Stochastic Structural Dynamics

Lecture-12

Random vibrations of sdof systems-4

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Recall

In general, for LTI systems, the knowledge of n^{th} order moment of input is adequate to determine the n^{th} order moment of the response process.

Stochastic steady state
Transients: nonstationarity
Steady state: stationarity

Conditions to be satisfied for existence of steady state

- System is damped
- Excitation is stationary

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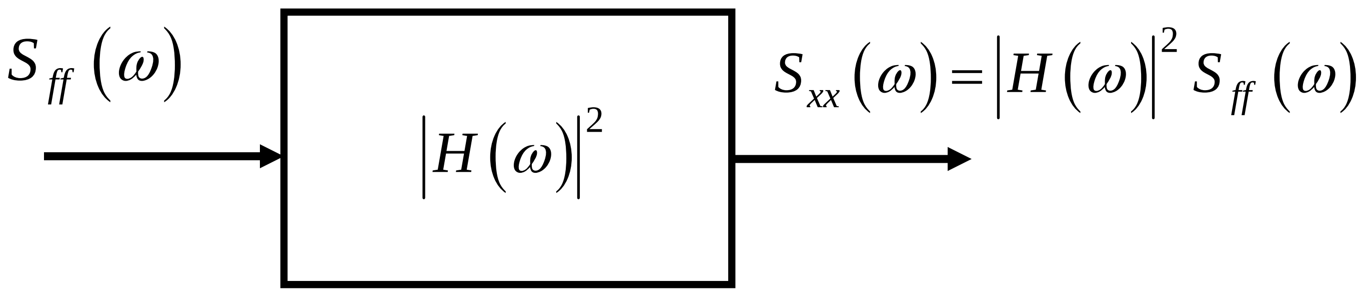
Frequency Domain I/O relations



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SDOF system under stationary random excitation

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$x(0) = 0; \dot{x}(0) = 0$$

$$\langle f(t) \rangle = 0; \langle f(t_1) f(t_2) \rangle = R_{ff}(t_2 - t_1)$$

$$x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

$$\Rightarrow$$

$$\langle x(t) \rangle = \int_0^t h(t-\tau) \langle f(\tau) \rangle d\tau = 0$$

$$\left\langle x(t_1)\,x(t_2)\right\rangle = \left\langle \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)\,f(\tau_1)\,h(t_2-\tau_2)\,f(\tau_2)\,d\tau_1 d\tau_2 \right\rangle$$

$$R_{xx}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)\left\langle f(\tau_1)\,f(\tau_2)\right\rangle d\tau_1 d\tau_2$$

$$= \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)R_{ff}(\tau_1,\tau_2)\,d\tau_1 d\tau_2$$

$$= \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)R_{ff}(\tau_2-\tau_1)\,d\tau_1 d\tau_2$$

Recall

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\exp(i\omega\tau)\,d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\exp(-i\omega\tau)\,d\omega$$

$$R_{ff}(\tau) = \frac{1}{\pi}\int_{0}^{\infty} S_{ff}(\Omega)\cos\Omega\tau\,d\Omega$$

$S_{ff}(\Omega)$: Physical PSD

$$
\begin{aligned}
R_{xx}\left(t_1,t_2\right) &= \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)R_{ff}\left(\tau_2-\tau_1\right)d\tau_1 d\tau_2 \\
&= \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)\left[\int_0^{\infty} S_{ff}\left(\Omega\right)\cos\Omega\left(\tau_1-\tau_2\right)d\Omega\right]d\tau_1 d\tau_2 \\
&= \int_0^{\infty} S_{ff}\left(\Omega\right)\left\{\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)\cos\Omega\left(\tau_1-\tau_2\right)d\tau_1 d\tau_2\right\}d\Omega \\
&= \int_0^{\infty} S_{ff}\left(\Omega\right)\mathcal{H}\left(\Omega,t_1,t_2\right)d\Omega
\end{aligned}
$$

Time dependent frequency response function

$$\mathscr{H}\left(\Omega, t_1, t_2\right) = \int_0^{t_1}\int_0^{t_2} h(t_1 - \tau_1) h(t_2 - \tau_2) \cos\Omega\left(\tau_1 - \tau_2\right) d\tau_1 d\tau_2$$

Questions

What is the nature of $\mathscr{H}\left(\Omega, t_1, t_2\right)$?

$$\lim_{\substack{t_1 \to \infty \\ t_2 \to \infty \\ t_2 - t_1 = \tau < \infty}} \mathscr{H}\left(\Omega, t_1, t_2\right) = ?$$

Do we recover steady state I/O relations?

Recall $h(t) = \dfrac{1}{m\omega_d}\exp(-\eta\omega t)\sin\omega_d t$

$$\mathscr{H}(\Omega,t_1,t_2) = \left|H(\Omega)\right|^2 [\cos\Omega(t_2-t_1) +$$

$$\exp(-\eta\omega t_1)\left\{\left(\cos\omega_d t_1 + \frac{\eta}{\sqrt{1-\eta^2}}\sin\omega_d t_1\right)\cos\Omega t_2 + \frac{\Omega}{\omega_d}\sin\omega_d t_1 \sin\Omega t_2\right\}$$

$$+\exp(-\eta\omega t_1)\left\{\left(\cos\omega_d t_2 + \frac{\eta}{\sqrt{1-\eta^2}}\sin\omega_d t_2\right)\cos\Omega t_1 + \frac{\Omega}{\omega_d}\sin\omega_d t_2 \sin\Omega t_1\right\}$$

$$+\exp\left[-\eta\omega(t_1+t_2)\right]\left\{\cos\omega_d t_1 \cos\omega_d t_2 + \frac{\eta^2\omega^2+\Omega^2}{\omega_d^2}\sin\omega_d t_1 \sin\omega_d t_2 + \frac{\eta}{\sqrt{1-\eta^2}}\sin\omega_d(t_1+t_2)\right\}]$$

$\mathscr{H}(\Omega,t_1,t_2) =$ Time dependent transfer function

$$\left|H(\Omega)\right|^2 = \frac{1}{m^2\left[\left(\omega^2-\Omega^2\right)^2 + \left(2\eta\omega\Omega\right)^2\right]}$$

$$\lim_{\substack{t_1 \to \infty \\ t_2 \to \infty \\ t_2 - t_1 = \tau < \infty}} \mathscr{H}\left(\Omega, t_1, t_2\right) = \left|H\left(\Omega\right)\right|^2 \cos\Omega\tau$$

$$\Rightarrow$$

$$\lim_{\substack{t_1 \to \infty \\ t_2 \to \infty \\ t_2 - t_1 = \tau < \infty}} R_{xx}\left(\tau\right) = \int_0^\infty S_{ff}\left(\Omega\right)\left|H\left(\Omega\right)\right|^2 \cos\Omega\tau d\tau$$

$$\Rightarrow$$

$$S_{xx}\left(\Omega\right) = \left|H\left(\Omega\right)\right|^2 S_{ff}\left(\Omega\right)$$

This agrees with the frequency domain I/O relation obtained earlier directly using the definition of PSD function.

$$S_{xx}(\Omega) = |H(\Omega)|^2 S_{ff}(\Omega)$$

$$\Rightarrow$$

$$\sigma_{xs}^2 = \int_0^\infty |H(\Omega)|^2 S_{ff}(\Omega)\, d\Omega$$

Example: $S_{ff}(\Omega) = \frac{I}{\pi}$ = Physical psd for white noise process

$$\Rightarrow$$

$$S_{xx}(\Omega) = \frac{I}{\pi}|H(\Omega)|^2 = \frac{(I/\pi)}{m^2\left[\left(\omega^2 - \Omega^2\right)^2 + \left(2\eta\omega\Omega\right)^2\right]}$$

$$\sigma_{xs}^2 = \int_0^\infty \frac{(I/\pi)}{m^2\left[\left(\omega^2 - \Omega^2\right)^2 + \left(2\eta\omega\Omega\right)^2\right]} d\Omega$$

$$\sigma_{xs}^2 = \int_0^\infty \frac{(I/\pi)}{m^2\left[\left(\omega^2-\Omega^2\right)^2+\left(2\eta\omega\Omega\right)^2\right]}\,d\Omega$$

