0.0000$$
$$0.0000 \quad 0.0000 \quad 1.9057];$$

Λ

Summary

- **Normal modes of vibration of a structure are special undamped free vibration solutions such that all points of the structure oscillate harmonically at the same frequency with the ratio of displacements at any two points being independent of time.**

- **Thus, for a structure vibrating in one of its modes, the phase difference between oscillations at any two points is either 0 or π.**

- **The frequencies at which normal mode oscillations are possible are called the natural frequencies.**

Summary (Continued)

- **Modal matrix is orthogonal to mass and stiffness matrices. This helps is diagonalising the mass and stiffness matrices.**

- **Undamped normal modes, in conjunction with proportional damping models, simplify vibration analysis procedures considerably.**

Frequency domain input - output relations

$$M\ddot{x}+C\dot{x}+Kx=f(t)$$

x → vector

Recall

$$X(\omega)=\int_{-\infty}^{\infty}x(\tau)\exp(i\omega\tau)\,d\tau$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\exp(-i\omega t)\,d\omega$$

scalar

Consider response as $t\to\infty$.

$$M\left\{\int_{-\infty}^{\infty}-\omega^{2}X(\omega)\exp(-i\omega t)\,d\omega\right\}+C\left\{\int_{-\infty}^{\infty}i\omega X(\omega)\exp(-i\omega t)\,d\omega\right\}+$$

$$K\left\{\int_{-\infty}^{\infty}X(\omega)\exp(-i\omega t)\,d\omega\right\}=\int_{-\infty}^{\infty}F(\omega)\exp(-i\omega t)\,d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \left\{ \left[-\omega^2 M + i\omega C + K \right] X(\omega) - F(\omega) \right\} \exp(-i\omega t)\, d\omega = 0$$

