Stochastic Structural Dynamics

Lecture-10

Random vibrations of sdof systems-2

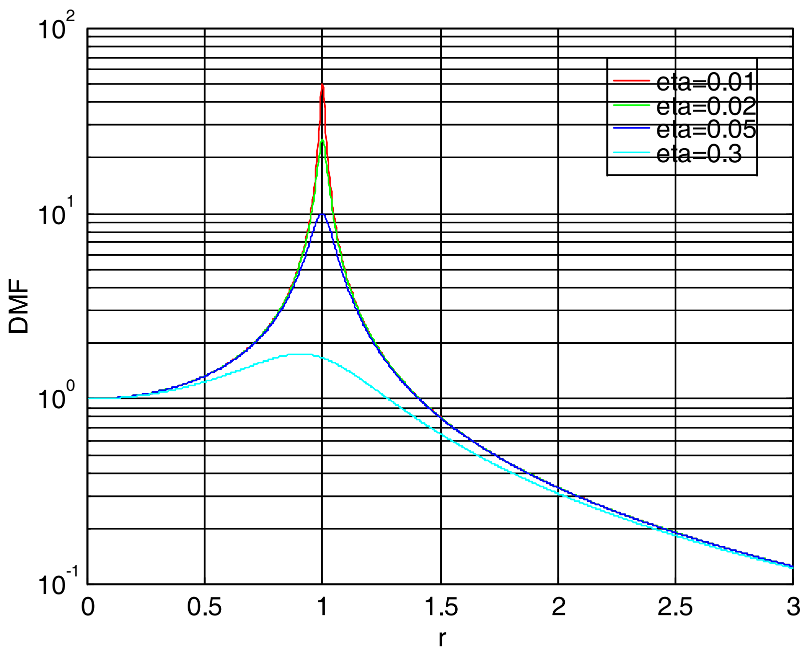
Dr C S Manohar
Department of Civil Engineering
Professor of Structural Engineering
Indian Institute of Science
Bangalore 560 012 India
manohar@civil.iisc.ernet.in

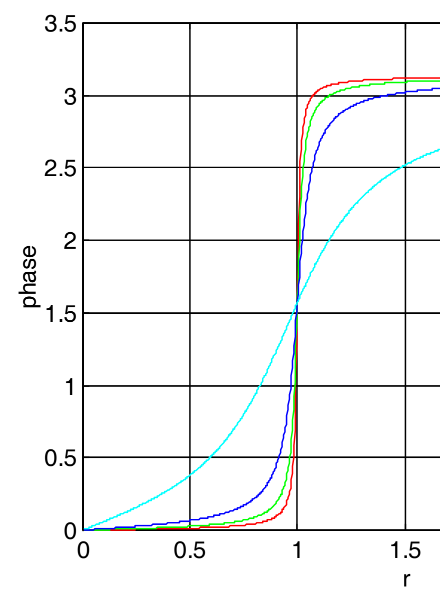
Recall

- SDOF systems under harmonic loads
    - Resonance
    - DMF and phase spectrum
    - Transient and steady state
- Indicial response function
- Impulse response function

$$\lim_{t\to\infty} x(t) \to \left(\frac{P}{K}\right) DMF\cos(\lambda t - \theta)$$

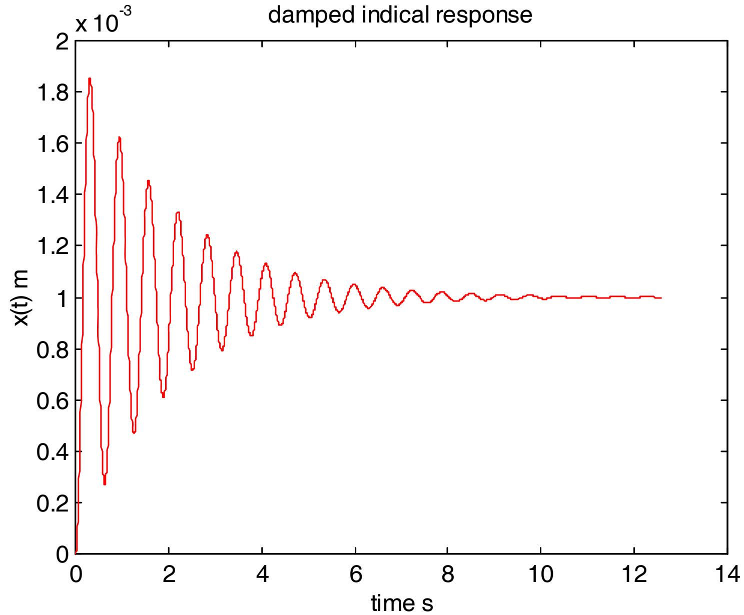
$$f(t) = P\cos\lambda t$$





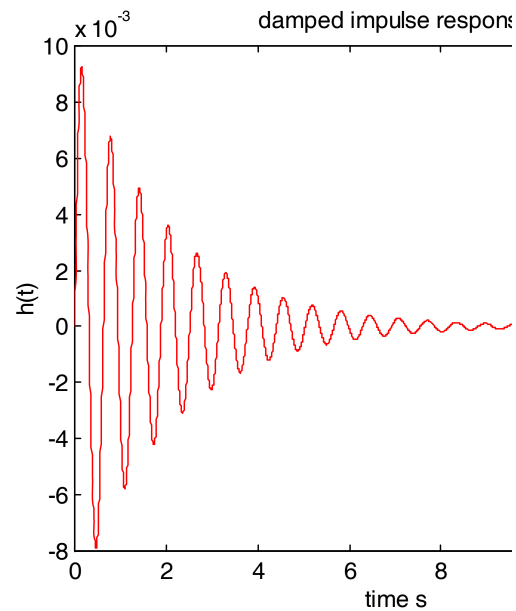
$$f(t) = U(t) \qquad x(t) = G(t)$$

$$f(t) = \delta(t) = \frac{dU}{dt} \qquad x(t) = h(t) = \frac{dG}{dt}$$



$$G(t)=\frac{1}{k}\left[1-\exp(-\eta\omega t)\left(\cos\omega_d t+\frac{\eta}{\sqrt{(1-\eta^2)}}\sin\omega_d t)\right)\right]$$