Exercise

Use residue theorem and evaluate the above integral and show that the result agrees with the one obtained already

Useful integrals

$$I_n = \int_{-\infty}^{\infty} \left|H_n(\omega)\right|^2 d\omega$$

$$H_n(\omega) = \frac{B_0 + (i\omega)B_1 + (i\omega)^2 B_2 + \cdots + (i\omega)^{n-1} B_{n-1}}{A_0 + (i\omega)A_1 + (i\omega)^2 A_2 + \cdots + (i\omega)^n A_n}$$

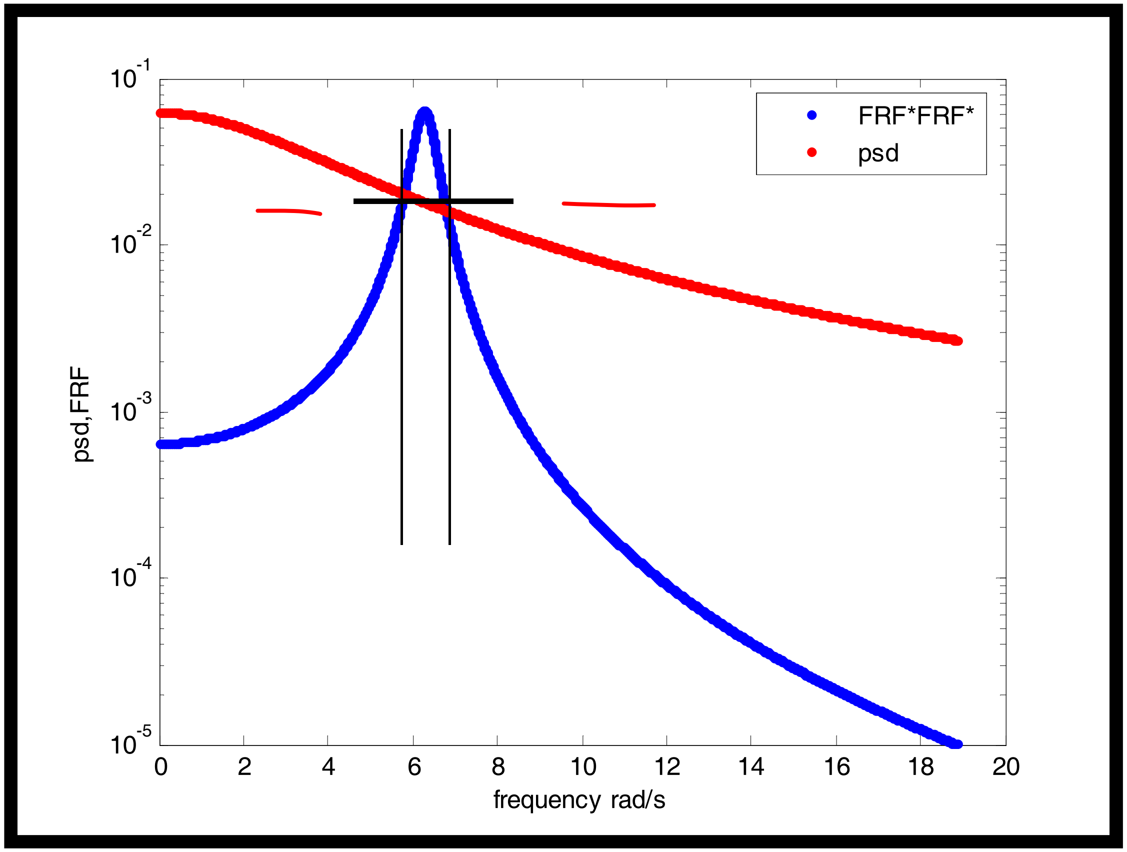
$n = 1$ $\quad H_1(\omega) = \dfrac{B_0}{A_0 + (i\omega)A_1}$ $\quad I_1 = \dfrac{\pi B_0^2}{A_0 A_1}$

$n = 2$ $\quad H_n(\omega) = \dfrac{B_0 + (i\omega)B_1}{A_0 + (i\omega)A_1 + (i\omega)^2 A_2}$ $\quad I_2 = \dfrac{\pi\left\{A_0 B_1^2 + A_2 B_0^2\right\}}{A_0 A_1 A_2}$

$n = 3$ $\quad H_n(\omega) = \dfrac{B_0 + (i\omega)B_1 + (i\omega)^2 B_2}{A_0 + (i\omega)A_1 + (i\omega)^2 A_2 + (i\omega)^3 A_3}$ $\quad I_3 = \dfrac{\pi\left\{A_0 A_3\left(2B_0 B_2 - B_1^2\right) - A_0 A_1 B_2^2 - A_2 A_3 B_0^2\right\}}{A_0 A_3\left(A_0 A_3 - A_1 A_2\right)}$

$\vdots \qquad \vdots \qquad \vdots$

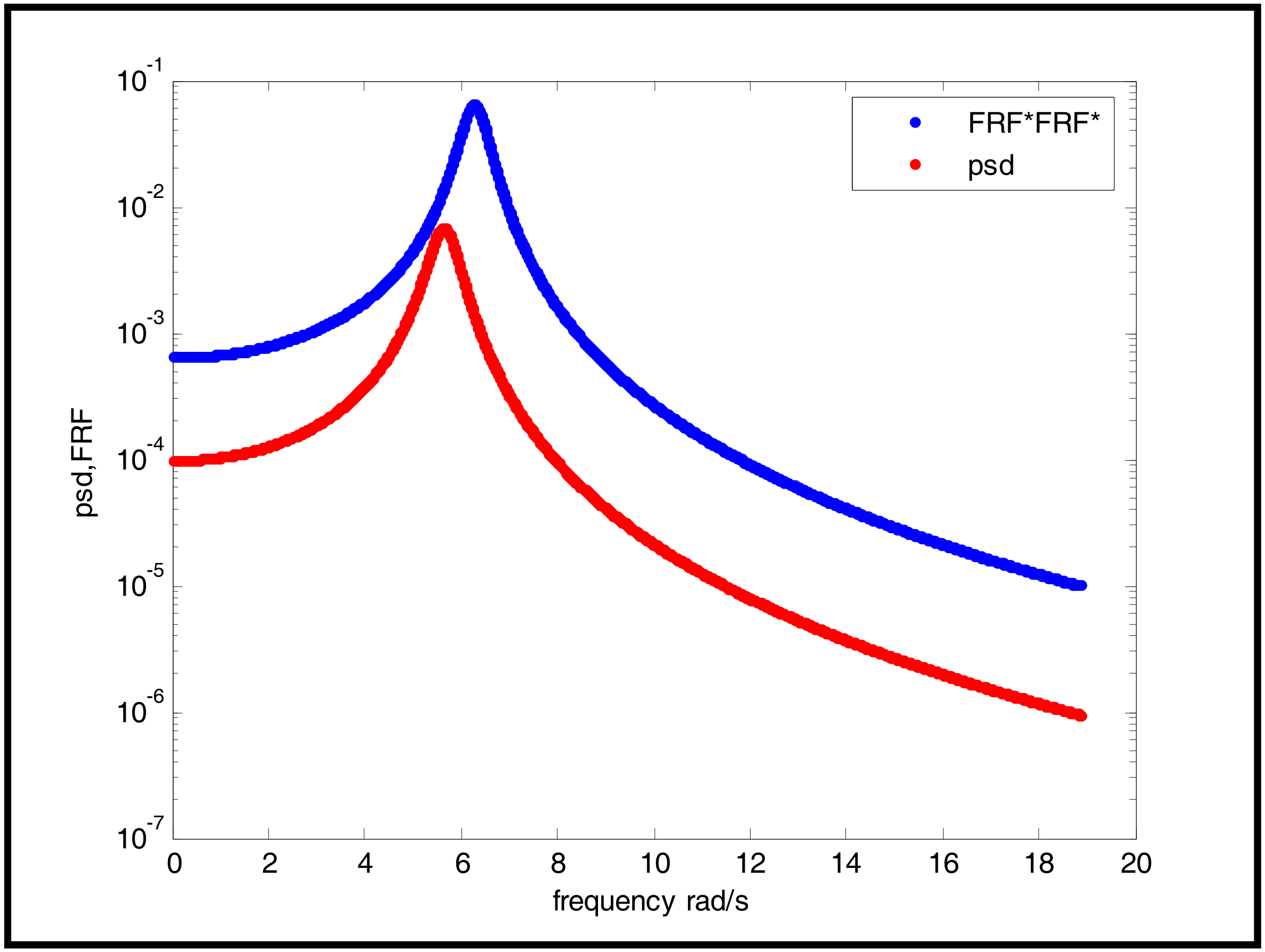
Approximation for broad band excitations.