$$X(\omega) = \left[-\omega^2 M + i\omega C + K \right]^{-1} F(\omega)$$

$$X(\omega) = H(\omega) F(\omega)$$

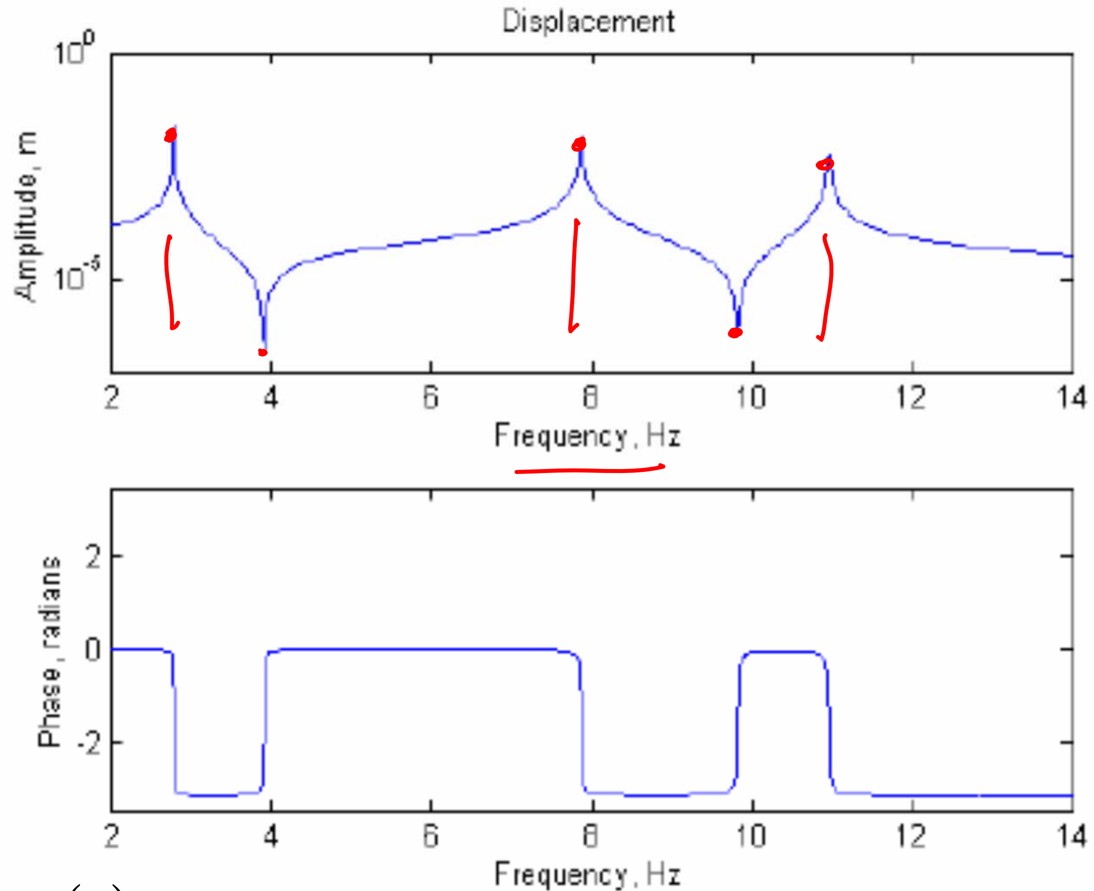
n x n

$$H(\omega) = \left[-\omega^2 M + i\omega C + K \right]^{-1}$$

$$\underset{(N\times 1)}{X(\omega)} = \underset{(N\times N)}{H(\omega)}\, \underset{(N\times 1)}{F(\omega)}$$

$$\underset{(N\times N)}{H(\omega)} = \text{Matrix of complex frequency response functions}$$

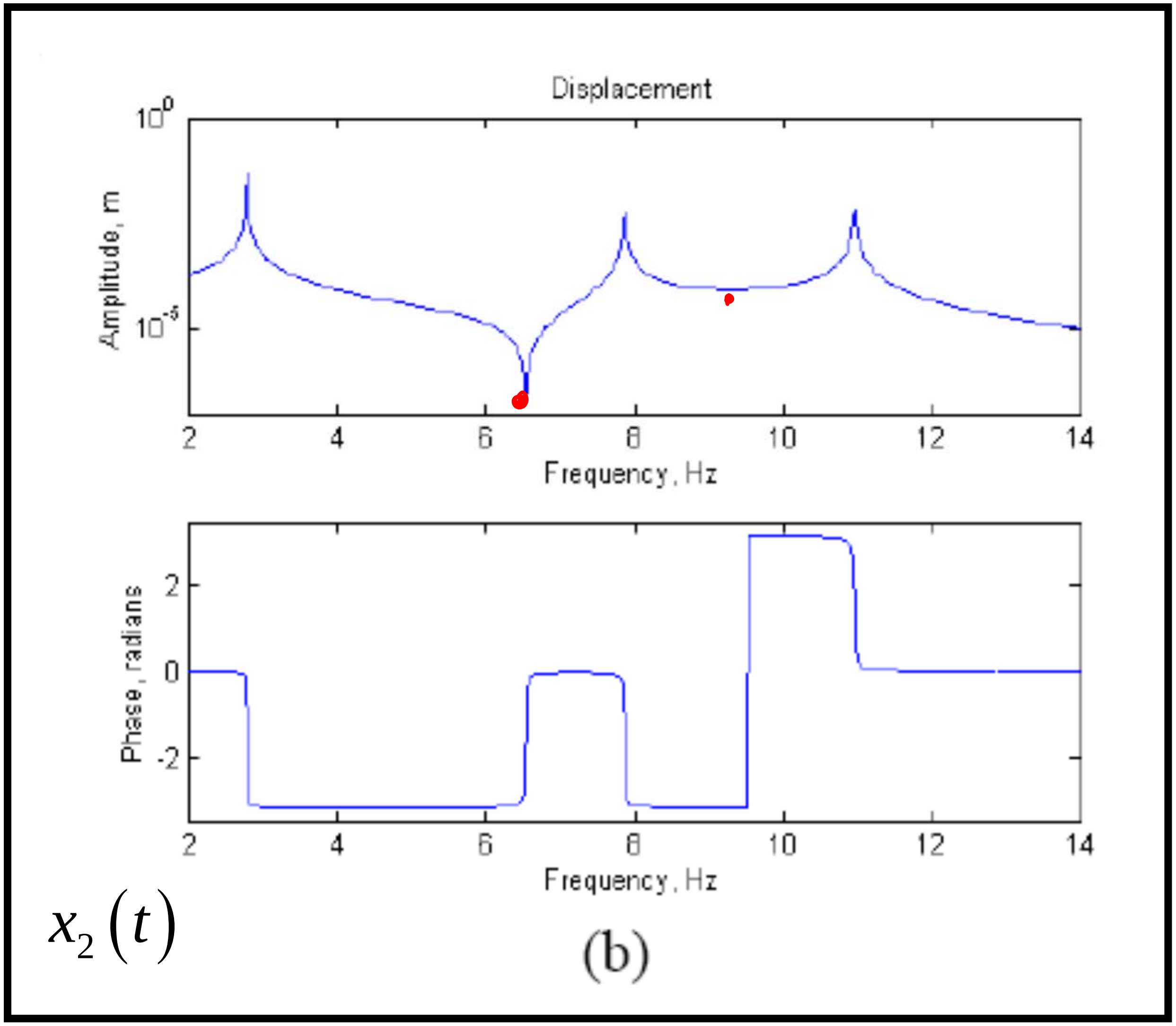
Frame in Example 1 under harmonic base motion