$$h(t)=\frac{dG}{dt}=\frac{1}{m\omega_d}\exp(-\eta\omega t)\sin\omega_d t$$



<u>Note:</u> the effect of applying impulse at *t=0* is equivalent to imparting an initial velocity at *t=0*

$$m\ddot{h}+c\dot{h}+kh=0$$

$$h(0)=0 \quad \dot{h}(0)=1/m$$

$$h(t)=\exp\left(-\eta\omega t\right)\left(A\cos\omega_d t+B\sin\omega_d t\right)$$

$$h(0)=0 \Rightarrow A=0$$

$$h(t)=B\exp\left(-\eta\omega t\right)\sin\omega_d t$$

$$\dot{h}\left(t\right)=B\left(-\eta\omega\right)\exp\left(-\eta\omega t\right)\sin\omega_d t+B\exp\left(-\eta\omega t\right)\omega_d\cos\omega_d t$$

$$\dot{h}\left(0\right)=\frac{1}{m}=B\omega_d$$

$$\Rightarrow h\left(t\right)=\frac{1}{m\omega_d}\exp\left(-\eta\omega t\right)\sin\omega_d t$$

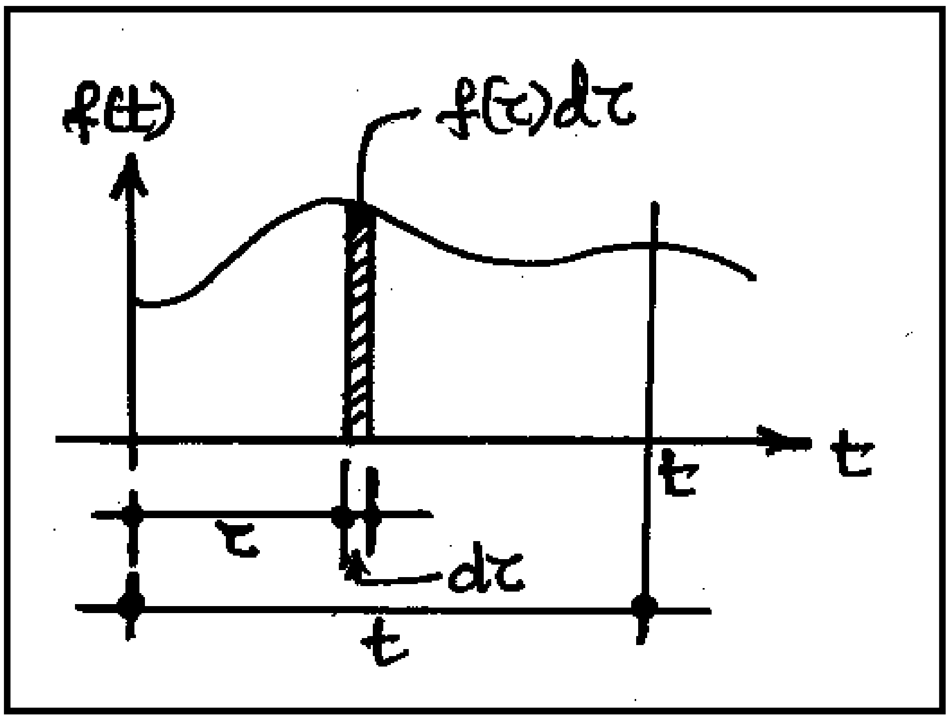
Generalization: n^{th} order differential equation

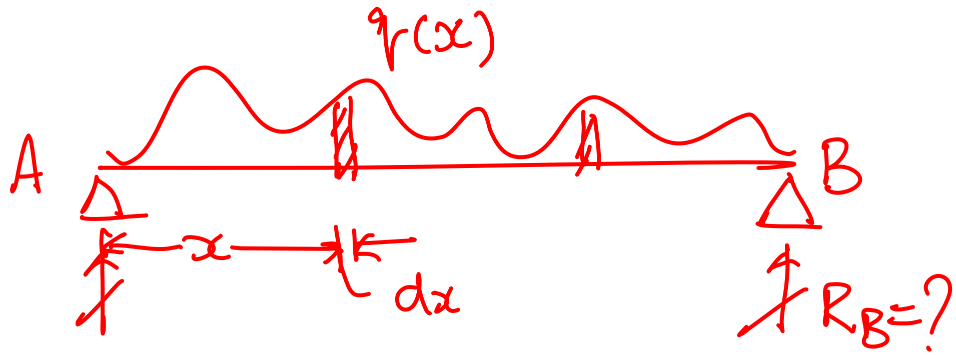
$$\frac{d^n h}{dt^n} + \alpha_{n-1} \frac{d^{n-1} h}{dt^{n-1}} + \cdots + \alpha_1 \frac{dh}{dt} + \alpha_0 h = 0$$

$$h(0) = 0; \frac{dh}{dt}(0) = 0; \cdots; \frac{d^{n-2} h}{dt^{n-2}}(0) = 0; \frac{d^{n-1} h}{dt^{n-1}}(0) = 1$$

Response to arbitrary excitation and Duhamel's integral

$$\boxed{\begin{aligned} & m\ddot{x} + c\dot{x} + kx = f(t) \\ & x(0) = x_0 \\ & \dot{x}(0) = \dot{x}_0 \end{aligned}}$$