$$\sigma_{xs}^2 = \int_0^\infty \left|H(\Omega)\right|^2 S_{ff}(\Omega)\, d\Omega$$

$$\approx S_{ff}(\omega_n) \int_0^\infty \left|H(\Omega)\right|^2 d\Omega$$

The approximation would not be acceptable here



SDOF system under non-stationary random excitation

$$m\ddot{x} + c\dot{x} + kx = e(t)\,f\left(t\right)$$

$$x\left(0\right) = 0;\ \dot{x}\left(0\right) = 0$$

$$\left\langle f\left(t\right)\right\rangle = 0;\left\langle f\left(t_1\right) f\left(t_2\right)\right\rangle = R_{ff}\left(t_2 - t_1\right)$$

$e(t) =$ Deterministic modulating (envelope) function

$$x\left(t\right) = \int_0^t h\left(t-\tau\right) e(\tau)\, f\left(\tau\right) d\tau$$

$$\Rightarrow$$

$$\left\langle x\left(t\right)\right\rangle = \int_0^t h\left(t-\tau\right) e(\tau) \left\langle f\left(\tau\right)\right\rangle d\tau = 0$$

$$\left\langle x(t_1)\,x(t_2)\right\rangle = \left\langle \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)e(\tau_1)\,f(\tau_1)\,h(t_2-\tau_2)e(\tau_2)\,f(\tau_2)\,d\tau_1 d\tau_2 \right\rangle$$

$$R_{xx}(t_1,t_2) = \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)e(\tau_1)e(\tau_2)\left\langle f(\tau_1)\,f(\tau_2)\right\rangle d\tau_1 d\tau_2$$

$$= \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)e(\tau_1)e(\tau_2)R_{ff}(\tau_1,\tau_2)\,d\tau_1 d\tau_2$$

$$= \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)e(\tau_1)e(\tau_2)R_{ff}(\tau_2-\tau_1)\,d\tau_1 d\tau_2$$

$$R_{xx}\left(t_1,t_2\right)=\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)e\left(\tau_1\right)e\left(\tau_2\right)R_{ff}\left(\tau_2-\tau_1\right)d\tau_1 d\tau_2$$

$$=\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)e\left(\tau_1\right)e\left(\tau_2\right)\left[\int_0^{\infty} S_{ff}\left(\Omega\right)\cos\Omega\left(\tau_1-\tau_2\right)d\Omega\right]d\tau_1 d\tau_2$$

$$=\int_0^{\infty} S_{ff}\left(\Omega\right)\left\{\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)e\left(\tau_1\right)e\left(\tau_2\right)\cos\Omega\left(\tau_1-\tau_2\right)d\tau_1 d\tau_2\right\}d\Omega$$

$$=\int_0^{\infty} S_{ff}\left(\Omega\right)\mathcal{H}\left(\Omega,t_1,t_2\right)d\Omega$$

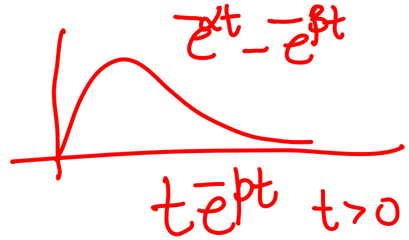
$$\mathcal{H}\left(\Omega,t_1,t_2\right)=\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)e\left(\tau_1\right)e\left(\tau_2\right)\cos\Omega\left(\tau_1-\tau_2\right)d\tau_1 d\tau_2$$

$$\sigma_x^2\left(t\right)=\int_0^{\infty} S_{ff}\left(\Omega\right)\mathcal{H}\left(\Omega,t\right)d\Omega$$

$$\mathcal{H}\left(\Omega,t\right)=\int_0^{t}\int_0^{t} h(t-\tau_1)h(t-\tau_2)e\left(\tau_1\right)e\left(\tau_2\right)\cos\Omega\left(\tau_1-\tau_2\right)d\tau_1 d\tau_2$$

$$\mathscr{H}\left(\Omega,t_1,t_2\right)=\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)e\left(\tau_1\right)e\left(\tau_2\right)\cos\Omega\left(\tau_1-\tau_2\right)d\tau_1 d\tau_2$$

Behavior of $\mathscr{H}\left(\Omega,t_1,t_2\right)$ for $t_1,t_2\to\infty$ depends on behavior of $e\left(t\right)$ for large times.



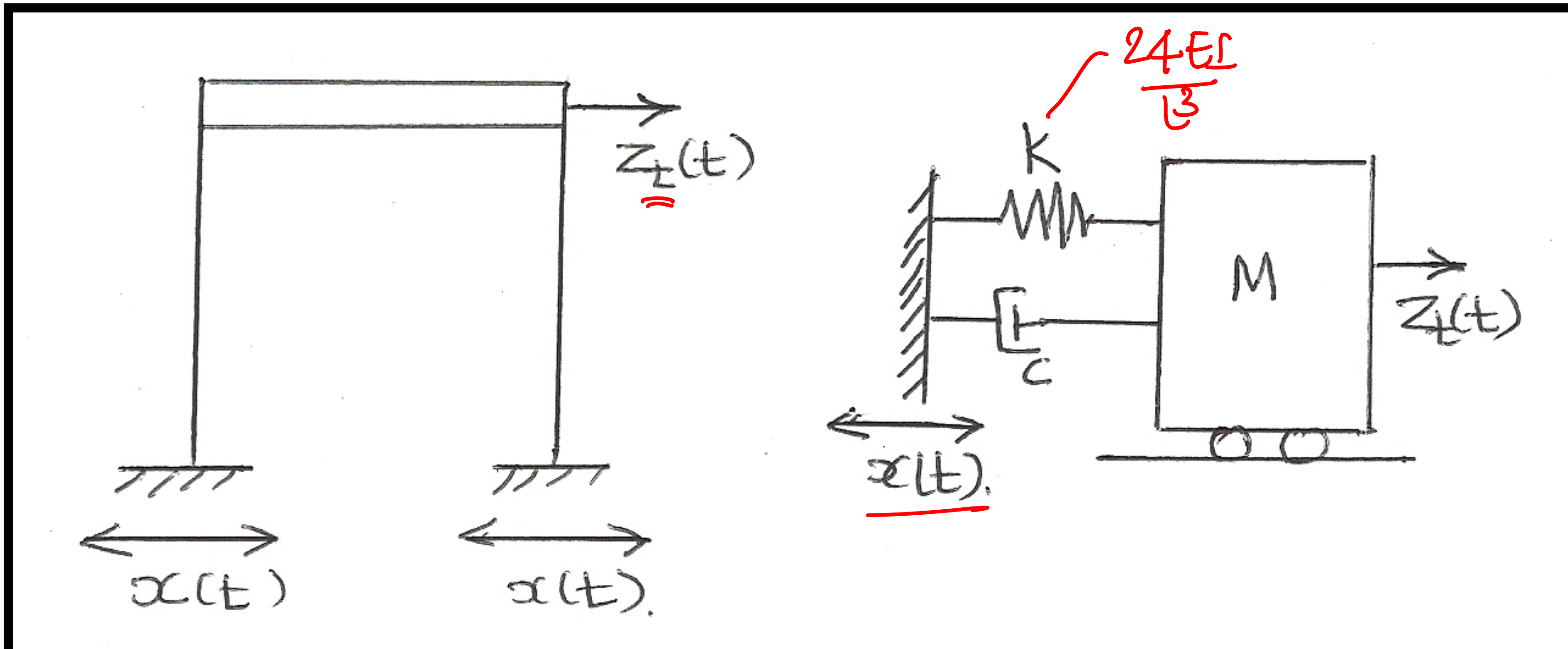
Clearly, if $\lim\limits_{t\to\infty} e\left(t\right)\to 0, \;\; \Rightarrow \lim\limits_{t\to\infty}\mathscr{H}\left(\Omega,t\right)\to 0$