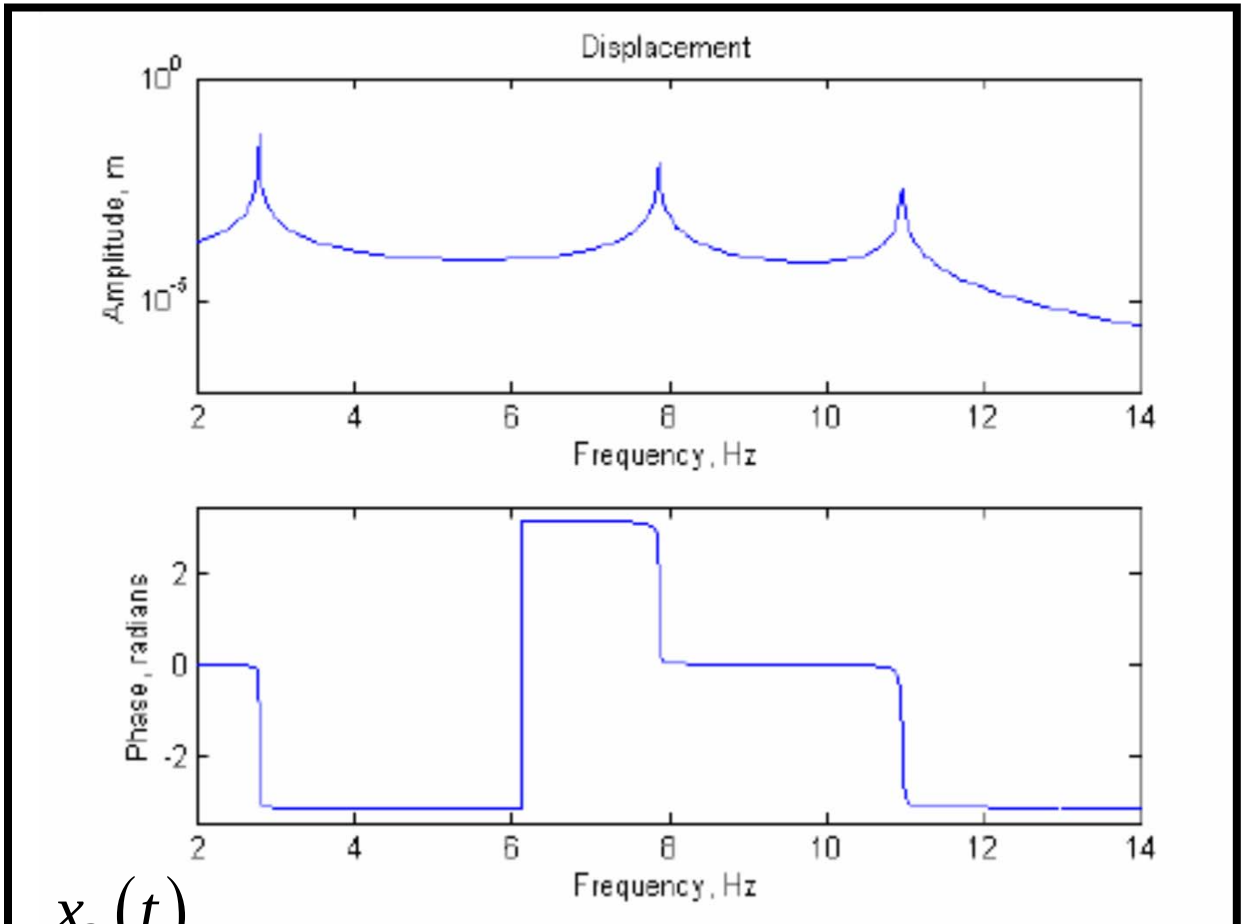
$x_1(t)$

(a)

Frame in Example 1 under harmonic base motion



Frame in Example 1 under harmonic base motion



(c)