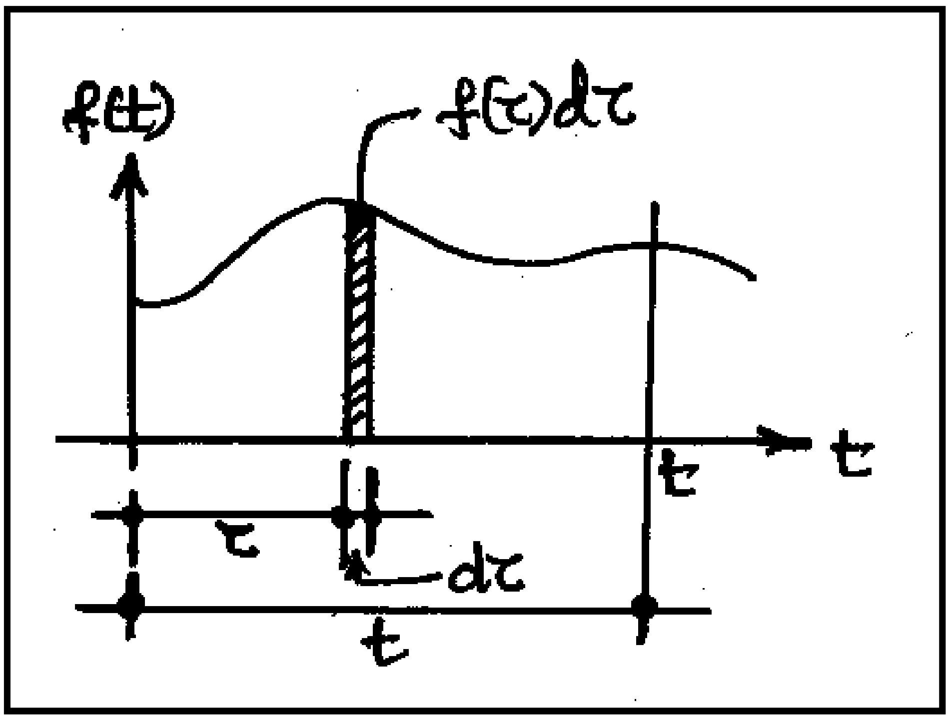


$$dR_B\, l = x\, q(x)\, dx$$

$$R_B = \frac{1}{l} \int_0 x\, q(x)\, dx$$

Duhamel's integral & response to arbitrary excitation

- **Approximate *f(t)* as a train of impulses**
- ***dx(t)*=response at t due to a single impulse at *t*=*τ* of magnitude *f(τ)d τ***
- ***X(t)*=total response due to all the impulses**



$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$x(0) = x_0; \quad \dot{x}(0) = \dot{x}_0$$

$$x(t) = CF + PI = x_{\text{cf}}(t) + x_{\text{pi}}(t)$$

$$dx(t) = h(t-\tau) f(\tau) d\tau$$

$$x_{\text{pi}}(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

$$x(t) = \exp(-\eta\omega t)\left[A\cos\omega_d t + B\sin\omega_d t\right] + \int_0^t h(t-\tau) f(\tau) d\tau$$

$h(t)$
Response at t
due an unit impulse
at $t=0$

Convolution integral
Duhamel's integral

$$x(t) = \exp(-\eta\omega t)\left[A\cos\omega_d t + B\sin\omega_d t\right] + \int_0^t h(t-\tau)\,f(\tau)\,d\tau$$

$$x(0) = x_0 = A$$

$$\dot{x}(t) = -\eta\omega\exp(-\eta\omega t)\left[A\cos\omega_d t + B\sin\omega_d t\right] +$$

$$\exp(-\eta\omega t)\left[-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t\right] + \frac{d}{dt}\int_0^t h(t-\tau)\,f(\tau)\,d\tau$$

Digress:

$$\frac{d}{dx}\int_{g(x)}^{q(x)} f(x,\tau)\,d\tau = \int_{g(x)}^{q(x)} \frac{\partial f(x,\tau)}{\partial x}\,d\tau + \frac{dq}{dx} f\left[x, q(x)\right] - \frac{dg}{dx} f\left[x, g(x)\right]$$

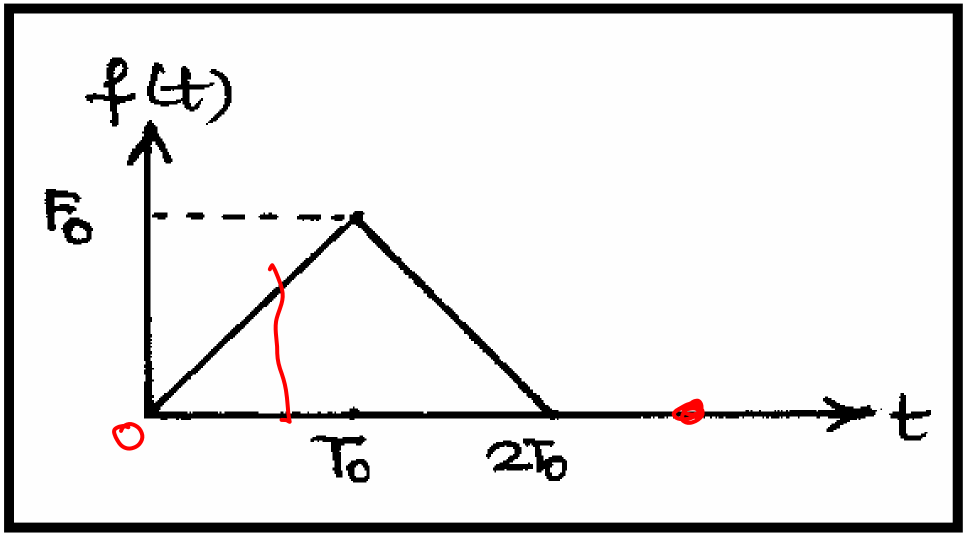
$$x(t) = \exp(-\eta\omega t)\left[x_0 \cos\omega_d t + \frac{\dot{x}_0 + \eta\omega x_0}{\omega\sqrt{(1-\eta^2)}}\cos\omega_d t\right]$$

$$+\int_0^t h(t-\tau)\,f(\tau)\,d\tau$$

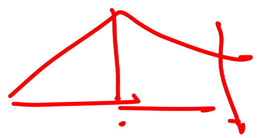
For systems starting from rest, Duhamel's integral provides the complete solution.

<u>Example:</u> A sdof system is excited by the force *f(t)* as shown. Assume that the system starts from rest. Write down the expression for the response valid for any time *t*.

$$
\begin{aligned}
F(t) &= \frac{F_0}{T_0} t \qquad 0 < t < T_0 \\
&= -\frac{F_0}{T_0} t + 2F_0 \qquad T_0 < t < 2T_0 \\
&= 0 \qquad t > 2T_0
\end{aligned}
$$



$$x(t) = \int_0^t \frac{F_0}{T_0} \tau h(t-\tau)\, d\tau \qquad 0 < t < T_0$$



$$= \int_0^{T_0} \frac{F_0}{T_0} \tau h(t-\tau)\, d\tau + \int_{T_0}^{t} \left\{ -\frac{F_0}{T_0}\tau + 2F_0 \right\} h(t-\tau)\, d\tau \qquad T_0 < t < 2T_0$$

$$= \int_0^{T_0} \frac{F_0}{T_0} \tau h(t-\tau)\, d\tau + \int_{T_0}^{2T_0} \left\{ -\frac{F_0}{T_0}\tau + 2F_0 \right\} h(t-\tau)\, d\tau \qquad t > 2T_0$$