$\Rightarrow$

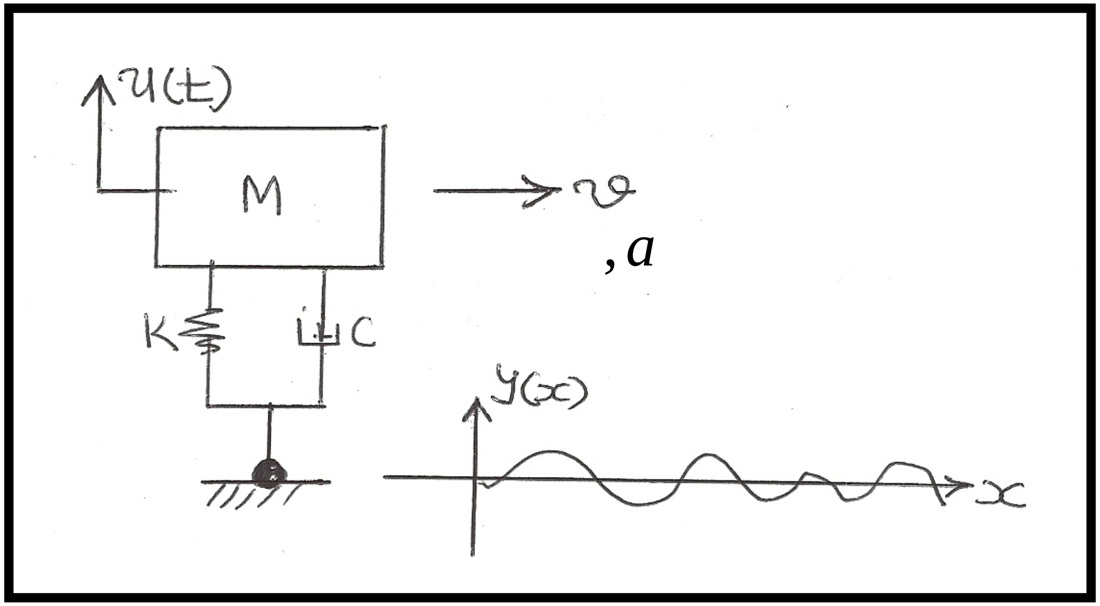
No steady state exists.

SDOF systems under random support motions

- Structural vibration during earthquakes
- Vehicles travelling on rough roads



Vehicle taxiing on rough road



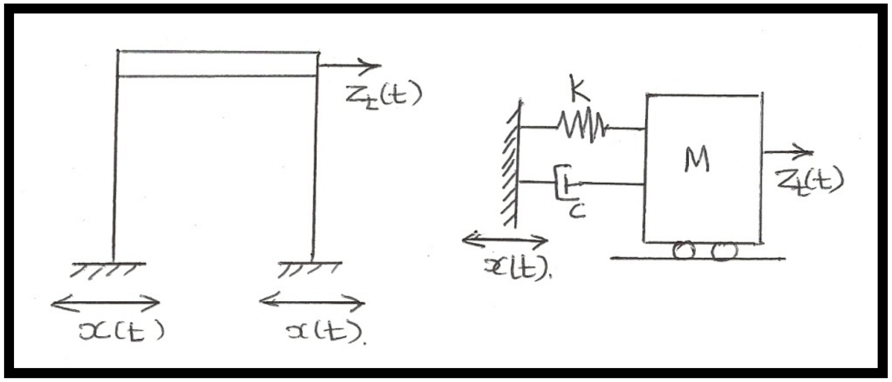
$$m\ddot{u} + c\frac{d}{dt}\left[u - y\left(x = vt + 0.5at^2\right)\right] + k\left[u - y\left(x = vt + 0.5at^2\right)\right] = 0$$

with $u(0)\,\&\,\dot{u}(0)$ specified.

$$\Rightarrow$$

$$m\ddot{u} + c\frac{du}{dt} + ku = c\frac{d}{dt}\left[y\left(x = vt + 0.5at^2\right)\right] + k\left[y\left(x = vt + 0.5at^2\right)\right]$$

$$= f(t)$$



$$m\ddot{z}_t + c\left[\dot{z}_t - \dot{x}\right] + k\left[z_t - x\right] = 0$$

$$z_t(0) = z_{t0};\ \dot{z}_t(0) = \dot{z}_{t0}$$

$$\ddot{z}_t + 2\eta\omega_n\dot{z}_t + \omega_n^2 z_t = -\left(2\eta\omega_n\dot{x} + \omega_n^2 x\right)$$

$$z = z_t - x$$

z_t = Total displacement

z = Relative displacement

x = Support displacement

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x}$$

$$z(0) = z_{t0} - x(0);\ \dot{z}(0) = \dot{z}_{t0} - \dot{x}(0)$$

$$\ddot{z} + 2\eta\omega_n\dot{z} + \omega_n^2 z = -\ddot{x}$$

Steady state Random vibration analysis
Analysis of total displacement

$$\ddot{z}_t + 2\eta\omega_n\dot{z}_t + \omega_n^2 z_t = -\left(2\eta\omega_n\dot{x} + \omega_n^2 x\right); z_t\left(0\right) = z_{t0}; \dot{z}_t\left(0\right) = \dot{z}_{t0}$$

$$z_t\left(t\right) = \exp\left(-\eta\omega t\right)\left[A\cos\omega_d t + B\sin\omega_d t\right] - \int_0^t h\left(t-\tau\right)\left[2\eta\omega_n\dot{x}\left(\tau\right) + \omega_n^2 x\left(\tau\right)\right]d\tau$$

Steady state

$$S_{z_t z_t}\left(\omega\right) = \left|H\left(\omega\right)\right|^2\left[\omega_n^4 + \left(2\eta\omega_n\omega\right)^2\right]S_{XX}\left(\omega\right)$$

$S_{XX}\left(\omega\right) =$ PSD of support displacement $x\left(t\right)$

$$\sigma_{z_t}^2 = \int_0^\infty \left|H\left(\omega\right)\right|^2\left[\omega_n^4 + \left(2\eta\omega_n\omega\right)^2\right]S_{XX}\left(\omega\right)d\omega$$

Steady state Random vibration analysis
Analysis of relative displacement

$$\ddot{z} + 2\eta\omega_n\dot{z} + \omega_n^2 z = -\ddot{x};\ z(0) = z_{t0} - x(0); \dot{z}(0) = \dot{z}_{t0} - \dot{x}(0)$$

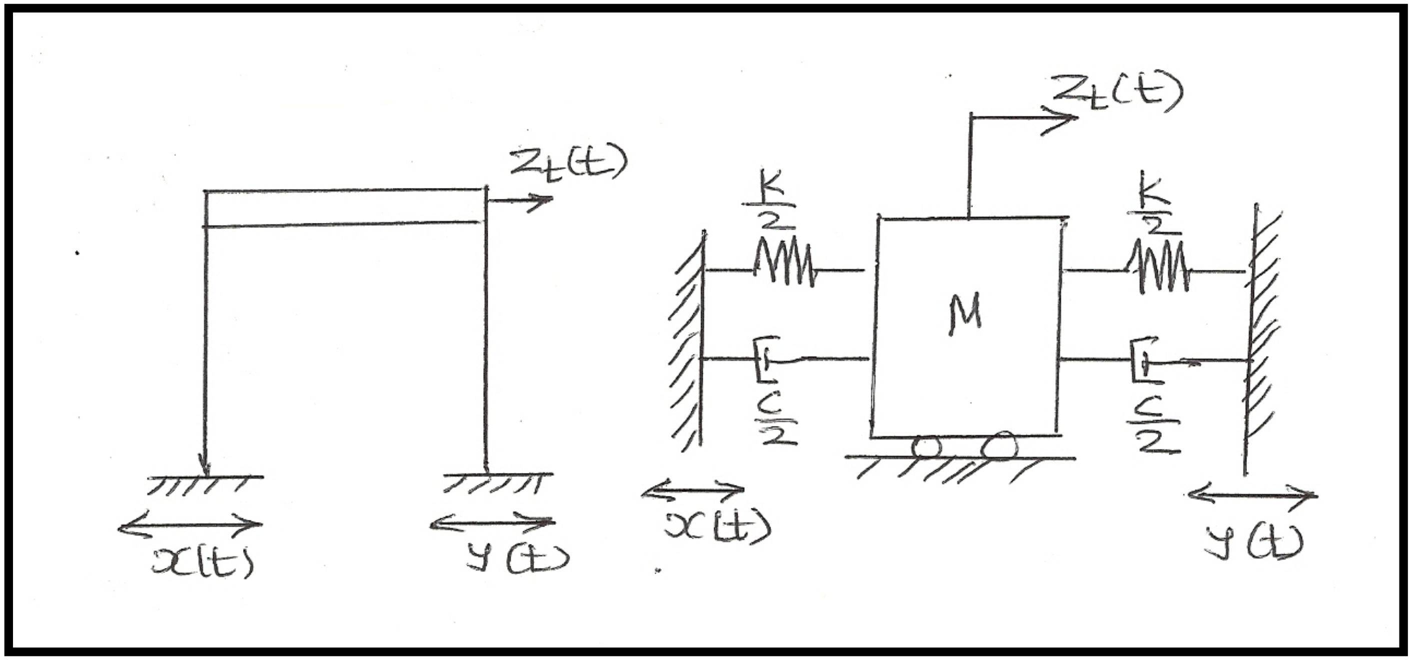
$$z(t) = \exp(-\eta\omega t)\left[A\cos\omega_d t + B\sin\omega_d t\right] - \int_0^t h(t-\tau)\, m\ddot{x}(\tau) d\tau$$

Steady state

$$S_{zz}(\omega) = \left|H(\omega)\right|^2 m^2 S_{\ddot{x}\ddot{x}}(\omega) = \left|H(\omega)\right|^2 \omega^4 m^2 S_{xx}(\omega)$$

$$\sigma_z^2 = \int_0^\infty \left|H(\omega)\right|^2 \omega^4 m^2 S_{xx}(\omega)\, d\omega$$

Doubly supported SDOF system under differential ground motions



What is relative displacement?

Total response=pseudo-dynamic response+dynamic response

$$m\ddot{z}_t + \frac{c}{2}\left[\dot{z}_t - \dot{x}\right] + \frac{c}{2}\left[\dot{z}_t - \dot{y}\right] + \frac{k}{2}\left[z_t - x\right] + \frac{k}{2}\left[z_t - y\right] = 0$$

$$m\ddot{z}_t + c\left[\dot{z}_t - \left(\frac{\dot{x}+\dot{y}}{2}\right)\right] + k\left[z_t - \left(\frac{x+y}{2}\right)\right] = 0$$

Pseudo-dynamic response

$$k\left[z_{ps} - \left(\frac{x+y}{2}\right)\right] = 0 \Rightarrow z_{ps} = \left(\frac{x+y}{2}\right)$$

Dynamic response

$$z(t) = z_t(t) - z_{ps}(t) = z_t(t) - \left(\frac{x+y}{2}\right)$$

$$\Rightarrow$$

$$m\ddot{z} + c\dot{z} + kz = -m\left(\frac{\ddot{x}+\ddot{y}}{2}\right)$$

Description of input

$\ddot{x}(t)$ & $\ddot{y}(t)$ are zero mean, stationary, Gaussian random processes with PSD matrix $S(\omega)$.

$$S(\omega)=\begin{bmatrix} S_{xx}(\omega) & S_{xy}(\omega) \\ S_{yx}(\omega) & S_{yy}(\omega) \end{bmatrix}$$

$$S_{xx}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle \left|x_T(\omega)\right|^2\right\rangle$$

$$S_{xy}(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle x_T(\omega)\,y_T^*(\omega)\right\rangle$$

$$S_{xy}^*(\omega)=\lim_{T\to\infty}\frac{1}{T}\left\langle x_T^*(\omega)\,y_T(\omega)\right\rangle = S_{yx}(\omega)$$

$$S_{xy}(\omega)=\left|S_{xy}(\omega)\right|\exp\left[-i\phi_{xy}(\omega)\right]$$

$$=\left|S_{xy}(\omega)\right|\left\{\cos\phi_{xy}(\omega)-i\sin\phi_{xy}(\omega)\right\}$$

$$S_{yx}(\omega)=S_{xy}^*(\omega)$$

$$=\left|S_{xy}(\omega)\right|\left\{\cos\phi_{xy}(\omega)+i\sin\phi_{xy}(\omega)\right\}$$

Force in the left spring

$$F = \frac{k}{2}\left[z_t(t) - x(t)\right]$$

$$= \frac{k}{2}\left[z + \frac{x+y}{2} - x\right]$$

$$= \frac{k}{4}\left[2z - (x-y)\right]$$

Define $g(t) = \frac{4F}{k} = \left[2z - (x-y)\right]$

Question

What is the psd of $g(t)$ and what is its variance?

$$\ddot{z} + 2\eta\omega_n\dot{z} + \omega_n^2 z = -\left(\frac{\ddot{x} + \ddot{y}}{2}\right)$$

$$\Rightarrow$$

$$z_T(\omega) = -H_0(\omega)\frac{1}{2}\left[-\omega^2 x_T(\omega) - \omega^2 y_T(\omega)\right]$$

$$= H_0(\omega)\frac{\omega^2}{2}\left[x_T(\omega) + y_T(\omega)\right]$$

$$H_0(\omega) = \frac{1}{\left(\omega_n^2 - \omega^2\right) + i2\eta\omega\omega_n}$$

$$S_{gg}(\omega) = \lim_{T\to\infty} \frac{1}{T} \left\langle \left| g_T(\omega) \right|^2 \right\rangle$$

$$g_T(\omega) = 2z_T(\omega) - \left[x_T(\omega) + y_T(\omega) \right]$$

$$z_T(\omega) = H_0(\omega) \frac{\omega^2}{2} \left[x_T(\omega) + y_T(\omega) \right]$$

$$\Rightarrow$$

$$S_{gg}(\omega) = S_{xx}(\omega) H_1(\omega) + S_{yy}(\omega) H_2(\omega) + \left| S_{xy}(\omega) \right| H_3(\omega)$$

$$H_1(\omega)=\left\{\frac{1}{\omega^4}+\frac{1}{\left(\omega_n^2-\omega^2\right)^2+\left(2\eta\omega\omega_n\right)^2}+\frac{2\left(\omega^2-\omega_n^2\right)}{\omega^2\left[\left(\omega_n^2-\omega^2\right)^2+\left(2\eta\omega\omega_n\right)^2\right]}\right\}\left|H_f(\omega)\right|^2$$

$$H_2(\omega)=\left\{\frac{1}{\omega^4}+\frac{1}{\left(\omega_n^2-\omega^2\right)^2+\left(2\eta\omega\omega_n\right)^2}-\frac{2\left(\omega^2-\omega_n^2\right)}{\omega^2\left[\left(\omega_n^2-\omega^2\right)^2+\left(2\eta\omega\omega_n\right)^2\right]}\right\}\left|H_f(\omega)\right|^2$$

$$H_3(\omega)=\left\{-\frac{2\cos\phi_{xy}(\omega)}{\omega^4}+\frac{2\cos\phi_{xy}(\omega)}{\left(\omega_n^2-\omega^2\right)^2+\left(2\eta\omega\omega_n\right)^2}+\frac{8\eta\omega\omega_n\sin\phi_{xy}(\omega)}{\omega^2\left[\left(\omega_n^2-\omega^2\right)^2+\left(2\eta\omega\omega_n\right)^2\right]}\right\}\left|H_f(\omega)\right|^2$$

$$\sigma_g^2=\int_0^\infty\left[S_{xx}(\omega)H_1(\omega)+S_{yy}(\omega)H_2(\omega)+\left|S_{xy}(\omega)\right|H_3(\omega)\right]d\omega$$