Generalization : n^{th} order differential equation

$$\frac{d^n x}{dt^n} + \alpha_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + \alpha_1 \frac{dx}{dt} + \alpha_0 x = f(t)$$

$$x(0) = x_0; \frac{dx}{dt}(0) = x_0^{(1)}; \cdots; \frac{d^{n-2}x}{dt^{n-2}}(0) = x_0^{(2)}; \frac{d^{n-1}h}{dt^{n-1}}(0) = x_0^{(n-1)}$$

Recall : definition of impulse response function

$$\frac{d^n h}{dt^n} + \alpha_{n-1}\frac{d^{n-1}h}{dt^{n-1}} + \cdots + \alpha_1 \frac{dh}{dt} + \alpha_0 h = 0$$

$$h(0) = 0; \frac{dh}{dt}(0) = 0; \cdots; \frac{d^{n-2}h}{dt^{n-2}}(0) = 0; \frac{d^{n-1}h}{dt^{n-1}}(0) = 1$$

$$x(t) = CF + PI$$

$$= \sum_{i=1}^{n} a_i x_i(t) + \int_0^t f(\tau) h(t-\tau)\, d\tau$$

SDOF system under harmonic loads

$$m\ddot{x}+c\dot{x}+kx=\underline{\exp\left(i\lambda t\right)}$$

$$\left(mH(-\lambda^2)m+ciH\lambda+K\right)e^{i\lambda t}=e^{i\lambda t}$$

$$H=\frac{1}{-m\lambda^2+i\lambda c+K}$$

$$\lim_{t\to\infty}x\left(t\right)=H\exp\left(i\lambda t\right)$$

$$\Rightarrow$$

$$H\left(m,c,k,\lambda\right)=\frac{1}{-m\lambda^2+ci\lambda+k} \quad //$$

$$\Rightarrow$$

$$H=\frac{1/m}{\left(\omega^2-\lambda^2\right)+i2\eta\omega\lambda}=\text{Frequency Response Function (FRF)}$$

Relationship between impulse response function (IRF) and frequency response response function (FRF)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(i\omega t)dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-i\omega t)d\omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)d\omega$$

$$x(t) = \int_{0}^{t} h(t-\tau) f(\tau) d\tau$$

$$= \int_{-\infty}^{t} h(t-\tau) f(\tau) d\tau \quad [\because f(t) = 0 \forall t < 0]$$

$$= \int_{-\infty}^{\infty} h(t-\tau) f(\tau) d\tau \quad [\because h(t) = 0 \forall t < 0]$$

Causal systems

$$x(t) = \int_{-\infty}^{\infty} h(t-\tau)\, f(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(t-\tau) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp\left(i\omega\tau\right) d\omega \right\} d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left\{ \int_{-\infty}^{\infty} h\left(t-\tau\right) \exp(i\omega\tau)\, d\tau \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left\{ \int_{-\infty}^{\infty} h\left(u\right) \exp\left[i\omega\left(t+u\right)\right] du \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) H(\omega) \exp(i\omega t)\, d\omega$$

$$\boxed{\Rightarrow X\left(\omega\right) = F(\omega) H(\omega)}$$

$Z = \frac{x}{y}$ $\log z = \log x - \log y$

Convolution in time domain

is equivalent to multiplication in

frequency domain

$$h(t) * f(t) = \int_0^t h(t-\tau) f(\tau)\, d\tau \Leftrightarrow \underline{H(\omega) F(\omega)}$$

One of the advantages of frequency domain analysis in linear vibration analysis

Consider

$$\boxed{\ddot{x} + 2\eta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{m}}$$

Introduce

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(i\omega t)d\omega$$

$$\dot{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega X(\omega)\exp(i\omega t)d\omega$$

$$\ddot{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\omega^2 X(\omega)\exp(i\omega t)d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty}\left[\left\{-\omega^2 + i2\eta\omega\omega_n + \omega_n^2\right\} X(\omega) - \frac{F(\omega)}{m}\right]\exp(i\omega t)\,d\omega = 0$$

$$\Rightarrow X(\omega) = \frac{F(\omega)/m}{(\omega_n^2 - \omega^2) + i2\eta\omega\omega_n} = F(\omega)H(\omega)$$

$$\text{where} \quad H(\omega) = \frac{1/m}{(\omega_n^2 - \omega^2) + i2\eta\omega\omega_n}$$

Furthermore, consider

$$\boxed{\begin{aligned}&\ddot{x}+2\eta\omega_n\dot{x}+\omega_n^2 x=\frac{\exp(i\omega t)}{m}\\&Let\quad x(t)=X(\omega)\exp(i\omega t)\\&\Rightarrow\quad x(t)=\frac{1/m}{(\omega_n^2-\omega^2)+i2\eta\omega\omega_n}\exp(i\omega t)\end{aligned}}$$

Finally

$$\boxed{\begin{aligned}&f(t)=\delta(t)\Rightarrow F(\omega)=1\\&Here\quad X(\omega)=H(\omega)\end{aligned}}$$

$$f(t) \Leftrightarrow F(\omega)$$
$$x(t) \Leftrightarrow X(\omega)$$
$$h(t) \Leftrightarrow H(\omega)$$

$$f(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

h(t)

Convolution

$$F(\omega) \longrightarrow \boxed{\text{LTI}} \longrightarrow X(\omega) = H(\omega) F(\omega)$$

H(ω)

Multiplication

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$$

$$\exp(i\omega t) \longrightarrow \boxed{\text{LTI}} \longrightarrow H(\omega)\exp(i\omega t)$$