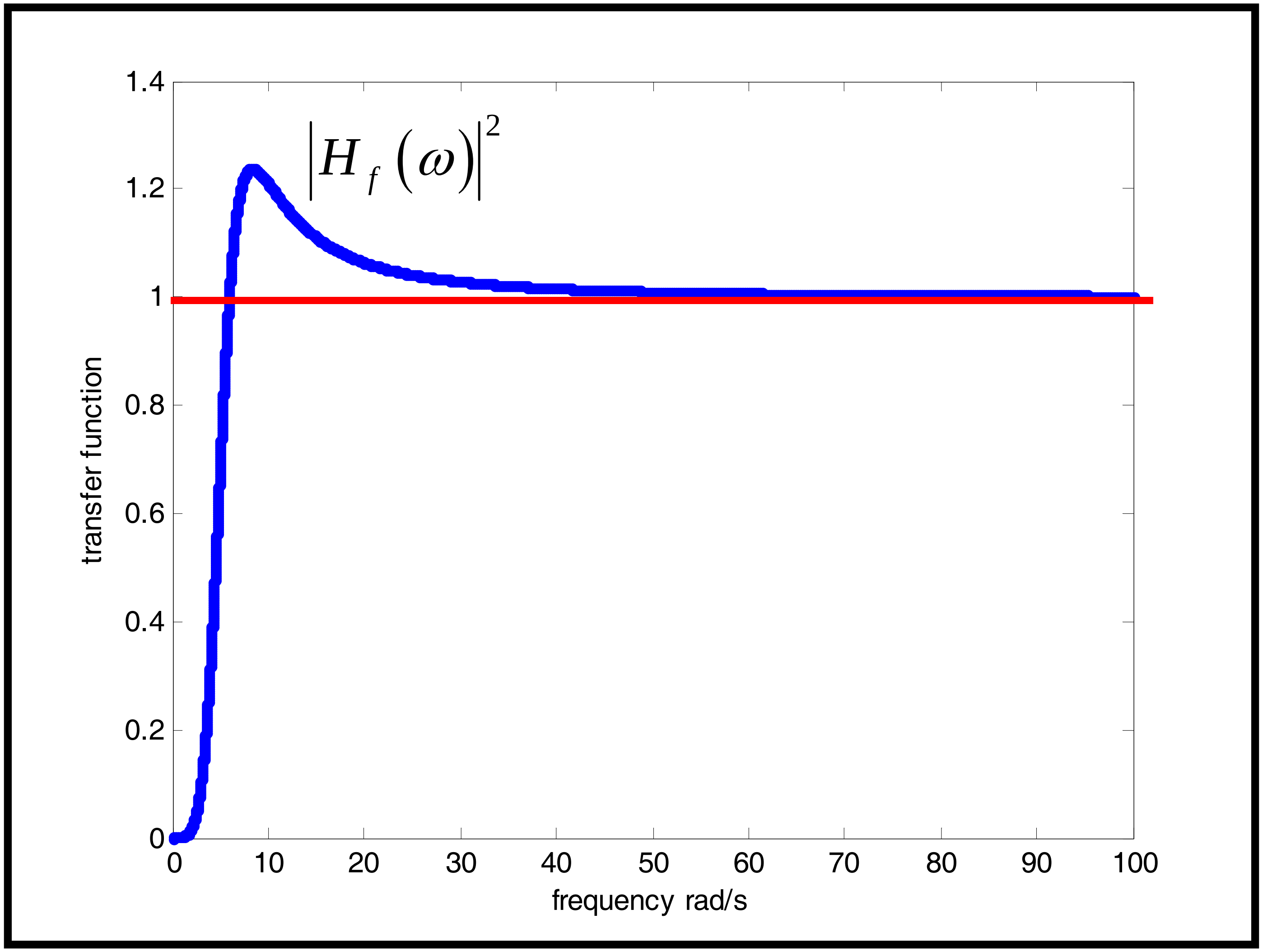
What is the role played by $\left|H_f(\omega)\right|^2$?

$$\left|H_f(\omega)\right|^2 = \frac{\left(\omega/\omega_f\right)^4}{\left[1-\left(\omega/\omega_f\right)^2\right]^2 + 4\varsigma_f^2\left(\omega/\omega_f\right)^2}$$

An artefact to remove singularity at ω=0 in the support displacement.

Typically

$\omega_f = 5.5$ rad/s; $\varsigma_f = 0.53$



1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
100
$\left|H_f(\omega)\right|^2$
transfer function
frequency rad/s

It can be shown that

$$H_1(\omega) = \frac{\left(2\omega^2 - \omega_n^2\right)^2 + \left(2\eta\omega\omega_n\right)^2}{\omega^4\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2\right]} \left|H_f(\omega)\right|^2$$

$$H_2(\omega) = \frac{\omega_n^2\left(\omega_n^2 + 4\eta^2\omega^2\right)}{\omega^4\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2\right]} \left|H_f(\omega)\right|^2$$

$\Rightarrow$

$H_1(\omega) \geq 0 \;\&\; H_2(\omega) \geq 0$

$$\sigma_g^2 = \int_0^\infty \left[S_{xx}(\omega)H_1(\omega) + S_{yy}(\omega)H_2(\omega) + \left|S_{xy}(\omega)\right|H_3(\omega)\right]d\omega$$

Exercise

Determine the nature of $\left|S_{xy}(\omega)\right|$ which produce

- the highest σ_g^2,
- the lowest σ_g^2.

Assume that $\phi_{XY}(\omega)$ is specified.

Note :

The optimal solutions are produced neither by fully coherent nor by incoherent motions.

Exercise

Show that

$$\sigma_g^2 = \sigma_{ps}^2 + \sigma_d^2 + \sigma_c^2$$

with

$$\sigma_{ps}^2 = \int_0^\infty \left\{ \frac{1}{\omega^4} \left[S_{xx}(\omega) + S_{yy}(\omega) - 2\cos\phi_{xy}(\omega) \left| S_{xy}(\omega) \right| \right] \right\} \left| H_f(\omega) \right|^2 d\omega$$

$$\sigma_d^2 = \int_0^\infty \left\{ \frac{1}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2} \right\} \left[S_{xx}(\omega) + S_{yy}(\omega) + 2\cos\phi_{xy}(\omega) \left| S_{xy}(\omega) \right| \right] \left| H_f(\omega) \right|^2 d\omega$$

$$\sigma_c^2 = \int_0^\infty \left\{ \frac{1}{\omega^2 \left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\eta\omega\omega_n\right)^2 \right]} \right\} \left[2\left(\omega^2 - \omega_n^2\right)\left(S_{xx}(\omega) - S_{yy}(\omega)\right) + 8\eta\omega\omega_n \sin\phi_{xy}(\omega) \left| S_{xy}(\omega) \right| \right] \left| H_f(\omega) \right|^2 d\omega$$

with

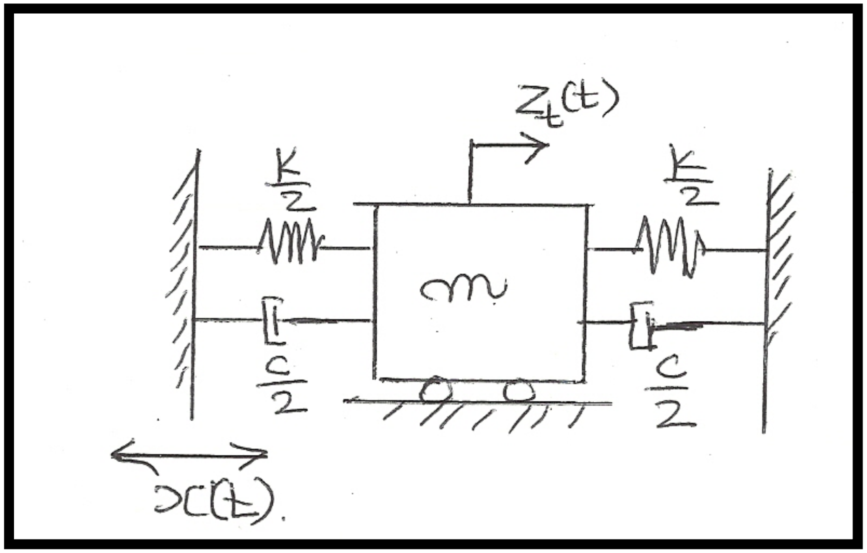
$\sigma_{ps}^2 =$ contribution from pseudo-dynamic component

$\sigma_d^2 =$ contribution from dynamic component

$\sigma_c^2 =$ contribution from correlation between pseudo-dynamic and dynamic components

Large mass concept

Can we replace a given system with support motions by a modified equivalent system in which support displacements as external forces?



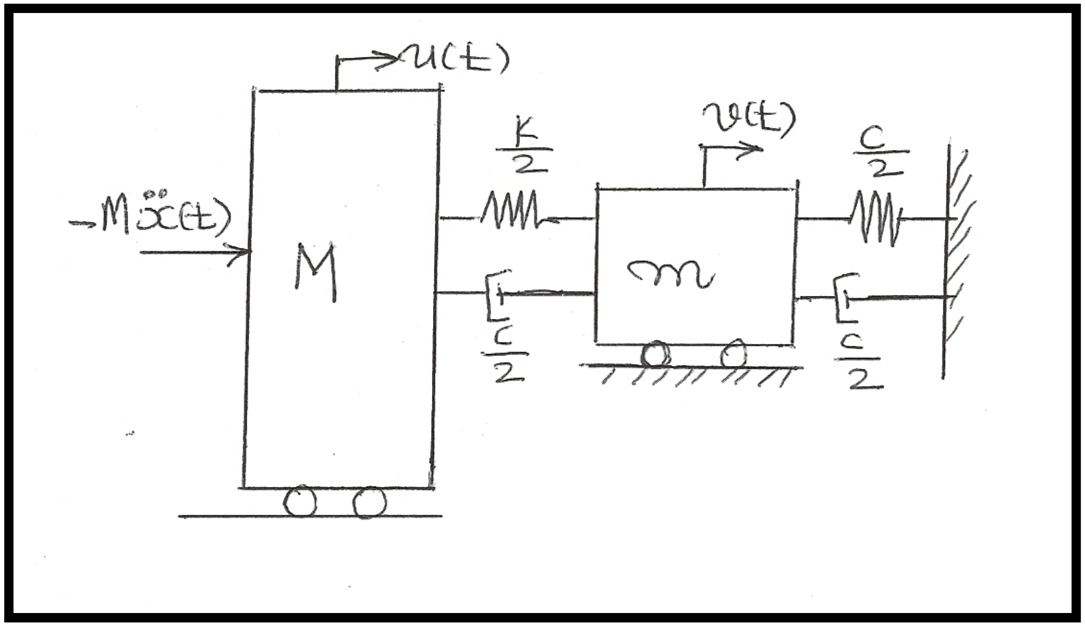
$$x(t) = \exp(i\Omega t)$$

$$m\ddot{z}_t + \frac{c}{2}(\dot{z}_t - \dot{x}) + \frac{c}{2}\dot{z}_t + \frac{k}{2}(z_t - x) + \frac{k}{2}z_t = 0$$

$$m\ddot{z}_t + c\dot{z}_t + kz_t = \frac{c}{2}\dot{x} + \frac{k}{2}x = \frac{1}{2}(i\Omega c + k)\exp(i\Omega t)$$

$$\lim_{t\to\infty} z_t(t) \to H(\Omega)\exp(i\Omega t)$$

$$H(\Omega) = \frac{\frac{1}{2}(i\Omega c + k)}{-m\Omega^2 + i\Omega c + k}$$



$$x(t) = \exp(i\Omega t)$$
$$\ddot{x}(t) = -\Omega^2 \exp(i\Omega t)$$

$$M\ddot{u} + \frac{c}{2}(\dot{u} - \dot{v}) + \frac{k}{2}(u - v) = M\Omega^2 \exp(i\Omega t)$$
$$m\ddot{v} + \frac{c}{2}(\dot{v} - \dot{u}) + \frac{c}{2}\dot{v} + \frac{k}{2}(v - u) + \frac{k}{2}v = 0$$
$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix} + \frac{c}{2} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} + \frac{k}{2} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} M\Omega^2 \exp(i\Omega t) \\ 0 \end{Bmatrix}$$

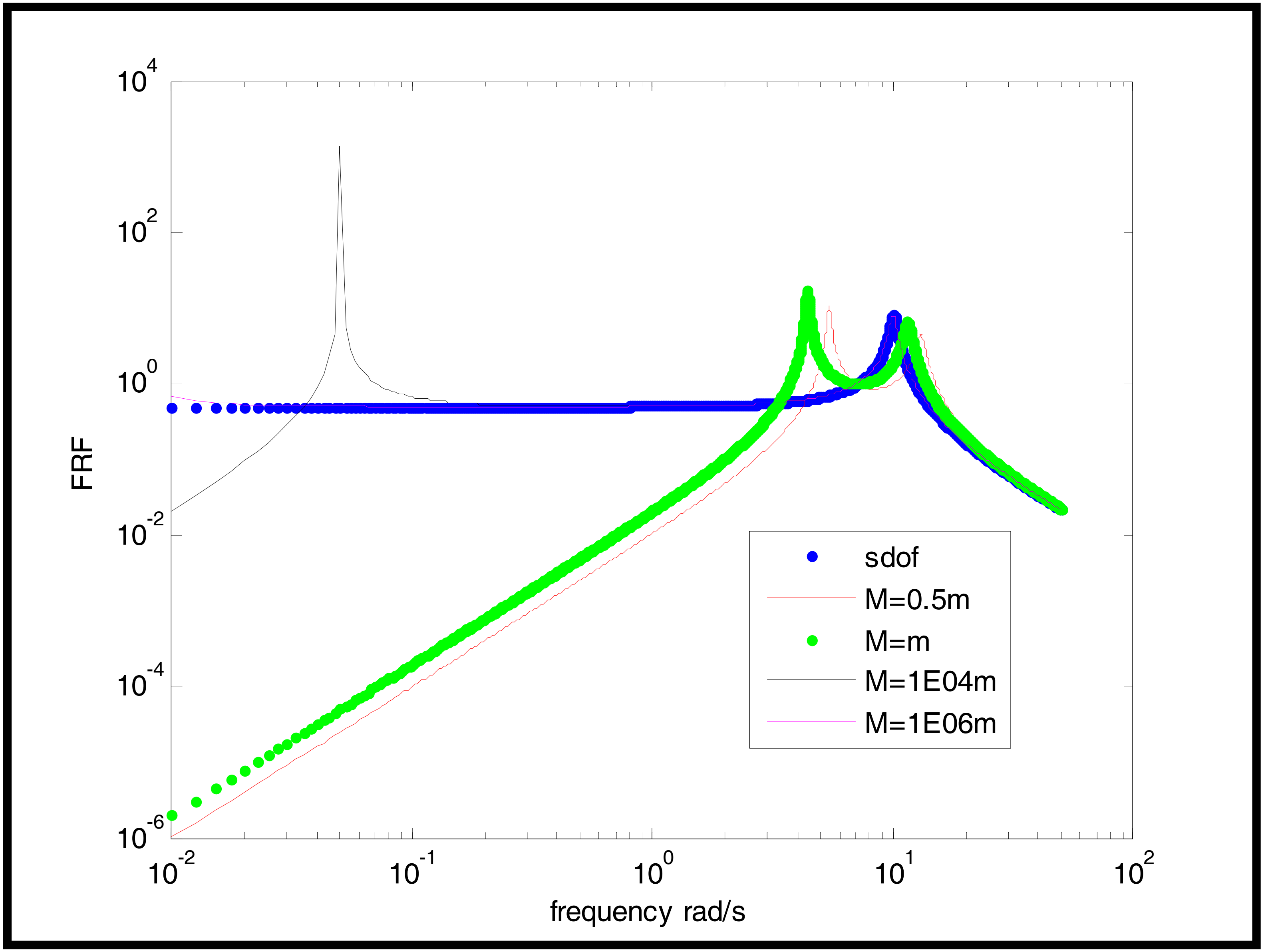
$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix} + c\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} + k\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} M\Omega^2 \exp(i\Omega t) \\ 0 \end{Bmatrix}$$

$$\lim_{t\to\infty}\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} U \\ V \end{Bmatrix}\exp(i\Omega t)$$