Input-output relations for linear time invariant systems

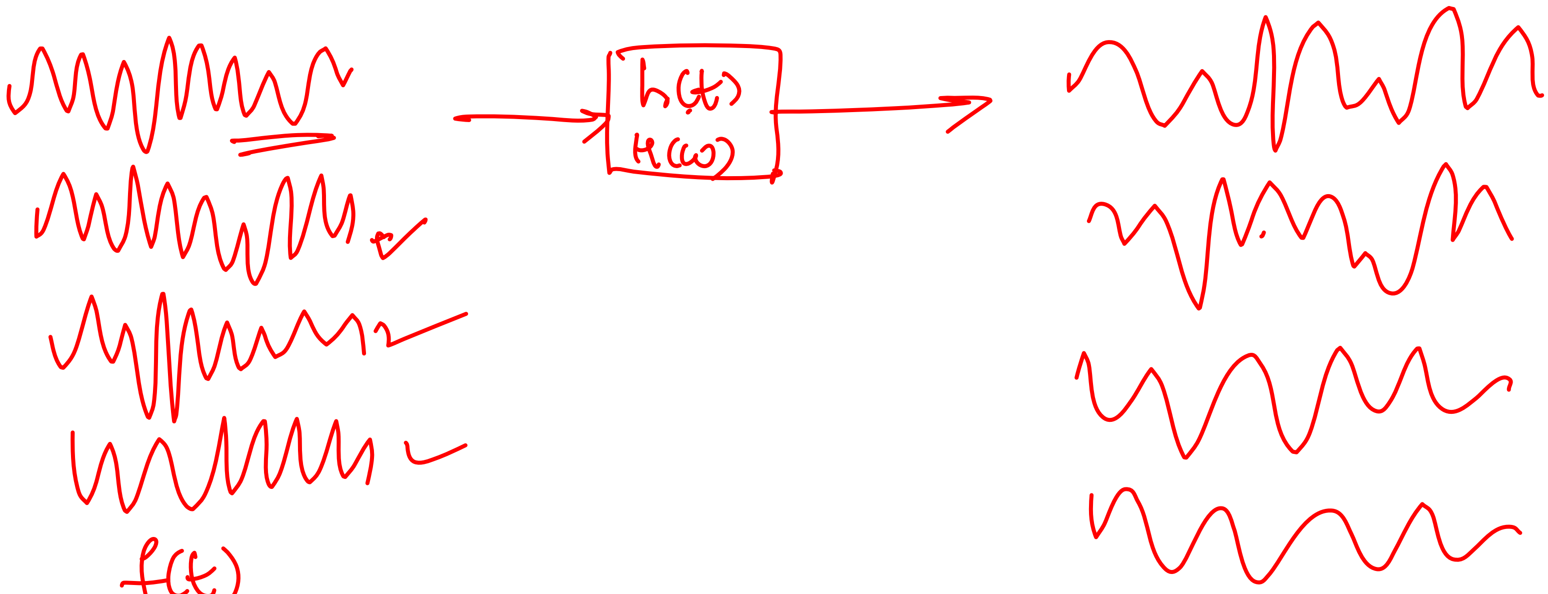
Randomly excited dynamical systems

$$\boxed{\begin{gathered} m\ddot{x} + c\dot{x} + kx = f(t) \\ x(0) = x_0; \dot{x}(0) = \dot{x}_0 \end{gathered}}$$

?

$f(t)$: a random process

- Completely specified
- Not necessarily stationary
- Not necessarily Gaussian



$f(t)$ RP

$x(t)$ RP

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$x(0) = x_0$$

$$\dot{x}(0) = \dot{x}_0$$

Problem of Uncertainty Propagation

Given complete description of $f(t)$ can we obtain the complete description of response process $x(t)$?

Samples of $f(t), x(0), \dot{x}(0)$

$h(t)$

$H(\omega)$

Samples of $x(t) = ?$

$$f(t), x(0), \dot{x}(0)$$

$$p_{\tilde{f}}(\tilde{\alpha}; \tilde{t})$$

$$m_f(t) = \langle f(t) \rangle$$

$$C_{ff}(t_1, t_2) =$$

$$\langle [f(t_1) - m_f(t_1)][f(t_2) - m_f(t_2)] \rangle$$

$\vdots$

$$x(t) = ?$$

$$p_{\tilde{x}}(\tilde{\beta}; \tilde{t}) = ?$$

$$m_x(t) = ?$$

$$C_{xx}(t_1, t_2) = ?$$

$\vdots$

Input output relations in time domain

$$x(t) = \exp(-\eta\omega t)\left[x_0 \cos\omega_d t + \frac{\dot{x}_0 + \eta\omega x_0}{\omega\sqrt{(1-\eta^2)}} \cos\omega_d t \right] + \int_0^t h(t-\tau) f(\tau) d\tau$$

Given the ensemble of $f(t)$ we can determine the ensemble of $x(t)$ using this relation

Propagation of uncertainty in inputs to the outputs follows laws of mechanics.

Mean response

$$\langle x(t)\rangle = \left\langle \exp(-\eta\omega t)\left[x_0\cos\omega_d t + \frac{\dot{x}_0 + \eta\omega x_0}{\omega\sqrt{(1-\eta^2)}}\cos\omega_d t\right]\right\rangle$$

$$+\left\langle \int_0^t h(t-\tau) f(\tau) d\tau \right\rangle$$

$$\langle x(t)\rangle = \exp(-\eta\omega t)\left[x_0\cos\omega_d t + \frac{\dot{x}_0 + \eta\omega x_0}{\omega\sqrt{(1-\eta^2)}}\cos\omega_d t\right] + \int_0^t h(t-\tau)\langle f(\tau)\rangle d\tau$$

$$\underline{m_x(t)} = \exp(-\eta\omega t)\left[x_0\cos\omega_d t + \frac{\dot{x}_0 + \eta\omega x_0}{\omega\sqrt{(1-\eta^2)}}\cos\omega_d t\right] + \int_0^t h(t-\tau)\underline{m_f(\tau)}\, d\tau$$

Knowledge of mean of the excitation process helps us to determine the mean of the response process.

Without loss of generality we will assume that the system starts from rest and Mean of *f(t)* is zero.

$$\Rightarrow m_X(t) = 0$$

$$x(t)=\int_0^t h(t-\tau)f(\tau)\,d\tau$$

$$\langle x(t_1)x(t_2)\rangle=\left\langle \int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)\underline{f(\tau_1)}h(t_2-\tau_2)\underline{f(\tau_2)}\,d\tau_1 d\tau_2\right\rangle$$

$$\Rightarrow R_{xx}(t_1,t_2)=\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_1-\tau_1)\underline{\underline{\langle f(\tau_1)f(\tau_2)\rangle}}\,d\tau_1 d\tau_2$$

$$\underline{\underline{R_{xx}(t_1,t_2)}}=\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_1-\tau_1)\underline{\underline{R_{ff}(\tau_1,\tau_2)}}\,d\tau_1 d\tau_2$$

Knowledge of autocovariance of the excitation process helps us to determine the autocovariance of of the response process.

$$R_{xx}(t_1,t_2)=\int_0^{t_1}\int_0^{t_2} h(t_1-\tau_1)h(t_2-\tau_2)R_{ff}(\tau_1,\tau_2)\,d\tau_1 d\tau_2$$

$$\text{Let } t_1=t_2=t$$

$$R_{xx}(t,t)=\sigma_x^2(t)=\int_0^{t}\int_0^{t} h(t-\tau_1)h(t-\tau_2)R_{ff}(\tau_1,\tau_2)\,d\tau_1 d\tau_2$$

- Knowledge of the variance of the input is not adequate to determine the variance of the output.

- Knowledge of autocovariance of the excitation process is needed to determine the variance of of the response process.

$$x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

$$\left\langle x(t_1) x(t_2) x(t_3) \right\rangle = \left\langle \int_0^{t_1}\int_0^{t_2}\int_0^{t_3} h(t_1-\tau_1) f(\tau_1) h(t_2-\tau_2) f(\tau_2) h(t_3-\tau_3) f(\tau_3) d\tau_1 d\tau_2 d\tau_3 \right\rangle$$

$$= \int_0^{t_1}\int_0^{t_2}\int_0^{t_3} h(t_1-\tau_1) h(t_2-\tau_2) h(t_3-\tau_3) \left\langle f(\tau_1) f(\tau_2) f(\tau_3) \right\rangle d\tau_1 d\tau_2 d\tau_3$$

Knowledge of third order moment of input is adequate
to determine the third order moment of the response process

In general for LTI systems
knowledge of nth order moment of input is adequate
to determine the nth order moment of the response process

Note: this is not true for nonlinear systems

Example

$$\dot{x} + \alpha x = f(t)$$

$\frac{1}{2}$ dof

$$x(0) = x_0$$

$$f(t) = \text{zero mean, Gaussian white noise}$$

$$\langle f(t) \rangle = 0; \langle f(t_1) f(t_2) \rangle = I_0 \delta(t_2 - t_1)$$

Impulse response function

$$\dot{x} + \alpha x = 0$$

$$x(0) = 1$$

$$x(t) = A \exp(-\alpha t) \Rightarrow$$

$$h(t) = \exp(-\alpha t)$$

$$\dot{x} + \alpha x = f(t)$$

$$\dot{x} + \alpha x = 0$$

$$x(t) = e^{st}$$

$$(s + \alpha) e^{st} = 0$$

$$x(0) = x_0$$

$$\Rightarrow$$

$$x(t) = a \exp(-\alpha t) + \int_0^t h(t - \tau) f(\tau)\, d\tau$$

$$x(0) = x_0 \Rightarrow a = x_0$$

Let $x_0 = 0$

$$x(t) = \int_0^t \exp\left[-\alpha (t - \tau)\right] f(\tau)\, d\tau$$

$$x(t)=\int_0^t \exp\left[-\alpha\left(t-\tau\right)\right]f\left(\tau\right)d\tau$$

$$\left\langle x(t)\right\rangle=\int_0^t \exp\left[-\alpha\left(t-\tau\right)\right]\left\langle f\left(\tau\right)\right\rangle d\tau=0$$

$$\left\langle x(t_1)x(t_2)\right\rangle=\int_0^{t_1}\int_0^{t_2} \exp\left[-\alpha\left(t_1-\tau_1\right)\right]\exp\left[-\alpha\left(t_2-\tau_2\right)\right]\left\langle f\left(\tau_1\right)f\left(\tau_2\right)\right\rangle d\tau_1 d\tau_2$$

$$=\int_0^{t_1}\int_0^{t_2} \exp\left[-\alpha\left(t_1-\tau_1\right)\right]\exp\left[-\alpha\left(t_2-\tau_2\right)\right]I_0\delta\left(\tau_1-\tau_2\right)d\tau_1 d\tau_2$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

$$=I_0\int_0^{t_2} \exp\left[-\alpha\left(t_1-\tau_2\right)\right]\exp\left[-\alpha\left(t_2-\tau_2\right)\right]d\tau_2$$

$$=I_0\exp\left[-\alpha\left(t_1+t_2\right)\right]\int_0^{t_2}\exp\left[2\alpha\tau_2\right]d\tau_2$$

$$R_{xx}(t_1,t_2) = I_0 \exp\left[-\alpha(t_1+t_2)\right]\int_0^{t_2} \exp\left[2\alpha\tau_2\right] d\tau_2$$

$$= I_0 \exp\left[-\alpha(t_1+t_2)\right]\left[\frac{\exp(2\alpha\tau)}{2\alpha}\right]_0^{t_2}$$

$$= \frac{I_0}{2\alpha}\left\{\exp\left[\alpha(t_2-t_1)\right]-\exp\left[-\alpha(t_1+t_2)\right]\right\}$$

$$\Rightarrow$$

$$\sigma_x^2(t) = \frac{I_0}{2\alpha}\left\{1-\exp\left[-2\alpha t\right]\right\}$$

What happens for large times?

$$R_{xx}(t_1,t_2) = \frac{I_0}{2\alpha}\left\{\exp\left[-\alpha(t_2-t_1)\right] - \exp\left[-\alpha(t_1+t_2)\right]\right\}$$

$$\lim_{\substack{t_1\to\infty \\ t_2\to\infty \\ (t_2-t_1)=\tau}} R_{xx}(t_1,t_2) \to \frac{I_0}{2\alpha}\exp\left[-\alpha|\tau|\right] = R_{xx}(\tau)$$

$$\Rightarrow \lim_{\substack{t_1\to\infty \\ t_2\to\infty \\ (t_2-t_1)=0}} \sigma_x^2 \to \frac{I_0}{2\alpha} \;//$$

Remarks

- For small times the response is a nonstationary random process and is dependent on initial conditions
- As time becomes large the response becomes a stationary random process.