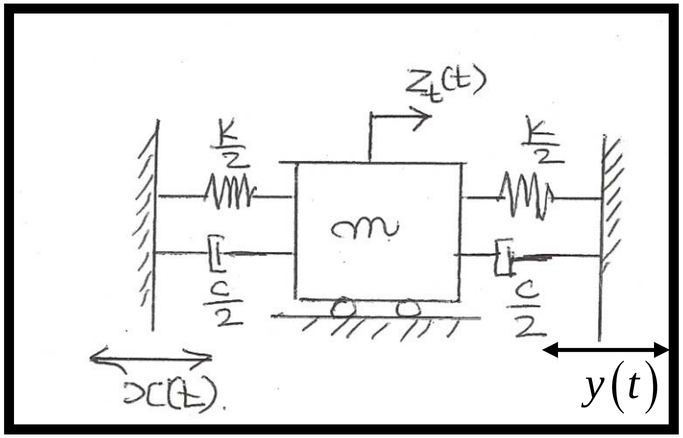
$$\Rightarrow$$

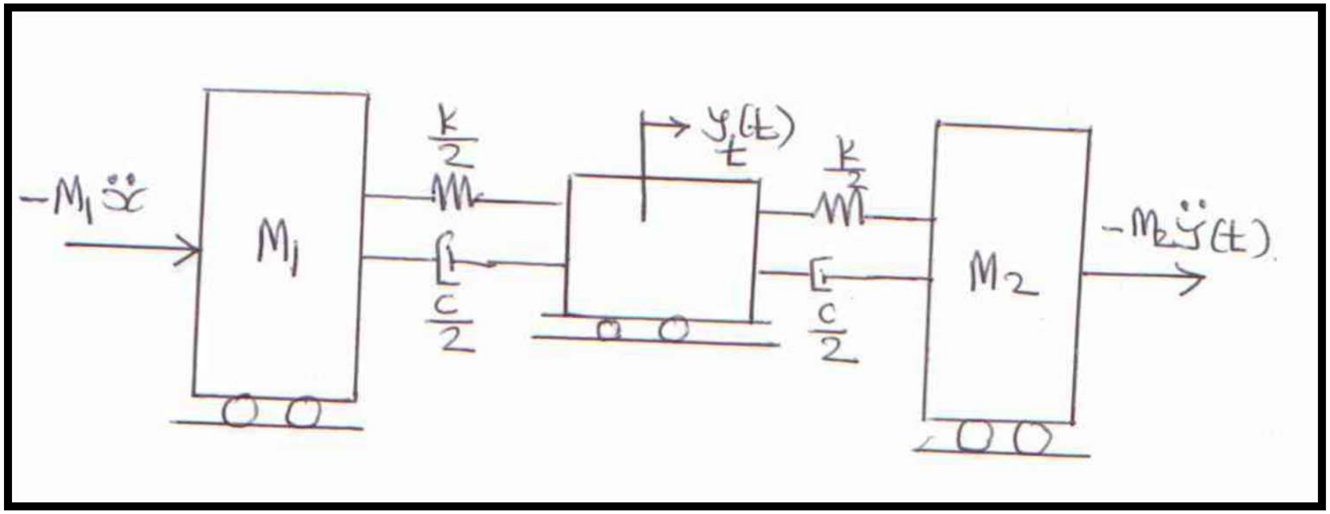
$$\begin{Bmatrix} U \\ V \end{Bmatrix} = \left[-\Omega^2\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix} + i\Omega c\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} + k\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}\right]^{-1}\begin{Bmatrix} M\Omega^2 \\ 0 \end{Bmatrix}$$

$$\lim_{M\to\infty} V(\Omega) \to H(\Omega)$$



FRF
frequency rad/s
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
sdof
M=0.5m
M=m
M=1E04m
M=1E06m





$$\lim_{\substack{M_1\to\infty\\M_2\to\infty}} y_t(t) \to z_t(t)$$

- Contribution from rigid body mode=pseudo-dynamic component
- Contribution from the elastic mode=dynamic component.

pdf of the response process (intutive arguement)

$$m\ddot{x} + c\dot{x} + kx = f(t); x(0) = 0; \dot{x}(0) = 0$$

Let $f(t)$ be a zero mean Gaussian random process

$$x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

$$x(t) \approx \sum_{n=1}^{N} h(t-\tau_n) f(\tau_n) \Delta\tau_n$$

$\Rightarrow$

$x(t)$ is obtained as a sum of Gaussian random variables

$\Rightarrow x(t)$ is Gaussian

Note

Rigorous proof that $x(t)$ is a Gaussian random process is possible using definition of Gaussian random variables in terms of log-characteristic functions and cumulants.

Exercise

Consider

$$\frac{dX}{dt} = \lim_{h \to 0} \frac{X(t+h) - X(t)}{h}$$

Let $X(t)$ be a zero mean Gaussian random process

Let $Y = X_2 - X_1$

$$X_2 = \frac{X(t+h)}{h} \;\&\; X_1 = \frac{X(t)}{h}$$

Show that Y is Gaussian and hence obtain $p_Y(y)$.

Examine the limit of $p_Y(y)$ as $h \to 0$.

pdf of the response process

$$m\ddot{x} + c\dot{x} + kx = f(t); x(0) = 0; \dot{x}(0) = 0$$

Let $f(t)$ be a zero mean Gaussian random process

$\Rightarrow x(t)$ is also a Gaussian random process.

$\Rightarrow$

$$p_x(x;t) = \frac{1}{\sqrt{2\pi\sigma_x^2(t)}} \exp\left[-\frac{1}{2}\left\{\frac{x - m_x(t)}{\sigma_x(t)}\right\}^2\right]; -\infty < x < \infty$$

$$p_{xx}(x_1, x_2; t_1, t_2) \sim N\left[0 \quad \begin{bmatrix} R_{xx}(t_1, t_1) & R_{xx}(t_1, t_2) \\ R_{xx}(t_1, t_2) & R_{xx}(t_2, t_2) \end{bmatrix}\right]$$

$\vdots$

$$p_{\tilde{x}}(\tilde{x}; \tilde{t}) \sim N\left[0 \quad [\boldsymbol{R}]\right]$$

Problem of reliability analysis

$$P\left[x(t) \le \alpha \forall t \in (0,T)\right] = ?$$

Select $\{t_i\}_{i=1}^{n} \in (0,T)$ such that $t_i = i\Delta t$ and $n\Delta t = T$.

Question: can we approximate the given probability by $\iint \cdots \int p_{\tilde{x}}(\tilde{x};\tilde{t})\, d\tilde{x}$ where the integration is carried over the region $\Omega = (x_1 \le \alpha) \cap (x_2 \le \alpha) \cap \cdots (x_n \le \alpha)$?

Not quite!

May be yes, as $n \to \infty$.

Even if this were to be acceptable, we still need to evaluate a multi-fold integral (with dimension $=n$ and set to become large) which by no means is a simple task.

How to proceed?

We need newer descriptions of $x(t)$.