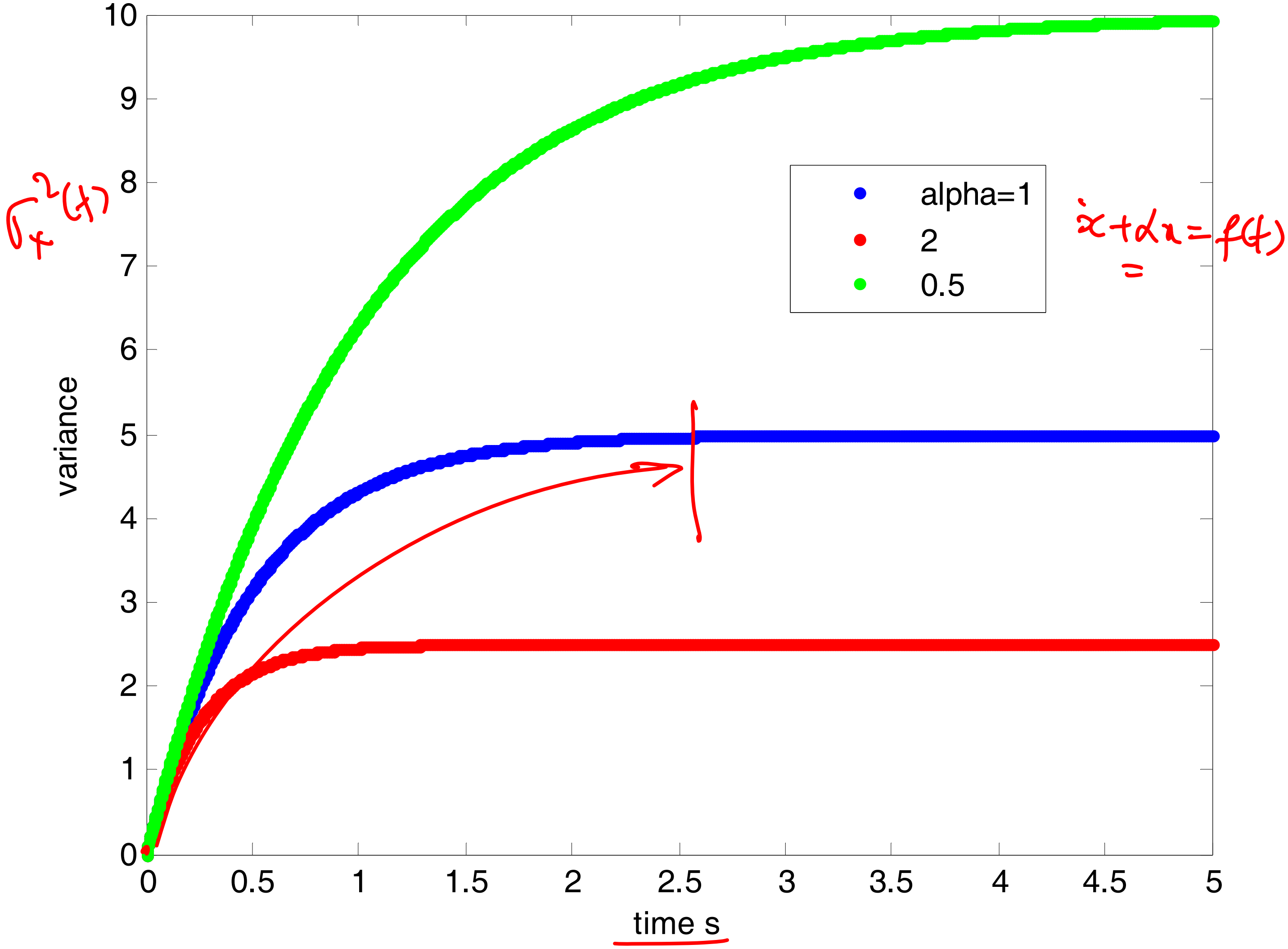
$$\text{Stochastic transient state} \equiv \text{Nonstationary response}$$

$$\text{Stochastic steady state} \equiv \text{stationary response}$$

$$\text{Mean} = 0$$

$$\text{Autocovariance is a function of time lag}$$

$$\text{Variance is time invariant}$$


$\sigma_x^2(t)$
variance
time s
alpha=1
2
0.5
$\dot{x}+\alpha x=f(t)$
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
10
9
8
7
6
5
4
3
2
1
0

Deterministic steady state versus stochastic steady state

Response under harmonic excitation

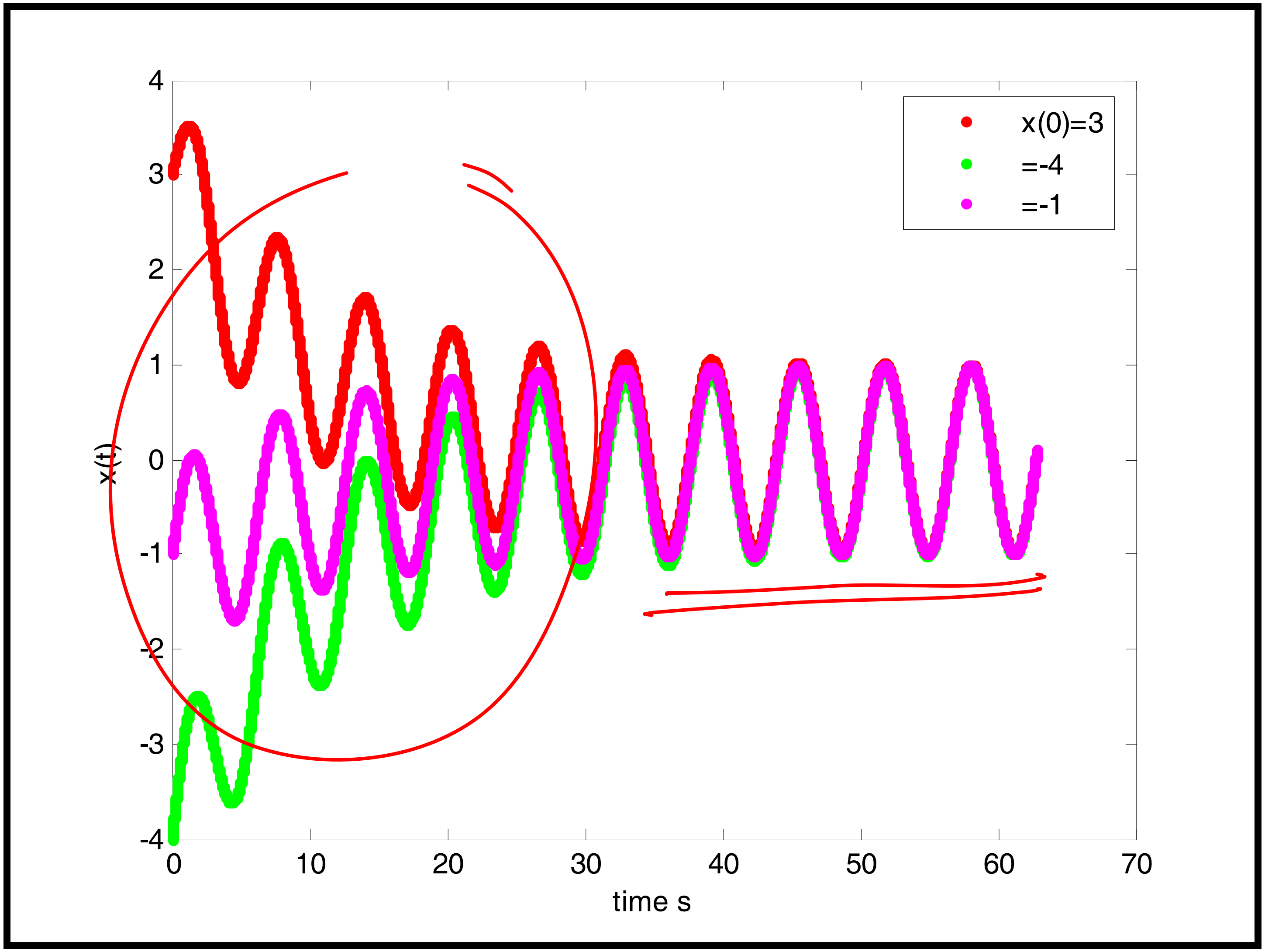
$$\dot{x} + \alpha x = \cos(\lambda t)$$

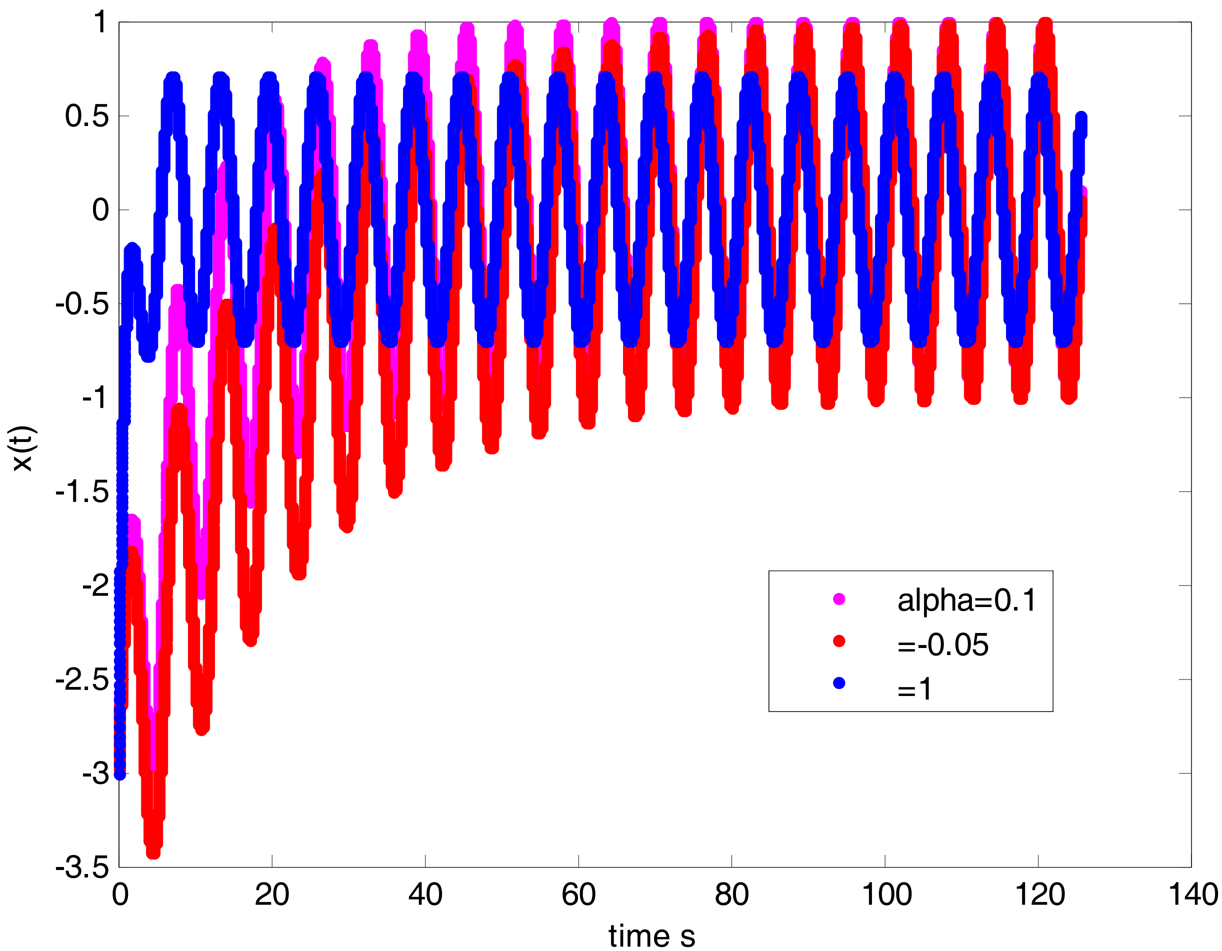
$$x(0) = x_0$$

$$x(t) = \left(x_0 - \frac{\alpha}{\alpha^2 + \lambda^2} \right) \exp(-\alpha t) + \frac{\alpha \cos \lambda t + \lambda \sin \lambda t}{\alpha^2 + \lambda^2}$$

For small times, response is aperiodic and depends on initial conditions.

For large times, response becomes periodic -harmonic at the driving frequency.


4
3
2
1
0
-1
-2
-3
-4
x(t)
0
10
20
30
40
50
60
70
time s
x(0)=3
=-4
=-1



x(t)
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
0
20
40
60
80
100
120
140
time s
alpha=0.1
=-0.05
=1